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Statistical Properties of Kumaraswamy Exponentiated Lomax Distribution

I.EL. Batal and A. Kareem
Department of Mathematical Statistics,
Institute of Statistical Studies and Research (ISSR), Cairo University, Giza, Egypt

Abstract: A new 5 parameter continuous distribution, the so-called Kumaraswamy exponentiated Lomax distribution that extends the exponentiated Lomax distribution and some other distributions is proposed and studied. Various structural properties of the new distribution are derived including expansions for the density function, explicit expressions for the moments, generating and quantile functions, researchers obtain the distribution of order statistics also the estimation of parameters is performed by maximum likelihood and least square estimator.

Key words: Kumarswamy exponentiated Lomax distribution, moment and generating functions, statistics, maximum likelihood estimation, hazard function

INTRODUCTION

In the univariate setup, the Lomax distribution is being widely used for stochastic modeling of decreasing failure rate life components. It also serves as a useful model in the study of labour turnover and queuing theory. Lomax (1954) or Pareto 2, distribution has been quite widely applied in a variety of contexts. Although, introduced originally for modeling business failure data, the Lomax distribution has been used for reliability modeling and life testing (Hassan and Al-Ghamdi, 2009) and applied to income and wealth distribution data (Harris, 1968), firm size (Corbelini *et al*., 2007) and queuing problems. It has also found application in the biological sciences and even for modeling the distribution of the sizes of computer files on servers (Holland *et al*., 2006). Some researchers, such as Bryson (1974) have suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed. Ahsanullah (1991) studied the record values of Lomax distribution. Balakrishnan and Ahsanullah (1994) introduced some recurrence relations between the moments of record values from Lomax distribution. The order statistics from non identical right-truncated Lomax random variables have been studied by Childs *et al*. (2001).

The 2 parameter Lomax distribution has the following Cumulative Distribution Function (CDF) and probability density function (pdf), respectively:

$$G(x, \lambda, \mu, \theta) = 1 - (1 + \lambda x)^{-\theta}, \lambda, \theta > 0, x \geq 0 \quad (1)$$

Where θ is the shape parameter and λ is the scale parameter and the:

$$g(x, \lambda, \theta) = \theta\lambda(1 + \lambda x)^{-(\theta+1)} \quad (2)$$

Recently, generalized the Lomax distribution by powering a positive real number α to the Cumulative Distribution Function (CDF) (Abdul-Moniem and Abdel-Hameed, 2012). This new family of distribution called Exponentiated Lomax Distribution (ELD) where the CDF and pdf of Exponentiated Lomax Distribution (ELD) as follows:

$$G(x, \lambda, \theta, \alpha) = \left[1 - (1 + \lambda x)^{-\theta}\right]^{\alpha}, \lambda, \theta > 0, x \geq 0 \quad (3)$$

The corresponding probability density function (pdf) is given by:

$$g(x, \lambda, \theta, \alpha) = \alpha\theta\lambda(1 + \lambda x)^{-(\theta+1)}\left[1 - (1 + \lambda x)^{-\theta}\right]^{\alpha-1} \quad (4)$$

The Kumaraswamy (Kw) distribution is not very common among statisticians and has been little explored in the literature. Its cdf is given by:

$$F(x, a, b) = 1 - (1 - x^{a})^{b}, 0 < x < 1 \quad (5)$$

Where, a>0 and b>0 are shape parameters and the probability density function:

Corresponding Author: I. EL. Batal, Department of Mathematical Statistics, Institute of Statistical Studies and Research (ISSR), Cairo University, Giza, Egypt

$$f(x,a,b) = abx^{a-1}\left(1-x^{a}\right)^{b-1} \quad (6)$$

Which can be unimodal, increasing, decreasing or constant, depending on the parameter values. It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However in a very recent study, Jones (2009) explored the background and genesis of this distribution and more importantly, made clear some similarities and differences between the beta and Kw distributions. However, the beta distribution has the following advantages over the Kum distribution, simpler formulae for moments and moment generating function (mgf), a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution by means of physical processes.

In this note, researchers combine the research of Kumaraswamy (1980) and Cordeiro and de Castro (2011) to derive some mathematical properties of a new model, called the Kumaraswamy Exponentiated Lomax (KEL) distribution which stems from the following general construction: If G denotes the baseline cumulative function of a random variable then a generalized class of distributions can be defined by:

$$F(x) = 1-\left[1-G(x)^{a}\right]^{b} \quad (7)$$

Where, a>0 and b>0 are 2 additional shape parameters. The Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form:

$$f(x) = abg(x)G(x)^{a-1}\left[1-G(x)^{a}\right]^{b-1} \quad (8)$$

The density family Eq. 7 has many of the same properties of the class of beta-G distributions (Eugene *et al*., 2002) but has some advantages in terms of tractability, since it does not involve any special function, such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special Kw-G distributions can be generated as follows: The Kw-normal distribution is obtained by taking G (x) in Eq. 4 to be the normal cumulative function. Analogously, the Kw-Weibull (Cordeiro *et al*., 2010), Kw-generalized gamma (De Pascoa *et al*., 2011), Kw-Birnbaum-Saunders (Saulo *et al*., 2012) and Kw-Gumbel (Cordeiro *et al*., 2012) distributions are obtained by taking G (x) to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively among several others. Hence, each new Kw-G distribution can be generated from a specified G distribution.

THE KUMARASWAMY EXPONENTIATED LOMAX DISTRIBUTION

In this study, researchers introduce the 5-parameter Kumaraswamy Exponentiated Lomax (KEL) distribution. by taking G (x) in Eq. 3 to be the cdf of exponentiated Lomax (KEL) distribution. Using Eq. 3 in 7, the cdf of the (KEL) distribution can be written as:

$$F_{KEL}(x,\lambda,\theta,\alpha,a,b) = 1-\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a}\right\}^{b} \quad (9)$$

Where, λ>0 is a scale parameter and the other positive parameters α, θ and b are shape parameters. The corresponding pdf and Hazard Rate Function (HRF), respectively:

$$f_{KEL}(x,\lambda,\theta,\alpha,a,b) = ab\theta\alpha\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a-1} \times \left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a}\right\}^{b-1} \quad (10)$$

And:

$$h(x,\lambda,\mu,\theta,a,b) = \frac{f(x,\lambda,\mu,\theta,a,b)}{\bar{F}(x,\lambda,\mu,\theta,a,b)} = \frac{ab\theta\alpha\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a-1}}{\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a}\right\}} \quad (11)$$

Researchers notice that the Kumaraswamy exponentiated Lomax Hazard Rate Function (HRF) does not involve any complicated function and it can be easily computed numerically. Moreover, it is quite flexible for modeling survival data. Plots of the Kumaraswamy Exponentiated Lomax (KEL) pdf and CDF and HRF for selected different parameter values are given in Fig. 1.

Special cases of the KEL distribution: The Kumaraswamy exponentiated Lomax is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the Kumaraswamy exponentiated Lomax distribution is explained in the following. If X is a random variable with cdf Eq. 9 then researchers have the following cases.

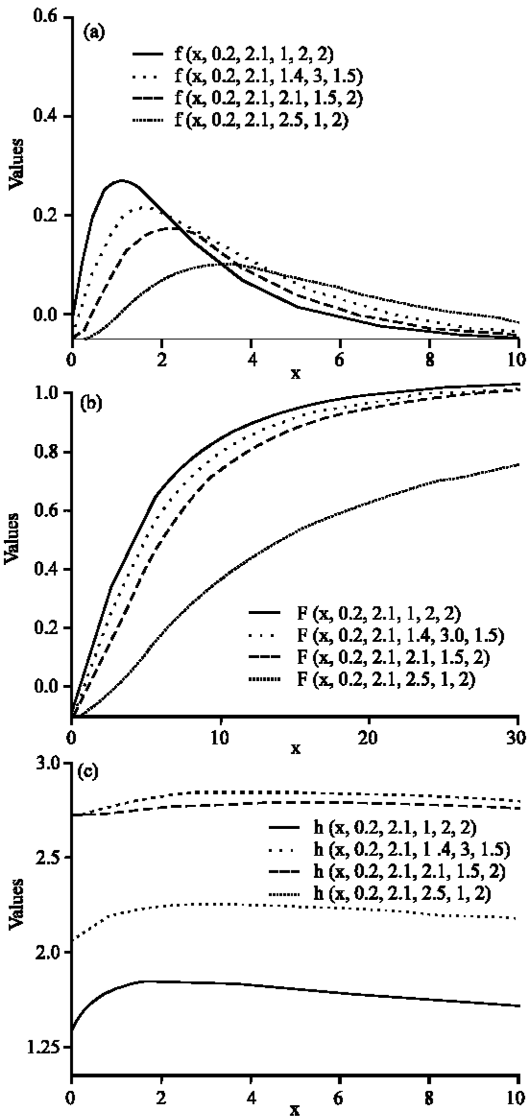

Fig. 1: Plotsof pdf, cdf and HRF of the KEL distribution for different values (λ, θ, α, a, b)

Special cases:

- If a = b = 1, then Eq. 9 reduces to the Exponentiated Lomax distribution (EL) which is introduced by Abdul-Moniem and Abdel-Hameed (2012)
- If λ = 1 researchers get the Kumaraswamy Exponentiated Pareto (KEP) distribution
- If a = b = λ = 1, researchers get the Exponentiated Pareto (KEP) distribution (Abdul-Moniem and Abdel-Hameed, 2012)
- If λ = a = 1, researchers get the Kumaraswamy Pareto distribution
- If a = b = λ = a = 1, researchers get the Pareto distribution
- If a = 1, researchers get the Kumaraswamy Lomax distribution
- If a = b = a = 1, researchers can obtain Lomax distribution (Abd-Elfattah *et al*., 2007)

Expansion for the cumulative and density functions: In this subsection, researchers present some representations of cdf, pdf of Kumaraswamy exponentiated Lomax. The mathematical relation given will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z|<1$ then:

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} z^i \tag{12}$$

Equation 9 becomes:

$$F(x,\lambda,\alpha,\theta,a,b)=1-\sum_{j=0}^{\infty}(-1)^j \binom{b}{j}\left[1-(1+\lambda x)^{-\theta}\right]^{\alpha aj}$$

Also, using the power series of Eq. 12, Eq. 10 becomes:

$$f(x,\lambda,\mu,\theta,a,b) = \sum_{j=0}^{\infty}(-1)^j \binom{b}{j} ab\theta\alpha\lambda(1+\lambda x)^{-(\theta+1)} \left[1-(1+\lambda x)^{-\theta}\right]^{\alpha a(j+1)-1} \tag{13}$$

Now using Eq. 12 in the last term of Eq. 13, researchers obtain

$$f(x,\lambda,\mu,\theta,a,b) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b}{j}\binom{\alpha a(j+1)-1}{i} ab\theta\alpha\lambda\binom{1+}{\lambda x}^{-\theta(i+1)-1}$$
$$= \omega_{ij}\ ab\theta\alpha\lambda(1+\lambda x)^{-\theta(i+1)-1} \tag{14}$$

Where:

$$\omega_{ij} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b}{j}\binom{\alpha a(j+1)-1}{i} \tag{15}$$

STATISTICAL PROPERTIES

In this study, researchers studied the statistical properties of the (KEL) distribution, specifically quantile function, moments and moment generating function.

Quantile function: The (KEL) quantile function, say x = Q (u) can be obtained by inverting Eq. 9. Researchers have:

$$x = Q(u) = \frac{1}{\lambda}\left[\left\{1-(1-u)^{\frac{1}{\alpha ab}}\right\}^{\frac{-1}{\theta}} - 1\right] \tag{16}$$

Researcher can easily generate X by taking u as a uniform random variable in (0, 1).

Moments: In this subsection, researchers discuss the rth moment for (KEL) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem: If X has (KEL) (Θ, x) Θ = (λ, θ, α, a, b), then the rth moment of X is given by the following:

$$\mu_r^{'}(x) = E(X^r) = \omega_{ij}\ \frac{ab\theta\alpha}{\lambda^r}\beta(\theta(i+1)-r, r+1) \quad (17)$$

Proof: Let, X be a random variable with density function Eq. 14. The rth ordinary moment of the (KEL) distribution is given by:

$$\begin{aligned}\mu_r^{'}(x) &= E(X^{r)} = \int_0^\infty x^r f(x,\Theta)dx \\ &= \omega_{ij} ab\theta\alpha\lambda \int_0^\infty x^r (1+\lambda x)^{-\theta(i+1)-1} dx\end{aligned} \quad (18)$$

Let, t = 1/1+λx then x = 1/λ (1/y-1), simplify Eq. 18, researchers get:

$$\mu_r^{'}(x) = \omega_{ij}\ \frac{ab\theta\alpha}{\lambda^r}\beta(\theta(i+1)-r, r+1)$$

Which completes the proof. Based on the first 4 moments of the (KEL) distribution, the measures of skewness A (Φ) and kurtosis k (Φ) of the (KEL) distribution can obtained as:

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^{\ 2}(\theta)\right]^{\frac{3}{2}}} \quad (19)$$

and:

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^2} \quad (20)$$

Moment generating function: In this subsection, researchers derived the moment generating function of distribution.

Theorem: If has distribution then the moment generating function has the following form:

$$M_x(t) = \sum_{r=0}^{\infty}\omega_{ij}\frac{ab\theta\alpha t^r}{\lambda^r r!}\left(\beta\theta(i+1) - r, r+1\right) \quad (21)$$

Proof: Researchers start with the well known definition of the moment generating function given by:

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_{KEL}(x,\Theta)dx$$

Since, the series expansion of is given by:

$$e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$$

Thus:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\int_0^\infty x^r f_{KEL}(x,\Theta)dx = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\mu_r(x) \quad (22)$$

Then, substituting from Eq. 17 into 22 researchers get:

$$M_x(t) = \sum_{r=0}^{\infty}\omega_{ij}\frac{ab\theta\alpha t^r}{\lambda^r r!}\left(\beta\theta(i+1) - r, r+1\right)$$

Which completes the proof. In the same way, the characteristic function of the KEL distribution becomes $\varphi_x(t) = M_x(i, t)$ where $I = \sqrt{-1}$ is the unit imaginary number.

DISTRIBUTION OF THE ORDER STATISTICS

In this study, researchers derive closed form expressions for the pdfs of the rth order statistic of the (KEL) distribution also the measures of skewness and kurtosis of the distribution of the rth order statistic in a sample of size n for different choices of n; r are presented in this study. Let $(x_1, x_2, ..., x_n)$ be a simple random sample from (KEL) distribution with cdf and pdf given by Eq. 9 and 10, respectively.

Let $x_1 \le x_2 \le, ..., \le x_n$ denote the order statistics obtained from this sample. Researchers now give the probability density function of $X_{r:n}$, say $f_{r:n}(x,\Phi)$ and the moments of $x_{r:n}$, r = 1, 2, ..., n. Therefore, the measures of skewness and kurtosis of the distribution of the $x_{r:n}$ are presented. The probability density function of $x_{r:n}$ is given by:

$$f_{r:n}(x,\Phi) = \frac{1}{B(r, n-r+1)}\left[F(x,\Phi)\right]^{n-r}\left[1 - F(x,\Phi)\right]^{n-r} f(x,\Phi) \quad (23)$$

Where, F (x, Φ) and f (x, Φ) are the cdf and pdf of the (KEL) distribution given by Eq. 9 and 10, respectively and Φ = (λ, θ, α, a, b). Since, $o<F(x, \Phi)^{n-r}$ for x>0 by using the binomial series expansion of $(1-F(x, \Phi)^{n-r}$ given by:

$$\left[1 - F(x,\Phi)\right]^{n-r} = \sum_{j=0}^{n-r}(-1)^j\begin{bmatrix} n-r \\ j \end{bmatrix}\left[F(x,\Phi)\right]^j \quad (24)$$

Researchers have:

$$f_{r:n}(x,\Phi)=\sum_{j=0}^{n-r}(-1)^{j}\begin{bmatrix}n-r\\ j\end{bmatrix}\left[F(x,\Phi)\right]^{r+j-1}f(x,\Phi) \quad (25)$$

Substituting from Eq. 9 and 10 into Eq. 25, researchers can express the kth ordinary moment of r_{th} the order statistics $x_{r:n}$ say $E(x_{r:n}^{k})$ as a liner combination of the kth moments of the (KEL) distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $x_{r:n}$ can be calculated.

LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATORS

In this study, researchers provide the regression based method estimators of the unknown parameters of the Kumaraswamy exponentiated Lomax which was originally suggested by Swain *et al.* (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, ..., Y_n$ is a random sample of size n from a distribution function G (.) and suppose $Y_{(i)}$, i = 1, 2, ..., n denotes the ordered sample. The proposed method uses the distribution of $GY_{(i)}$. For a sample of size n, researchers have:

$$E\left(G\left(Y_{(j)}\right)\right)=\frac{j}{n+1},V\left(G\left(Y_{(j)}\right)\right)=\frac{j(n-j+1)}{(n+1)^{2}(n+2)}$$

and:

$$Cov\left(G\left(Y_{(j)}\right)\right),\left(G\left(Y_{(k)}\right)\right)=\frac{j(n-j+1)}{(n+1)^{2}(n+2)},\text{ for } j<k \quad (26)$$

Johnson, Kotz and Balakrishnan. Using the expectations and the variances, 2 variants of the least squares methods can be used.

Method 1 (least squares estimators): Obtain the estimators by minimizing:

$$\sum_{j=1}^{n}\left(G\left(Y_{(j)}\right)-\frac{j}{n+1}\right)^{2} \quad (27)$$

With respect to the unknown parameters. Therefore, in case of KEL distribution the least squares estimators of λ, θ, α a and b, say and λ_{LSE}, θ_{LSE}, α_{LSE}, b_{LSE}, respectively can be obtained by minimizing:

$$\sum_{j=1}^{n}\left[1-\left\{1-\left[1-\left(1+\lambda x_{(j)}\right)^{-\theta}\right]^{\alpha a}\right\}^{b}-\frac{j}{n+1}\right]^{2} \quad (28)$$

With respect to λ, θ, α a and b.

Method 2 (weighted least squares estimators): The weighted least squares estimators can be obtained by minimizing:

$$\sum_{j=1}^{n}w_{j}\left(G\left(\left(Y_{(j)}\right)-\frac{j}{n+1}\right)^{2} \quad (29)$$

With respect to the unknown parameters where:

$$w_{j}=\frac{1}{V\left(G\left(Y_{(j)}\right)\right)}=\frac{(n+1)^{2}(n+2)}{j(n-j+1)}$$

Therefore in case of KEL distribution, the weighted least squares estimators of λ, θ, α, a and b, say λ_{WLSE}, θ_{WLSE}, α_{WLSE}, a_{WLSE} and b_{WLSE}, respectively can be obtained by minimizing:

$$\sum_{j=1}^{n}w_{j}\left[1-\left\{1-\left[1-\left(1+\lambda x_{(j)}\right)^{-\theta}\right]^{\alpha a}\right\}^{b}-\frac{j}{n+1}\right]^{2}$$

With respect to the unknown parameters only.

MAXIMUM LIKELIHOOD ESTIMATION

In this study, researchers derive the non-linear equations for find the Maximum Likelihood Estimation (MLE) and inference of the parameters for the KEL distribution. The Maximum Likelihood Estimation is one of the most widely used estimation method for finding the unknown parameters. Here, researchers find the estimators for the KEL. Let $X_1, X_2, ..., X_n$ be a random sample from X~KEL(λ, θ, α, a, b)^T with observed values $x_1, x_2, ..., x_n$ and let Φ = (λ, θ, α, a, b)^T be the vector of the model parameters. The log likelihood function of Eq. 10 is defined as:

$$Log(L)=\begin{Bmatrix}n\log a+n\log b+n\log\theta+n\log\alpha+\\ n\log\lambda-(\theta+1)\sum_{i=1}^{n}\log(1+\lambda x_{i})+\\ (\alpha a-1)\sum_{i=1}^{n}\log\left[1-(1+\lambda x_{i})^{-\theta}\right]+\\ (b-1)\sum_{i=1}^{n}\log\left[1-\left[1-(1+\lambda x_{i})^{-\theta}\right]^{\alpha a}\right]\end{Bmatrix} \quad (30)$$

The score vector is U(Φ) = (∂L/∂λ, ∂L/∂θ, ∂L/∂α, ∂L /∂α, ∂L/∂b)^T where, the components corresponding to the model parameters are calculated by differentiating Eq. 31. By setting $z_i = (1+\lambda x_i)^{-\theta}$ researchers obtain:

$$\frac{\partial \text{Log}(L)}{\partial\lambda} = \left\{ \frac{n}{\lambda} - (\theta+1)\sum_{j=1}^{n}\frac{x_i}{Z_i^{-\theta}} + (\alpha a-1)\sum_{j=1}^{n}\frac{\theta x_i z_i^{\left(1+\frac{1}{\theta}\right)}}{[1-z_i]} + (b-1)\sum_{i=1}^{n}\frac{\alpha a[1-z_i]^{\alpha a-1}\theta x_i z_i^{\left(1+\frac{1}{\theta}\right)}}{\left[1-[1-z_i]^{\alpha a}\right]} \right\} \tag{31}$$

$$\frac{\partial \text{Log}(L)}{\partial\lambda} = \frac{n}{\theta} - \sum_{i=1}^{n}\log z_i + (\alpha a-1)\sum_{j=1}^{n}\frac{z_i \log z_i}{[1-z_i]} - (b-1)\sum_{i=1}^{n}\frac{\alpha a[1-z_i]^{\alpha a-1} z_i \log z_i}{\left[1-[1-z_i]^{\alpha a}\right]} \tag{32}$$

$$\frac{\partial \text{Log}(L)}{\partial\lambda} = \frac{n}{\alpha} + \alpha\sum_{i=1}^{n}\log(1-z_i) + (b-1)\sum_{i=1}^{n}\frac{\alpha(1-z_i)^{\alpha a}\log(1-z_i)}{1-(1-z_i)^{\alpha a}} \tag{33}$$

$$\frac{\partial \text{Log}(L)}{\partial\lambda} = \frac{n}{\alpha} + \alpha\sum_{i=1}^{n}\log(1-z_i) - (b-1)\sum_{i=1}^{n}\frac{\alpha(1-z_i)^{\alpha a}\log(1-z_i)}{1-(1-z_i)^{\alpha a}} \tag{34}$$

$$\frac{\partial \text{Log}(L)}{\partial\lambda} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-z_i)^{\alpha a} \tag{35}$$

Researchers can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear Eq. 31 and 35 to zero and solve them simultaneously.

CONCLUSION

Researchers introduce a new distribution, so-called the Kumaraswamy Exponentiated Lomax (KEL) distribution and study some of its general mathematical and statistical properties. The new model contains, especially sub-models such the Kumaraswamy Exponentiated Pareto (KEP) distribution, Kumaraswamy Lomax distribution, Kumaraswamy Pareto (KP) distribution and Kumaraswamy Lomax (KL) distributions. The estimation of parameters is approached by the method of least squares estimators and maximum likelihood estimations.

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