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On Size Baised Gamma Distribution

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Abstract: In this study, researchers have considered a Size Biased Gamma Distribution (SBGMD), a particular case of the weighted gamma distribution, taking the weights as the variate values. Moments of the distribution are derived. Also, the estimates of the parameters of Size Biased Gamma Distribution (SBGMD) are obtained by employing different methods. A Bayes' estimator of Size Biased Gamma Distribution (SBGMD) has also been obtained by using non-informative and gamma prior distributions.

Key words: Gamma distribution, size-biased gamma distribution, non-informative, Bayes' estimator, maximum likelihood estimator, moment estimator

INTRODUCTION

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size biased. Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, these were later formalized in a unifying theory (Rao, 1965). Van Deusen (1975) was first to apply them in connection with sampling wood cells. Johnson *et al.* (1995) quotes from Laplace in which the latter obtains a gamma distribution as the posterior distribution of the precision constant gives the values of n independent normal variables with mean zero and standard deviation σ. The gamma distribution appeared again in 1900 (Pearson), as the appropriate distribution for the Chi-square statistics used for various tests in contingency tables. Dennis and Patil (1984) used stochastic differential equations to arrive at a weighted gamma distribution as the stationary Probability Density Function (PDF) for the stochastic population model with predation effects. Gove (2003) reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry. Sandal (1964) derived the method of estimation of parameters of the gamma distribution. Mir and Ahmad (2009), also discussed some of the discrete size biased distributions.

These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non-experimental, non-replicated and non-random categories.

If the random variable X has distribution f (x; θ) with unknown parameter θ then the corresponding size biased distribution is of the form:

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu'_c} \quad (1)$$

$$\mu'_c = \int x^c f(x;\theta)dx \quad \text{For continuous series}$$
$$\mu'_c = \sum_{i=1}^{n} x^c f(x;\theta)dx \quad \text{For discrete series} \quad (2)$$

When c = 1 and 2, researchers get the size biased and area biased distributions, respectively.

In this study, a Size Biased Gamma Distribution (SBGMD) is defined. Moments about origin are obtained. The estimates of the parameters of Size Biased Gamma Distribution (SBGMD) are obtained by employing the moments, maximum likelihood and Bayesian method of estimation.

GAMMA DISTRIBUTION

The gamma distribution is used as a lifetime model Thom (1958) though not, nearly as much as the Weibull distribution. It does fit a widely variety of lifetime adequately, besides failure process models that leads to

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it. It also arises in some situations involving the exponential distribution. A continuous random variable X is said to be Gamma Distribution (GD) with two parameters α, β and its probability density function is given by:

$$f(x;\alpha,\beta) = \frac{\alpha^{\beta}}{\Gamma\beta} e^{-\alpha x} x^{\beta-1} = 0, \quad \text{otherwise}; \alpha > 0, \beta > 0;\ 0 < x < \infty \tag{3}$$

Researchers know that:

$$\mu_1' = \int_0^{\infty} x f(x;\theta) dx = \frac{\beta}{\alpha},\ \mu_2 = \frac{\beta}{\alpha^2} \tag{4}$$

SIZE BIASED GAMMA DISTRIBUTION

A Size Biased Gamma Distribution (SBGMD) is obtained by applying the weights x^c where c = 1 to the gamma distribution. Researchers have from Eq. 3 and 4:

$$\mu_1' = \int_0^{\infty} x f(x;\theta) dx = \frac{\beta}{\alpha}$$

$$\int_0^{\infty} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} x^{\beta}\, dx = 1$$

$$\int_0^{\infty} f(x;\alpha,\beta+1) = 1$$

Where, f (x; α,β+1) represents a probability density function. This gives the Size Biased Gamma Distribution (SBGMD) as:

$$f(x;\alpha,\beta+1) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} x^{\beta} = 0, \quad \text{otherwise}; \alpha > 0, \beta \geq 0;\ 0 < x < \infty \tag{5}$$

Special cases

Case 1: When $\beta = 0$, then Size Biased Gamma Distribution (SBGMD) Eq. 5 reduces to Exponential Distribution (EPD) as:

$$f(x;\alpha) = \alpha e^{-\alpha x};\ 0 < x < \infty \tag{6}$$

Case 2: When $\beta = 1$, then Size Biased Gamma Distribution (SBGMD) Eq. 5 reduces to Size Biased Exponential Distribution (SBEPD) as:

$$f(x;\alpha,1) = \alpha^2 x e^{-\alpha x};\ 0 < x < \infty \tag{7}$$

MOMENTS OF SIZE BIASED GAMMA DISTRIBUTION

The rth moment of size biased gamma distribution (Eq. 5) about origin is obtained as:

$$\begin{aligned}\mu_r' &= \int_0^{\infty} x^r f(x;\alpha,\beta+1) dx \\ &= \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \int_0^{\infty} e^{-\alpha x} x^{\beta+r} dx \\ &= \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \frac{\Gamma(\beta+r+1)}{\alpha^{\beta+r+1}}\end{aligned} \tag{8}$$

Using mentioned earliar relation the mean and variance of the SBGMD are given as:

$$\mu_1' = \frac{\beta+1}{\alpha} \tag{9}$$

$$\mu_2' = \frac{\beta+1}{\alpha^2} \tag{10}$$

ESTIMATION OF PARAMETERS

In this study, researchers discuss the various estimation methods for size biased gamma distribution and verifying their efficiencies.

METHODS OF MOMENTS

In the method of moments replacing the population mean and variance by the corresponding sample mean and variance, researchers get the following estimates:

$$\hat{\alpha} = \frac{\bar{x}}{s^2} \tag{11}$$

$$\hat{\beta} = \frac{\bar{x}^2}{s^2} - 1 \tag{12}$$

METHOD OF MAXIMUM LIKELIHOOD

Let, $x_1, x_2, x_3, ..., x_n$ be a random sample from the size biased gamma distribution then the corresponding likelihood function is given as:

$$L(x;\alpha,\beta+1) = \frac{\alpha^{n\beta+n}}{\left[\Gamma(\beta+1)\right]^n} e^{-\alpha\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x^{\beta} \tag{13}$$

The log likelihood of Eq. 13 can be written as:

$$\log L = (n\beta + n)\log\alpha - n\log\Gamma(\beta+1) - \alpha\sum_{i=1}^{n} x_i + \beta\sum_{i=1}^{n}\log x_i$$

The corresponding likelihood equations are given as:

$$\frac{\partial \log L}{\partial \alpha} = \frac{n\beta + n}{\alpha} - \sum_{i=1}^{n} x_i = 0$$
$$\hat{\alpha} = \frac{\beta+1}{\bar{x}} \quad (14)$$

$$\frac{\partial \log L}{\partial \beta} = n\log\alpha - n\frac{\Gamma'(\beta+1)}{\Gamma(\beta+1)} - \sum_{i=1}^{n} x_i = 0 \quad (15)$$

Solving the Eq. 15 equation gives the MLE estimate of β.

BAYESIAN ESTIMATION OF PARAMETER OF SIZE BIASED GAMMA DISTRIBUTION (SBGMD)

The likelihood function of SBGMD is given as:

$$L(x;\alpha,\beta+1) = \frac{\alpha^{n\beta+n}}{n\Gamma(\beta+1)} e^{-\alpha\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\beta}$$
$$L(x;\alpha,\beta+1) = \frac{\alpha^{n\beta+n}}{k} e^{-\alpha y} \prod_{i=1}^{n} x_i^{\beta} \quad (16)$$

Where:

$y = \sum_{i=1}^{n} x_i$

$k = n\Gamma(\beta+1)$

Since, $0<x<\infty$, therefore researchers assume that prior information about α when β is known. Researchers know that g (α) proportional to $1/\alpha$. Thus, $g(\alpha) = 1/\alpha$; $\alpha>0$. The posterior distribution is given by:

$$\prod(\alpha/y) = \frac{Lg(\alpha)}{\int_0^{\infty} Lg(\alpha)d\alpha}$$

Using mentioned earliar relation, the Bayes' estimator of α is given as:

$$\hat{\alpha} = \frac{n\beta + n}{y} = \frac{(\beta+1)}{\bar{x}}$$

This coincides with moment and maximum likelihood estimator.

CONCLUSION

The estimates of the parameters of Size Biased Gamma Distribution (SBGMD) are obtained by employing different estimation methods. A Bayes' estimator of Size Biased Gamma Distribution (SBGMD) has also been obtained by using non-informative and gamma prior distributions.

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