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A Δ-Convergence Theorem in CAT (0) Spaces

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Abstract: In this study, researchers introduce a new iterative process for two non-expansive map satisfying the condition (II) in CAT (0) space and establish Δ-convergence theorem for the proposed process under condition.

Key words: Non-expansive mapping, CAT (0) spaces, fixed point, Hilbert space, Δ-convergence, Iran

INTRODUCTION

The study of CAT (0) spaces was initiated by Kirk (2003). He shows that every non-expansive single-valued mapping defined on a bounded closed convex subset of a complete CAT (0) space always has a fixed point. The fixxed point theorems in CAT (0) spaces has applications in graph theory, biology and computer science (Bartolini *et al*., 1999; Dhompongsa and Panyanak, 2008; Espinola and Kirk, 2006; Park, 2010). Dhompongsa *et al*. (2005, 2007) obtained some convergence theorems for different iterations for non-expansive single-valued mappings in CAT (0) spaces. Many researchers introduced and studied kinds of iterative for single and multi-valued mappings in Hilbert spaces (Kirk, 2004; Laowang and Panyanak, 2010; Panyanak, 2007; Razani and Salahifard, 2010; Sastry and Babu, 2005).

The purpose of this research is to study the iterative scheme define as follow: Let, C be a closed convex subset of a complete CAT (0) space and $T_1, T_2 : C \rightarrow C$ be two non-expansive mappings with $F(T_1) \cap F(T_2) \neq \phi$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$:

$$\begin{aligned} z_n &= \gamma_n T_1 x_n \oplus (1-\gamma_n) x_n \\ y_n &= \beta_n T_2 x_n \oplus (1-\beta_n) x_n \\ x_n + 1 &= \alpha_n T_1 y_n \oplus (1-\alpha_n) T_2 z_n \end{aligned} \quad (1)$$

For all $n \geq 1$ where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(a, b) \subset (0, 1)$. Researchers show that the sequence $\{x_n\}$ is Δ-convergence to a common fixed point T_1 and T_2.

CAT (0) SPACES

Let, X, d be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map γ from a closed interval $[0, l] \subset R$ to X such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t), \gamma(t')) = |t-t'|$, for all $t, t' \in [0, l]$. In particular is an isometry and $d(x, y) = l$. The image γ is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by x, y. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y, for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1\text{-}x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1\text{-}x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1\text{-}x_3) = \Delta(\bar{x}_1 - \bar{x}_3)$ in the euclidean plane R^2 such that $d_{R^2}(\bar{x}, \bar{y}) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a CAT (0) space (Bridson and Hafliger, 1999) if all geodesic triangles of appropriate size satisfy the following comparison ax-iom. Let, Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT (0) inequality if for all $x, y \in \Delta$ and all comparison points $(\bar{x}, \bar{y}) \in \bar{\Delta}$ $d(x, y) \leq d_{R^2}(\bar{x}, \bar{y})$. It is known that in a CAT (0) space, the distance function is convex (Bridson and Hafliger, 1999). Complete CAT (0) spaces are often called Hadamard spaces. Finally, researchers observe that if x, y_1, y_2 are points of a CAT (0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$ which researchers will denote by $\frac{y_1 \oplus y_2}{2}$ then the CAT (0) inequality implies:

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2 \quad (2)$$

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A geodesic metric space is a CAT (0) space if and only if it satisfies inequality Eq. 2 (which is known as the CN inequality).

Let X be a complete CAT (0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$ set:

$$r(x,\{x_n\}) = \text{Lim sup}_{n\to\infty} d(x,x_n)$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by:

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}$$

And the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the:

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$$

Also, a sequence $\{x_n\}$ in a metric space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, researchers write $\Delta\text{-Lim}_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Two mappings T_1, T_2: $C \to X$ is said to satisfy condition (II) if there exist a non-decreasing function f: $[0, \infty] \to [0, \infty]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that:

$$\sum_{i=1}^{2} d(x_i, T_i x) \geq f(d(x, F(T_1) \cap F(T_2))$$

The following lemma will be useful for proving the main results in this study.

Lemma 1: Let, (X, d) be a CAT (0) space (Dhompongsa *et al*., 2005). For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that:

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1\text{-}t)d(x, y)$$

Researchers use the notation $(1\text{-}t)\, x \oplus ty$ for the unique z.

Lemma 2: Let, (X, d) be a CAT (0) space (Dhompongsa *et al*., 2005). Then:

$$d((1-t)x \oplus ty,\ z)^2 \leq (1-t)d(x,\ z)^2 + td(y,\ z)^2 - t(1-t)d(x,\ y)^2$$

For all $t \in [0, 1]$ and $x, y, z \in X$.

Lemma 3: Every bounded sequence in a complete CAT (0) space always has a Δ-convergent subsequence (Kirk and Panyanak, 2008).

Lemma 4: If C is a closed convex subset of complete CAT (0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C (Dhompongsa *et al*., 2007).

Lemma 5: Let, C be a closed convex subset of a complete CAT (0) space X and let T: $C \to X$ be a non-expansive mapping. Then conditions $\{x_n\}$ Δ-converges to x and $\text{Lim}_{n\to\infty} d(x_n, Tx_n) = 0$, imply $x \in C$ and $Tx = x$ (Dhompongsa *et al*., 2005).

Δ-CONVERGENCE THEOREM

Here, the main result is presented.

Theorem 1: Let C be a non-empty closed convex subset of a complete CAT (0) space X and T_1, T_2: $C \to C$ be non-expansive mappings with $F(T_1) \cap F(T_2) \neq \phi$. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by Eq. 1. Then $\lim_{n\to\infty} d(x_n, x^*)$ exists, for all $x^* \in F(T_1) \cap F(T_2)$.

Proof: Let, $x^* \in F(T_1) \cap F(T_2)$ researchers have:

$$d(z_n, x^*) = d(\gamma_n T_1 x_n \oplus (1-\gamma n)x_n, x^*) \leq \gamma_n d(T_1 x_n, x^*) + (1-\gamma_n)d(x_n, x^*)$$

Then:

$$d(z_n, x^*) \leq \gamma_n d(x_n, x^*) + (1-\gamma_n)d(x_n, x^*) = d(x_n, x^*) \quad (3)$$

On the other hand:

$$d(y_n, x^*) = d(\beta_n T_2 x_n \oplus (1-\beta_n)x_n, x^*) \leq \beta_n d(T_2 x_n, x^*) + (1-\beta_n)\, d(x_n, x^*)$$

$$d(y_n, x^*) \leq \beta_n d(x_n, x^*) + (1-\beta_n)d(x_n, x^*) = d(x_n, x^*) \quad (4)$$

By Eq. 3 and 4, researchers have:

$$\begin{aligned} d(x_n+1, x^*) &= d(\alpha_n T_1 y_n \oplus (1-\alpha_n)T_2 z_n, x^*) \\ &\leq \alpha_n d(T_1 y_n, T_1 x^*) + (1-\alpha_n)d(T_2 z_n, T_2 x^*) \\ &\leq \alpha_n d(y_n, x^*) + (1-\alpha_n)d(z_n, x^*) \\ &= d(x_n, x^*) \end{aligned}$$

Consequently, $d(x_n+1, x^*) \leq d(x_n, x^*)$. Then $d(x_n, x^*) \leq d(x_1, x^*)$ for all $n \geq 1$. This implies that $\{d(x_n, x^*)\}$ is bounded and decreasing. Hence, $\lim_{n\to\infty} d(x_n, x^*)$ exists.

Theorem 2: Let, C be a non-empty closed convex subset of a complete CAT (0) space X and T_1, T_2: $C \to C$ be non-expansive mappings and satisfying condition (II) with $\{F(T_1) \cap (T_2) \neq \phi$. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by Eq. 1. Then the sequence $\{x_n\}$ convergence to common fixed point $F(T_1) \cap F(T_2)$.

Proof: Let, $x^* \in F(T_1) \cap F(T_2)$:

$$d(z_n, x^*)^2 = d(\gamma_n T_1 x_n \oplus (1-\gamma_n) x_n, x^*)^2 \leq \gamma_n d(T_1 x_n, x^*)^2 + (1-\gamma_n) d(x_n, x^*)^2 - \gamma_n (1-\gamma_n) d(T_1 x_n, x_n)^2 = \gamma_n d(T_1 x_n, T_1 x^*)^2 + (1-\gamma_n) d(x_n, x^*)^2 - \gamma_n (1-\gamma_n) d(T_1 x_n, x_n)^2$$

Hence:

$$d(z_n, x^*)^2 \leq d(x_n, x^*)^2 - \gamma_n (1-\gamma_n) d(T_1 x_n, x^*)^2 \tag{5}$$

Also:

$$d(y_n, x^*)^2 = d(\beta_n T_2 x_n \oplus (1-\beta_n) x_n, x^*)^2 \leq \beta_n d(T_2 x_n, x^*)^2 + (1-\beta_n) d(x_n, x^*)^2 - \beta_n (1-\beta_n) d(T_2 x_n, x_n)^2 \leq \beta_n d(T_2 x_n, x^*)^2 + (1-\beta_n) d(x_n, x^*)^2 - \beta_n (1-\beta_n) d(T_2 x_n, x_n)^2$$

Hence:

$$d(y_n, x^*)^2 \leq d(x_n, x^*)^2 - \beta_n (1-\beta_n) d(T_2 x_n, x^*)^2 \tag{6}$$

On the other hand, researchers have:

$$d(x_n + 1, x^*)^2 = d(\alpha_n T_1 y_n \oplus (1-\alpha_n) T_2 z_n, x^*)^2 \leq \alpha_n d(T_1 y_n, x^*)^2 + (1-\alpha_n) d(T_2 z_n, x^*)^2 - \alpha_n (1-\alpha_n) d(T_1 y_n, T_2 z_n)^2 \leq \alpha_n d(y_n, x^*)^2 + (1-\alpha_n) d(z_n, x^*)^2 - \alpha_n (1-\alpha_n) d(T_1 y_n, T_2 z_n)^2$$

By Eq. 5 and 6, researchers have:

$$d(x_n + 1, x^*)^2 \leq \alpha_n (d(x_n, x^*)^2 - \beta_n (1-\beta_n) d(T_2 x_n, x^*)^2) + (1-\alpha_n)(d(x_n, x^*)^2 - \gamma_n (1-\gamma_n) d(T_1 x_n, x^*)^2) - \alpha_n (1-\alpha_n) d(T_1 y_n, T_2 z_n)^2 = d(x_n, x^*)^2 - \alpha_n \beta_n d (T_2 x_n, x_n)^2 - \gamma_n (1-\gamma_n)(1-\alpha_n) d(T_1 x_n, x_n)^2 - \alpha_n (1-\alpha_n) d(T_1 y_n, T_2 z_n)^2 \leq d(x_n, x^*)^2 - \alpha_n \beta_n d (T_2 x_n, x_n)^2 - \gamma_n (1-\gamma_n)(1-\alpha_n) d(T_1 x_n, x_n)^2$$

Since, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[a, b] \subset (0, 1)$, researchers have:

$$a(1-b)^2\ d(T_1 x_n, x_n)^2 + a^2 (1-b) d(T_2 x_n, x_n)^2 \leq \alpha_n \beta_n d(T_2 x_n, x_n)^2 + \gamma_n (1-\gamma_n)(1-\alpha_n) d(T_1 x_n, x_n)^2 \leq d(x_n, x^*)^2 - d(x_n + 1, x^*)^2$$

And so:

$$\sum_{n=1}^{\infty} \alpha (1-b)^2 d(T_1 x_n, x_n) < \infty$$

And:

$$\sum_{n=1}^{\infty} \alpha^2 (1-b) d(T_2 x_n, x_n) < \infty$$

This implies that:

$$\lim_{n\to\infty} d(T_1 x_n, x_n) = 0, \lim_{n\to\infty} d(T_2 x_n, x_n) = 0 \tag{7}$$

Now by condition (II), researchers conclude:

$$\lim_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$$

That is, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ and sequence pk in $F(T_1) \cap F(T_2)$ such that:

$$d(x_{nk}, pk) < \frac{1}{2^k} \text{ for all } k$$

Since:

$$d(x_{nk+1}, pk) \leq d(x_{nk}, pk) < \frac{1}{2^k}$$

It follow that:

$$d(pk+1, pk) \leq d(x_{nk+1}, pk+1) + d(x_{nk+1}, pk) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$$

This implies that the sequence pk is cauchy and then is convergence to $p \in C$ since:

$$d(pk, T_i p) = d(T_i pk, T_i p) \leq d(pk, p) \text{ for } i \in \{1, 2\}$$

And $pk \to p$ as $k \to \infty$, it follow that $d(p, T_i p) = 0$ for all I = 1, 2 and thus, $p \in F(T_1) \cap F(T_2)$. Therefore, $\{x_{nk}\}$ convergence strongly to p. Since, $\lim_{n\to\infty} d(x_n, p)$ exists, then $\{x_n\}$ convergence strongly to p, common fixed point T_1 and T_2.

Theorem 3: Let C be a non-empty closed convex subset of a complete CAT (0) space X and T_1, T_2: $C \to C$ be non-expansive mappings and satisfying condition (II) with $F(T_1) \cap F(T_2) \neq \phi$. Suppose $x_1 \in C$ and $\{x_n\}$ is defined by Eq. 1. Then $\{x_n\}$, Δ-converges to common fixed point of T_1 and T_2.

Proof: Let $W\omega(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Researchers claim that $W\omega(x_n) \subset F(T_1) \cap F(T_2)$. Let $u \in W\omega(x_n)$ then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that A $(\{x_n\}) = \{u\}$. By lemma and there exist a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n$ = $v \in C$, also Eq. 7 implies that $\lim_n d(v_n, T_1v_n) = 0$, $\lim_n d(v_n, T_2v_n) = 0$; then $v \in F(T_1) \cap F(T_2)$. Now, researchers claim that u = v. Suppose not since, $v \in F(T_1) \cap F(T_2)$, theorem $\lim_{n\to\infty} d(x_n, v)$ exist. From the uniqueness of asymptotic centers, researchers have:

$$\begin{aligned}\lim_{n\to\infty}\sup d(v_n, v) &< \lim_{n\to\infty}\sup d(v_n, u)\\ &\le \lim_{n\to\infty}\ \sup d(u_n, u)\\ &< \lim_{n\to\infty}\ \sup d(u_n, v)\\ &= \lim_{n\to\infty}\ \sup d(x_n, v)\\ &= \lim_{n\to\infty}\ \sup d(v_n, v)\end{aligned}$$

A contradiction and hence, $u = v \in F(T_1) \cap F(T_2)$. To show that $\{x_n\}$ Δ-converges to common fixed point of T_1 and T_2 it sufficient to show that $W\omega$ (x_n) consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By lemma, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n = v \in C$. Let A $(\{u_n\}) = \{u\}$ and A $(\{x_n\}) = \{x\}$. Researchers have seen that u = v and $v \in F(T_1) \cap F(T_2)$. Researchers complete the proof the uniquness of asymptotic centers:

$$\begin{aligned}\lim_{n\to\infty}\ \sup d(v_n, v) &< \lim\ \sup_n d(v_n, x)\\ &\le \lim_{n\to\infty}\ \sup d(x_n, x)\\ &< \lim_{n\to\infty}\ \sup d(x_n, v)\\ &= \lim_{n\to\infty}\ \sup d(v_n, v)\end{aligned}$$

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