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Optimum Plan for Step-Stress Accelerated Life Testing Model under Type-I Censored Samples

N. Chandra and Mashroor Ahmad Khan
Department of Statistics, Ramanujan School of Mathematical Sciences,
Pondicherry University, 605014 Pudhucherry, India

Abstract: A simple step-stress accelerated life test plan is developed for two parameters Lomax distribution under type-I censoring. The scale parameter of the lifetime distribution at any constant stress level is assumed to be a log linear function of the stress level. The maximum likelihood function is derived for step-stress accelerated life testing model with type-I censoring. The Fisher information matrix and the asymptotic variance are obtained. The optimum plan is proposed by minimizing the asymptotic variance of the maximum likelihood estimators of the mean life at a specified design stress with respect to stress change time. A numerical study, is carried out to justify the proposed optimum plan based on the simulated data and the confidence interval of the model parameters that based on asymptotic normality of the maximum likelihood estimates are also obtained.

Key words: Accelerated life testing, asymptotic variance, cumulative exposure model, fisher information matrix, Lomax distribution, maximum likelihood function

INTRODUCTION

Accelerated Life Testing (ALT) is generally practiced in product life testing and analysis. ALT is used to shorten the period between product design and market release time and to improve the product performance and reliability. Now-a-days, the human made products/equipments are well designed and highly sophisticated. The products are computers, washing machine, refrigerator, electronic camera, cell mobile, etc. These products/equipments are giving good services to customers with satisfaction. The researchers are facing problem in getting real life data of such types of highly reliable products for reliability prediction. It is very difficult to obtain a reasonable amount of test data under normal use condition. Thus, ALT is preferable to get failure time data more quickly of highly reliable product at a higher than usual level of stress (e.g., temperature, voltage, pressure, vibration, humidity, cycling conditions, etc.). Failure time data collected from ALT at higher stresses are then extrapolated to the design stress to estimates the lifetime distribution.

The problems of designing optimum plan and making inferences have been studied by more than a few researchers in the field of step-stress ALT was firstly introduced by Wayne (1980) that allow test conditions to change during testing. In step-stress ALT, a unit is placed on test at an initial low stress and if the unit does not fail in a pre-specified time τ, stress on it is raised and held for a specified time. Further, stress is repeatedly increased until the test unit fails or the censoring time is reached. If there is a single change of stress, the ALT is called a simple step-stress test.

The cumulative exposure model defined by Nelson (1990) for simple step-stress testing with stresses X_1 and X_2 is:

$$F(t)=\begin{cases} F_1(t), & 0\le t<\tau \\ F_2(t-\tau+s), & \tau\le t<\infty \end{cases} \tag{1}$$

Where:
- $F_1(t)$ = The cumulative distribution function of the failure time at stress X_i
- τ = The time to change stress
- s = The solution of $F_1(\tau) = F_2(s)$

In past few years in the step-stress ALT planning problems, a general optimization criterion is to minimize the asymptotic variance of the Maximum Likelihood Estimate (MLE) of the logarithm of mean life or some percentile of life at a specified stress level with the exponential exposure model. Miller and Nelson (1983) developed two optimum plans, time step test and failure step test and verified that these both plan having the same asymptotic variance. Bai *et al*. (1989) extended the Miller and Nelson (1983) research for censored data and they used the nomographs techniques for obtaining the optimum times. Khamis and Higgins (1996) derived the

Corresponding Author: N. Chandra, Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, 605014 Pudhucherry, India

optimum 3-step SSALT by considering the existence of quadratic stress-life relation for type-I censored data. Khamis (1997) generalized the optimum plans for m-step SSALT design with k stress variables by assuming complete knowledge of the stress-life relationship with multiple stress variables. These studies were based on the assumption that the failure time follows exponential distribution because of its simplicity. Khamis and Higgins (1998) proposed an alternative to Weibull step-stress Model based on time-transformation of the exponential Model. Alhadeed and Yang (2002) obtained the optimal design for the simple SSALT using the Khamis-Higgins (K-H) Model. They assumed constant shape parameter and a log-linear life-stress relationship between the scale parameter and the stress. Alhadeed and Yang (2005) derived a simple SSALT plan for optimal time of changing stress level using log-normal distribution. Fard and Li (2009) studied the optimum SSALT design for reliability prediction using K-H Model for Weibull distribution. Chandra and Khan (2012) also developed a new optimum SSALT plan for Rayleigh distribution using K-H Model. Srivastava and Shukla (2008) derived optimum simple SSALT plan for log-logistic distribution with type-I censoring. Ebrahem and Al-Masri (2007) obtained the optimum times of changing stress levels under a log-logistic cumulative exposure model. Hassan and Al-Ghamdi (2009) has developed the optimum plan for SSALT for Lomax distribution for the complete samples.

In this study, researchers have developed an optimum plan for simple step-stress accelerated life test with a pre-specified censoring time by minimizing asymptotic variance of the maximum likelihood estimators of the model parameters at a design stress.

PROPOSED MODEL AND ASSUMPTIONS

The Lomax distribution is a special case of second kind of the Pareto distribution, proposed by Lomax (1954). It is a useful model in a wide variety of socioeconomic, as well as lifetime contexts. It has been used in the analysis of income data and business failure data. Also, it has been used to provide a good model in biomedical problems. It may describe the lifetime of a decreasing failure rate component and has been recommended by Bryson (1974), as a heavy tailed alternative to the exponential distribution. In reliability theory, it appears as a mixture of the one parameter exponential distribution.

The probability density function (pdf) of a random variable that has the Lomax distribution is given by:

$$F(t,\theta,\lambda)=\frac{\lambda}{\theta}\left(1+\frac{t}{\theta}\right)^{-(\lambda+1)},\ t>0,\theta>0 \quad (2)$$

Under any constant stress, the time to failure of a test unit follow a Lomax distribution with distribution function:

$$F_i(t,\theta,\lambda)=1-\left(1+\frac{t}{\theta_i}\right)^{-\lambda},\ t>0,\theta>0 \quad (3)$$

In the case, of the simple step-stress Lomax cumulative exposure model, the cumulative distribution function is:

$$F(t)=\begin{cases}1-\left(1+\dfrac{t}{\theta_1}\right)^{-\lambda}, & 0\le t<\tau\\ 1-\left(1+\dfrac{t-\tau}{\theta_2}+\dfrac{\tau}{\theta_1}\right)^{-\lambda}, & \tau\le t<T\end{cases} \quad (4)$$

From Eq. 1:

$$s=\tau\left(\frac{\theta_2}{\theta_1}\right)$$

The corresponding pdf of Eq. 4 is given as:

$$F(t)=\begin{cases}\dfrac{\lambda}{\theta_1}\left(1+\dfrac{t}{\theta_1}\right)^{-(\lambda+1)}, & 0\le t<\tau\\ \dfrac{\lambda}{\theta_2}\left(1+\dfrac{t-\tau}{\theta_2}+\dfrac{\tau}{\theta_1}\right)^{-(\lambda+1)}, & 0\le t<\infty\end{cases} \quad (5)$$

The following assumptions are made:

- Testing is done at stresses X_1 and X_2 where $X_1<X_2$
- The scale parameter θ_i at stress level i, i = 0, 1, 2 is a log-linear function of stress, i.e.:

$$\log(\theta_i)=\beta_0+\beta_1X_i \quad (6)$$

Where, β_0 and β_1 are unknown parameters depending on the nature of the product and method of the test.

- The lifetimes of test units are independent and identically distributed
- The test is preceded as follows. All n units are initially placed on low stress X_1 and run until time τ when the stress is changed to high stress X_2 and the test is continued until all units fail or until a pre-specified censoring time T_c

MAXIMUM LIKELIHOOD ESTIMATION AND OPTIMUM PLAN

Let t_{ij}, i = 1, 2; j = 1, 2, 3, ..., n_i, be the observed failure time of a test unit j under stress level i where n_i denotes the number of units failed at the stress level i. The likelihood function is given as:

$$L(\theta_1,\theta_2,\lambda)=\prod_{j=1}^{n_1}\frac{\lambda}{\theta_1}\left(1+\frac{t_{1j}}{\theta_1}\right)^{-\lambda-1}\prod_{j=1}^{n_2}\frac{\lambda}{\theta_2}\left(1+\frac{t_{2j}-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{-\lambda-1}\left(1+\frac{T_C-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{-\lambda n_C} \tag{7}$$

Taking the logarithm of the likelihood Eq. 7 then:

$$\begin{aligned}\log L(\theta_1,\theta_2,\lambda)&=n\log\lambda-n_1\log\theta_1-n_2\log\theta_2-(\lambda+1)\sum_{j=1}^{n_1}\log\left(1+\frac{t_{1j}}{\theta_1}\right)-(\lambda+1)\\&\sum_{j=1}^{n_2}\log\left(1+\frac{t_{2j}-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)-\lambda n_C\log\left(1+\frac{T_C-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)\end{aligned} \tag{8}$$

Where; $n = n_1+n_2+n_c$. From the assumption 2, the log-likelihood function becomes:

$$\begin{aligned}\log L(\beta_1+\beta_2,\lambda)&=n\log\lambda-n_1(\beta_0+\beta_1X_1)-n_2(\beta_0+\beta_1X_2)-\lambda n_C\log\left\{1+(T_C-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}\right\}-\\&(\lambda+1)\sum_{j=1}^{n_1}\log\left(1+t_{1j}e^{-(\beta_0+\beta_1X_1)}\right)-(\lambda+1)\sum_{j=1}^{n_1}\log\left\{1+(t_{2j}-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}\right\}\end{aligned} \tag{9}$$

The maximum likelihood estimates of parameter β_0 and β_1 are obtained by solving these two Eq. 10 and 11 as given:

$$\begin{aligned}\frac{\partial\log L}{\partial\beta_0}&=-n+\lambda n_C\frac{(T_C-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}}{\left\{1+(T_C-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}\right\}}+(\lambda+1)\sum_{j=1}^{n_1}\frac{t_{1j}e^{-(\beta_0+\beta_1X_1)}}{\left(1+t_{1j}e^{-(\beta_0+\beta_1X_1)}\right)}+\\&(\lambda+1)\sum_{j=1}^{n_2}\frac{(t_{2j}-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}}{\left\{1+(t-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}\right\}}=0\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial\log L}{\partial\beta_1}&=\lambda n_C\frac{X_2(T_C-\tau)e^{-(\beta_0+\beta_1X_2)}+X_1\tau e^{-(\beta_0+\beta_1X_1)}}{\left\{1+(T_C-\tau)e^{-(\beta_0+\beta_1X_2)}\tau e^{-(\beta_0+\beta_1X_1)}\right\}}+(\lambda+1)\sum_{j=1}^{n_1}\frac{X_1t_{1j}e^{-(\beta_0+\beta_1X_1)}}{\left(1+t_{1j}e^{-(\beta_0+\beta_1X_1)}\right)}-\\&(n_1X_1+n_2X_2)+(\lambda+1)\sum_{j=1}^{n_2}\frac{X_2(t-\tau)e^{-(\beta_0+\beta_1X_2)}+X_1\tau e^{-(\beta_0+\beta_1X_1)}}{\left\{1+(t-\tau)e^{-(\beta_0+\beta_1X_2)}+\tau e^{-(\beta_0+\beta_1X_1)}\right\}}=0\end{aligned} \tag{11}$$

The fisher information matrix is obtained by taking expected values of second and mixed derivatives of log-likelihood function with respect to β_0 and β_1 is given as:

$$F=n\frac{\lambda}{(\lambda+2)}\begin{bmatrix}I_1 & (I_1-I_2)X_1+I_2X_2\\(I_1-I_2)X_1+I_2X_2 & (I_1-I_2)X_1^2+I_2X_2^2\end{bmatrix}$$

Where:

$$I_1=1-\frac{1}{\left(1+\frac{T-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{(\lambda+2)}}$$

$$I_2=\left\{\frac{1}{\left(1-\frac{\tau}{\theta_1}\right)^{\lambda+2}}-\frac{1}{\left(1+\frac{T_C-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{\lambda+2}}\right\}\left(1+\frac{\tau}{\theta_1}\right)$$

The asymptotic variance of the maximum likelihood estimate of log of mean life at design stress is then given by:

$$\begin{aligned}nAVar\left(\log\hat{\theta}_0\right)&=nAVar\left(\hat{\beta}_0+\hat{\beta}_1X_0\right)\\&=\frac{\xi^2I_1+(1+2\xi)I_2}{(I_1-I_2)I_2}\end{aligned} \tag{12}$$

Optimum test plan can be obtained by minimizing with respect to the change time τ, the asymptotic variance of the maximum likelihood estimator of log mean life at design stress given in Eq. 12.

CONFIDENCE INTERVAL

For construct confidence intervals for model parameters, researchers will use the asymptotic normality of the maximum likelihood estimates:

Table 1: Simulated failure time data under step-stress model

Step-stress	Failure times
$X_1 = 0.5$	0.143, 0.254, 0.369, 0.449, 1.245, 1.469, 3.318, 3.746, 4.802, 5.118, 5.392, 5.476, 5.771, 5.777
$X_2 = 1.2$	10.267, 12.335, 12.355, 14.540, 16.219, 17.467, 18.114, 19.482

$$\left(\hat{\beta}_0, \hat{\beta}_1, \hat{\lambda}\right) \sim N\left\{\left(\beta_0, \beta_1, \lambda\right), F^{-1}\right\}$$

Thus, a (1-α) 100% confidence interval for the model parameter is given by:

$$\text{Estimates} \pm Z_{(1+\alpha)} SE_{(\text{Estimates})} \tag{13}$$

Where, SE (Standard Error of MLEs) which is the square root of the diagonal element of inverse of Fisher information matrix and $Z_{(1+\alpha/2)}$ is the $(1+\alpha/2)$ percentile of the standard normal distribution.

NUMERICAL STUDY

In the numerical study, researchers first assumes the initial values $\beta_0 = 1$, $\beta_1 = 2$, $\lambda = 1.5$, $X_0 = 0.2$, $X_1 = 0.5$, $X_2 = 1.2$ and $T_c = 20$ to calculate optimum value of τ using Mathematica 8.0 Software form Eq. 12. Hence, researchers obtained $\tau^* = 5.94$. Then, researchers simulated 35 failure time observations from Lomax cumulative exposure model given in Eq. 4 which are presented in Table 1 and using the same set of simulated observations, researchers estimated the following points.

- The MLE's of the parameters are $\hat{\beta}_0 = 1.8674$, $\hat{\beta}_1 = 0.6215$
- The inverse of the observed Fisher information matrix is:

$$\hat{F}^{-1} = \begin{bmatrix} 0.3651 & -0.4018 \\ -0.4018 & 0.5787 \end{bmatrix}$$

- A 95% confidence interval for $\hat{\beta}_0$ is (0.6831, 3.0517)
- A 95% confidence interval for $\hat{\beta}_1$ is (-0.8695, 2.1125)

CONCLUSION

Accelerated life testing is one of the interesting areas of reliability studies which help in getting life time data of reliable products/equipments in short period of time with reducing the cost and manpower. In ALT, the step stress scheme is most appropriate tool for life testing of highly reliable units which allow applying the amount of stresses in different steps to induce early failures.

In this study, researchers developed optimum plan for step stress model under type-I censoring for Lomax distribution. Researchers, also focus to obtain the optimum stress changing time when experimenter needs to change the steps, i.e., from step one to two. The pattern of stress changing times from one to another are presented for the varied values of θ_1 with λ, as well as possible variation in censoring time T_c with λ are given in Table 2 and 3, respectively. It observed that from Table 2 and 3, researchers can conclude that the stress changing time of life testing units are reduces either increases stress (X_1) or censoring time (T_c) with the possible increase in shape parameter of the model. This is justifying that SSALT is recommended of high reliability products/equipments because no longer wait in life tests of such units.

Table 2: Optimum time of stress changing when T = 20, $\theta_2 = 29.96$, $\xi = 0.3$

	λ				
θ_1	1	1.5	2	2.5	3
2	1.8892	1.5455	1.2984	1.1137	0.9715
4	3.6367	3.0150	2.5537	2.2021	1.9276
6	5.1421	4.3201	3.6901	3.1999	2.8109
8	6.4576	5.4973	4.7365	4.1304	3.6428
10	7.9471	6.8807	5.9977	5.2719	4.6748

Table 3: Optimum time of changing stress when $\theta_1 = 7.38$, $\theta_2 = 29.96$, $\xi = 0.3$

	λ				
T	1	1.5	2	2.5	3
20	5.6057	4.8288	4.2024	3.6955	3.2818
25	6.1676	5.2120	4.4668	3.8803	3.4125
30	6.6407	5.5173	4.6661	4.0133	3.5022
35	7.0206	5.7490	4.8119	4.1053	3.5613
40	7.3308	5.9308	4.9195	4.1705	3.6015
45	7.5734	6.0659	4.9966	4.2154	3.6281

RECOMMENDATIONS

In further research, researchers may use >2 stress levels, for example a three step stress model and obtain the optimum times of changing stress levels. Researchers may allow progressive type I or II censored data. Also, researchers may obtain different types of interval estimation for example large sample confidence intervals, bootstrap confidence intervals and compare them in terms of their lengths and coverage probabilities.

NOMENCLATURE

$X_{0\text{-}2}$ = Stresses (design, low and high)
ξ = Extrapolation amount where, $\xi = (X_1\text{-}X_0)/(X_2\text{-} X_1)$
n = Number of test units
n_i = Number of failed units at stress X_i, i = 0, 1, 2
n_c = Number of censored units
t_{ij} = Failure time of test unit j at stress X_i, i = 1, 2, j = 1, 2, ..., n_i
θ_i = Mean life at X_i, i = 0, 1, 2
τ = Stress changing point
τ^* = Optimum time of changing stress
T_c = Censoring time

F_i (t) = cdf of Lomax distribution with mean θ_i, i = 1, 2
F (t) = cdf of a test unit under step-stress
λ = Shape parameter
$\beta_{0\text{-}1}$ = Parameters of log-linear relationship between stress X_i and mean life β_i

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