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Solution of Boundary Value Problems by Lanczos-Canonical Method

[1]Murtala M. Ahmad, [2]M.R. Odekunle, [3]A.U. Bawa and [1]S. Samuel
[1]Department of Mathematics, Statistics and Computer Science,
Taraba State University, PMB 1167, Jalingo, Nigeria
[2]Department of Mathematics, Modibbo Adama University of Technology,
[3]Department of Computer Science, Power Holdings Company of Nigeria, Yola, Nigeria

Abstract: Canonical polynomials were used as basis functions in Lanczos-reduction method to obtain what we call Lanczos-canonical reduction method. The scheme was implemented for the case N = 4 on elliptic convection diffusion problem. The results were compared with that of Lanczos-Chebyshev and were found to be better.

Key words: Elliptic convection diffusion problem, Lanczos-reduction method, canonical polynomials, scheme, functions

INTRODUCTION

Most schemes of problems of Partial Differential Equation in Engineering, Science and Management are difficult to solve analytically hence, we seek numerical solution. Consider the equation (Mitchell and Wait, 1978):

$$\begin{aligned} &\nabla^2 u + 2\alpha u_w = 0 \\ &u(w,z) = u(w,z) = 0 \\ &u(w,z) = \xi \end{aligned} \qquad (1)$$

This can be interpreted as a diffusion term $\nabla^2 u$ together with convection by a flow with velocity 2α parallel to the w-axis.

In recent years, part of the efforts of mathematicians in the area of ordinary and partial differential equations have been so much as to construct solutions of particular differential equation but rather to develop efficient and accurate techniques that are suitable for treating classes of equations with the aid of computers (Boyce and DiPrima, 1986).

The approximating properties of Chebyshev polynomials attracted a lot of interest and rapid development in the 1930's as can be seen in the research of Courant and Hilbert (1931), Sommerfield (1935) and Van der Pol (1935). Towards the middle of that decade, Lanczos, a co-worker of Albert Einstein, studied some application of interpolation and expansions in Chebyshev polynomials to problems of Mathematical Physics. He pointed out that the Fourier series expansion in the solution of practical problems is limited by the fact that the integrals giving the coefficients of the expansion are in general not adapted to actual evaluation and so he proposed alternative techniques. One of them is the so called Tau method (Lanczos, 1938). This method moved the domain of application of interpolation and economization from sphere of analytic function to that of approximating function (Onumanyi, 1981).

Firstly, Lanczos (1938) introduced an approximation technique called the Tau method to solve differential equations of simple form. Ortiz (1969) develop two approaches to the Tau method by Lanczos (1938) to treat more complex problems and the two methods are recursive Tau and operational Tau. El-Daou *et al*. (1993) studied the two methods of Ortiz (1969) and found that those approaches are equivalent. Crisci and Russo (1983) extended the recursive Tau method for certain linear systems of ordinary differential equation's and this extension for the operational Tau method which is known as a realization of the recursive one was first discussed by Abadi (1988) and then by Liu and Pan (1999). Later, Ortiz (1969) considered the subdivision of the interval of integration and the Tau method is applied on each subinterval then it is called step by step Tau. Onumanyi (1981) introduced a scheme on the boundary value problems with the Tau method.

Chen (1979, 1981) used a reduction method based on the Lanczos Tau method by Fox (1962) as applied to functional fitting and solution of ordinary differential equation to solve self-equilibrating end load problems in hollow cylinders. It is, however evident that the accuracy

Corresponding Author: Murtala Maigari Ahmad, Department of Mathematics, Statistics and Computer Science, Taraba State University, PMB 1167, Jalingo, Nigeria

of this numerical method is bounded by the error of the reduction method. But, due to the elimination of the $\tau(y)$ parameters involved in the reduction process of Chen (1981), it is not immediately obvious how an error estimate can be established.

In the development of the scheme, Odekunle (1992) solved the boundary value problems with the Lanczos-Chebyshev and Lanczos-Legendre method which improved the result with little effort.

This research investigates the use of canonical polynomial develop by Onumanyi (1981) on the boundary value problems.

MATERIALS AND METHODS

The reduction method: According to Odekunle (1998), if we have an unknown function u (x, y) in two dimensions say that satisfies the linear partial differential equation:

$$Lu(x,y) = \eta \ \text{in} \xi \tag{2}$$

And the boundary condition:

$$Lu(x,y) = \xi(x,y) \ \text{on} \ \xi \tag{3}$$

where $\mathfrak{R}$ is the boundary enclosing the region $\mathfrak{S}$. Researchers then approximate u(x, y) by a finite sum of products using:

$$\bar{u}(x,y) \approx u(x,y) = \sum_{i=0}^{N} P_i(x) Q_i(y) \tag{4}$$

Lanczos τ-method involves the replacement of one of the two functions by an approximate polynomial of the form:

$$P_i(x) = x^i \tag{5}$$

And must be bounded by the lines x = ±1 in the x-direction. The problem is then slightly perturbed to become:

$$Lu(x,y) = \tau_1(y) C_{N-2}(x) + \tau_2(y) C_{N-1}(x) + \eta \tag{6}$$

Subject to:

$$Lu(x,y) = \xi(x,y),\ x = \pm 1 \tag{7}$$

where $\tau_1(y)$ and $\tau_2(y)$ are arbitrary and $C_j(x)$ is the jth order canonical polynomial in the range x ∊ (−1, 1), unlike in the case of Odekunle (1992, 1998) where they are either Chebyshev or Legendre polynomials. Equating the powers of x in Eq. 6 and the boundary conditions, Eq. 7 give N+4 equations with N+4 unknown $\tau_1(y)$, $\tau(y)$, $Q_1(y)$,..., $Q_N(y)$. The arbitrary functions are eliminated to give a set of N ordinary differential equations in N-unknowns $Q^v_1(y)$,..., $Q^v_N(y)$ where v is the order of the resulted ordinary differential equations.

Definition: We say $Q_j(x)$ is a canonical polynomial of second order is defined by an operator L as:

$$L = \frac{d^2}{dx} + \frac{d}{dx} + 1 \tag{8}$$

Researchers derive the basis $Q_j(x)$ as (Onumanyi, 1981):

$$Q_j(x) = x^j - jQ_{j-1}(x) - j(j-1)Q_{j-2}(x),\ j \geq 0 \tag{9}$$

DERIVATION OF THE METHOD

We seek an approximation of the form:

$$u(x,y) = \sum_{i=0}^{N} g_i(y) x^i \approx \bar{u}(x,y) \tag{10}$$

So that:

$$L\bar{u}(x,y) = \sum_{i=0}^{N} \left[g_i''(y) + 2g_i''(y) + (i+1)(i+2) g_{i+2}(y) \right] x^i \tag{11}$$

And by Eq. 6:

$$L\bar{u}(x,y) = \tau_1(y) Q_{N-1}(x) + \tau_2(y) Q_N(x) \tag{12}$$

Denoting:

$$Q_N(x) = \sum_{i=0}^{N} C_i^N x^i, -1 \leq x \leq 1 \tag{13}$$

Then (Eq. 2, 3) becomes:

$$L\bar{u}(x,y) = \sum \left[\tau_1(y) C_{i-1}^{N-2} + \tau_2(y) C_{i-1}^{N-1} \right] x^i \tag{14}$$

Comparing the right hand sides of Eq. 11, 14 and get an equation which is use to obtain four equations and the last two equations are obtained from the conditions.

$$\sum_{i=0}^{N} \begin{bmatrix} g_i''(y) + \pi g_i''(y) + (i+1)(i+2) \\ g_{i+2}(y) - \tau_1(y) C_{i-1}^{N-2} - \tau_2(y) C_{i-1}^{N-1} \end{bmatrix} x^i = 0 \tag{15}$$

From which we obtain:

$$\begin{aligned}
&g_i^{''}(y)+\pi g_i^{'}(y)+(i+1)(i+2)g_{i+2}(y)-\\
&\tau_1(y)C_{i-1}^{N-2}-\tau_2(y)C_{i-1}^{N-1}=0\\
&i=1,\ldots,N-2, N=3,4,\ldots\\
&g_{N-1}^{''}(y)+2g_{N-1}^{'}(y)-\tau_1(y)C_{N-1}^{N-1}-\tau_2(y)C_{N-1}^{N}=0 \quad (16)\\
&g_i^{''}(y)+2g_i^{'}(y)-\tau_2(y)C_N^N=0\\
&g_1(y)-g_2(y)+g_3(y)+\ldots+(-1)^N g_N(y)=0\\
&g_1(y)+g_2(y)+g_3(y)+\ldots+g_N(y)=0
\end{aligned}$$

Where the last two equations of 16 are obtained from the conditions:

$$u(-1,y)=u(1,y)=0 \tag{17}$$

Eliminating $\tau_1(y)$ and $\tau_2(y)$ from the four equations gives N second order differential equations in N unknowns $g_i(y)$ (i = 1, 2, ..., N). Using the boundary conditions:

$$\begin{bmatrix} u(x_1,-1)=\sum_{i=0}^{N} g_i(-1)x_1^i \\ u(x_1,1)=\sum_{i=0}^{N} g_i(1)x_1^i=\left(\frac{\pi}{2}\right)^2-x^2 \end{bmatrix} \tag{18}$$

Leads to N simultaneous algebraic equations in N unknown x_1 (i = 0, 1, ..., N) are the collocation points of $C_N(x)$. This can be solved by any of the mathematical software, i.e., Matlab or scientific workplace.

Numerical example: Consider the equation (Mitchell and Wait, 1978):

$$\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial w^2}+2\alpha\frac{\partial u}{\partial w}=0 \tag{19}$$

Subject to the following boundary conditions:

$$\begin{bmatrix} u\left(\frac{-\pi}{2},w\right)=0,\ \left(|w|<\frac{\pi}{2}\right) \\ u\left(z,\frac{-\pi}{2}\right)=0,\ \left(|z|\le\frac{\pi}{2}\right) \\ u\left(z,\frac{\pi}{2}\right)=\left[\frac{\pi}{2}\right]^2-z^2,\ \left(|z|\le\frac{\pi}{2}\right) \end{bmatrix} \tag{20}$$

Valid is the region $-\pi/2\le z, w\le\pi/2$. The problem is first mapped into an interval occupying $-1\le x\le 1$ by the linear transformation, so that the problem becomes that of solving:

$$\nabla^2 u+\pi u_y=0,\ -1<x,\ y<1 \tag{21}$$

Subject to:

$$\left.\begin{aligned} &u(\pm1,y)=0,(|y|\le1)\\ &u(x,-1)=0,(|x|\le1),\\ &u(x,-1)=\frac{\pi^2}{4}(1-x^2)\end{aligned}\right\} \tag{22}$$

We shall illustrate the method above by solving numerically the convention-diffusion equation for the case N = 4 which is the smallest possible value of N.

In the following and subsequent appearance elsewhere, the C_i, i = 1, 2, ... N are constants of integration. From Eq. 16, we obtain:

$$\begin{aligned}
&g_1^{''}(y)+\pi g_1^{'}(y)+2g_3(y)-6\tau_2=0\\
&g_2^{''}(y)+\pi g_2^{'}(y)+6g_4(y)+2\tau_1=0\\
&g_3^{''}(y)+\pi g_3^{'}(y)-\tau_1+3\tau_2=0\\
&g_4^{''}(y)+\pi g_4^{'}(y)-\tau_2=0\\
&g_1(y)-g_2(y)+g_3(y)-g_4(y)=0\\
&g_1(y)+g_2(y)+g_3(y)+g_4(y)=0
\end{aligned} \tag{23}$$

Eliminating $\tau_1(y)$ and $\tau_2(y)$ from Eq. 22, we obtain the system of differential equations.

$$\begin{aligned}
&g_1^{''}(y)+\pi g_1^{'}(y)+2g_2(y)-6g_4^{''}(y)-6\pi g_4^{'}(y)=0\\
&g_2^{''}(y)+\pi g_2^{'}(y)+2g_3^{''}(y)+2\pi g_3^{'}(y)+6g_4^{''}(y)+\\
&6\pi g_4^{'}(y)+6g_4(y)=0\\
&g_3^{''}(y)+\pi g_3^{'}(y)-2g_3(y)+6g_4^{''}(y)+6\pi g_4^{'}(y)=0\\
&2g_3^{''}(y)+2\pi g_3^{'}(y)+5g_4^{''}(y)+5\pi g_4^{'}(y)+6g_4(y)=0
\end{aligned} \tag{24}$$

Solving, we obtain:

$$\begin{aligned}
&g_1(y)=-4.00009598C_1e^{-3.1054y}\text{Cos}0.41631y-\\
&8.944275376C_1e^{-3.1054y}\text{Sin}0.41631y+\\
&8.944275372C_2e^{-3.1054y}\text{Cos}0.41631y-\\
&4.00009599C_2e^{-3.1054y}\text{Sin}0.41631y-\\
&4.0000875C_3e^{-0.036194y}\text{Cos}0.41631y+\\
&8.944246626C_3e^{-0.036194y}\text{Sin}0.41631y-\\
&8.944246628C_4e^{-0.036194y}\text{Cos}0.41631y-\\
&4.0000875C_4e^{-0.036194y}\text{Sin}0.41631y-\\
&1.000000001C_5e^{0.542826416y}-C_6e^{-3.68441907y}+C_7+C_8e^{-\pi y}
\end{aligned} \tag{25}$$

Table 1: Approximate solutions to the convection-diffusion problem with $\alpha = 1$ using Lanczos-Canonical and Lanczos-Chebyshev reduction method when N = 4

x	y	Lanczos-Canonical	Error in Lanczos-Canonical	Lanczos-Chebyshev	Error in Lanczos-Chebyshev	Exact
0.0	0.5	1.828415138	0.007702629	1.875517295	0.054804486	1.820712509
0.5	0.5	1.394111717	0.083415453	1.496333013	0.185636449	1.310696264
0.0	0.0	1.311279518	0.001324030	1.273844293	0.036106195	1.309950488
0.5	0.0	1.000614078	0.069918523	1.041980112	0.111284457	0.930695555
0.0	-0.5	0.832351921	0.022707448	0.815571039	0.039488630	0.855059669
0.5	-0.5	0.666532913	0.061104476	0.683371597	0.077943160	0.605428437

$$\begin{aligned} g_2(y) = {} & 1.999822144C_1e^{-3.1054y}Cos0.41631y - \\ & 26.83291766C_1e^{-3.1054y}Sin0.41631y + \\ & 26.83291768C_2e^{-3.1054y}Cos0.41631y + \\ & 1.999822124C_2e^{-3.1054y}Sin0.41631y - \\ & 1.999509585C_3e^{-0.036194y}Cos0.41631y - \\ & 26.83419658C_1e^{-0.036194y}Sin0.41631y - \\ & 26.83419658C_4e^{-0.036194y}Cos0.41631y + \\ & 1.999509584C_4e^{-0.036194y}Sin0.41631y - \\ & 2.000000001C_5e^{0.542826416y} + \\ & 1.999999995C_6e^{-3.68441907y} + C_7 + C_8e^{-\pi y} \end{aligned} \tag{26}$$

$$\begin{aligned} g_3(y) = {} & -3.999952018C_1e^{-3.1054y}Cos0.41631y - \\ & 4.47213769C_2e^{-3.1054y}Cos0.41631y + \\ & 4.472137686C_1e^{-3.1054y}Sin0.41631y - \\ & 3.999952016C_2e^{-3.1054y}Sin0.41631y - \\ & 3.999956252C_3e^{-0.036194y}Cos0.41631y + \\ & 4.472123311C_4e^{-0.036194y}Cos0.41631y - \\ & 4.472123312C_3e^{-0.036194y}Sin0.41631y - \\ & 3.999956252C_4e^{-0.036194y}Sin0.41631y + \\ & C_5e^{-0.542826416y} + C_6e^{-3.68441907y} \end{aligned} \tag{27}$$

$$\begin{aligned} g_4(y) = {} & C_1e^{-3.1054y}Cos0.41631y + \\ & C_2e^{-3.1054y}Sin0.41631y + C_3e^{-0.036194y} \\ & Cos0.41631y + C_4e^{-0.036194y}Sin0.41631y \end{aligned} \tag{28}$$

RESULTS AND DISCUSSION

For N = 4 in Table 1, shows the results obtained by using the new method to solve the problem. It can be observed by comparing columns three and four in Table 1 that result obtained by this new method is better than the Lanczos-Chebyshev.

CONCLUSION

Although, the study was limited to convection diffusion equation, it can also be expanded to partial differential equations of higher order, since the accuracy as compared with the result of Odekunle (1992) has been displayed and proved successful in the convection diffusion equation.

In view of the economical use of the trial functions in the Lanczos-Canonical method, the extension to time dependent, boundary layer effect and non-linear problems is worth knowing.

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