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A Fixed Point Result on Cone Matrix Spaces

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Abstract: In this study, we are exploring fixed point theory by using complete cone metric space and sequentially compact cone metric space and proved some fixed point results.

INTRODUCTION

Using modification of metric space by Huang and Zhang (2007) defined cone metric spaces by substituting an ordered normed space for the real numbers and proved some fixed point theorems of contractive mappings on cone metric spaces. In general, theory of cone metric space is used for contractive type mappings and fixed point on his theory has been developed by many mathematicians (Abbas and Jungck, 2008; Dhanorkar and Salunke, 2011; Abbas and Rhoades, 2009; Abdeljawad and Karapinar, 2009; Altun *et al*., 2010; Altun and Durmaz, 2009; Azam *et al*., 2008; Arshad *et al*., 2009; Azam and Arshad, 2009). In this study, we are expanding fixed point results by using contractive condition given in man result.

Preliminary notes: Huang and Zhang (2007) defined following cone metric space.

Definition 2.1: Let E always be a real Banach space and P a subset of E. P is called a cone if:

- P is closed, non-empty and $P \neq 0$
- $ax+by \in P$ for all $x, y \in P$ and non-negative real numbers a, b
- $P \cap (-P) = 0$

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \leq P$. $x<y$ will stand for $x \leq y$ and $x \neq y$ while $x \ll y$ will stand for $y-x \in intP$ where intP denotes the interior of P(1).

Definition 2.2: The cone P is called normal if there is a number $M>0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies:

$$\| x \| \leq\sim M \| y \|$$

The least positive number satisfying above is called the normal constant of P(1). It is clear that $M \geq 1$. In the following, let E be a normed linear space, P be a cone in E satisfying $int(P) \neq \emptyset$ and '$\leq$' denote the partial ordering on E with respect to P.

Definition 2.3: Let X be a non-empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies:

- $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, y) \leq d(x, z)+d(z, y)$ for all $x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space (Huang and Zhang, 2007).

Example 2.4: Let $E = R^2$, $P = (x, y)$ E: $x, y \geq 0$, X = R and $d: X \times X \to E$ defined by $d(x, y) = (|x- y|, \alpha|x-y|)$ where $\alpha \geq 0$ is constant. Then (X, d) is a cone metric space (Huang and Zhang, 2007).

Definition 2.5; Huang and Zhang (2007): Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then:

- $\{x_n\}_{n \geq 1}$ is said to converge to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$
- $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$
- (X, d) is called a complete cone metric space if every Cauchy sequence is convergent in X

Definition 2.6: Let (X, d) be a cone metric space and $B \subseteq X$. A point b in B is called an interior point of B whenever there exists a point p, $0 \ll p$ such that:

$$N(b, p) \subseteq B$$

where, $N(b, p): = \{y \in X: d(y, b) \ll p\}$. A subset $A \subseteq X$ is called open if each element of A is an interior point of A.

MAIN RESULTS

In this study, a common fixed point theorem is proved on a cone metric space under a contractive condition.

Theorem 2.7: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Let the mappings $T: X \to X$ satisfy the contractive condition:

$$d(Tx, Ty) \leq M(x, y), \text{ for all } x, y \in X \quad (1)$$

where, $M(x, y) = \max \{\alpha_1 d(x, y), \alpha_2 d(x, Tx), \alpha_3 d(y, Ty), \alpha_4[d(x, Ty)+d(y, Tx)]\}$ with $\alpha_1, \alpha_2, \alpha_3<1$ and $2\alpha_4<1$. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n(x)\}$ converges to a fixed point.

Proof: Choose $x_0 \in X$:

- $x_1 = Tx_0$
- $x_2 = Tx_1 = T^2x_0$
- $x_3 = Tx_2 = T^3x_0$
- $x_n = Tx_{n-l} = T^nx_0$
- $x_{n+1} = Tx_n = T^{n+1}x_0$

We have $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \max\{\alpha_1 d(x_n, x_{n-1}), \alpha_2 d(x_n, Tx_n), \alpha_3 d(x_{n-1}, Tx_{n-1}), \alpha_4[d(x_n, Tx_{n-1})+d(x_{n-1}, Tx_n)]\} = \max \{\alpha_1 d(x_n, x_{n-1}), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x_{n-1}, x_n), \alpha_4[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]\} = \max \{\alpha_1 d(x_n, x_{n-1}), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x_{n-1}, x_n), \alpha_4 d(x_{n-1}, x_{n+1})\} \leq \alpha_4 d(x_{n-1}, x_{n+1}) \leq \alpha_4[d(x_{n-1}, x_n)+ d(x_n, x_{n+1})$ gives $d(x_{n+1}, x_n) \leq \alpha_4\ d(x_{n-1}, x_n)$. Taking $k = \max \{\alpha_1, \alpha_3, \alpha_4/1-\alpha_4\}$:

- $\therefore d(Tx_n, Tx_{n+1}) \leq kd(Tx_{n-1}, Tx_n)$
- $\leq k^2 d(Tx_{n-2}, Tx_{n-1})$
- $\leq k^3 d(Tx_{n-3}, Tx_{n-2})$
- $\leq k^4 d(Tx_{n-4}, Tx_{n-3})$
- $\leq k^n d(Tx_0, Tx_1)$

So, for n>m, $d(x_n, x_m) \leq d(x_n, x_{n-1})+d(x_{n-1}, x_{n-2})+d(x_{n-2}, x_{n-3})+, ...,+d(x_{m+1}, x_m) \leq (k^{n-1}+k^{n-2}+k^{n-3}+, ..., +k^m)d(x_1, x_0) \leq km/1-k\ d(x_1, x_0)$. We get: $||d(x_n, x_m)|| \leq k^m/(1-k)K||d(x_1, x_0)||$.

This implies $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in X and (X, d) is cone metric space. By the completeness of X, there exists x in X such that $x_n \to x^*$ as $n \to \infty$. $d(Tx^*, x^*) \leq d(Tx_n, Tx^*)+d(Tx_n, x^*) \leq kd(x_n, x^*)+d(x_{n+1}, x^*)||d(Tx^*, x^*) \leq K(k||d(x_n, x^*)||+||d(x_{n+1}, x^*)||) \to 0$. Hence, $||d(Tx^*, x^*)|| = 0$. This implies $Tx^* = x^*$. So, x^* is a fixed point of T.

Uniqueness: Now if y^* is another fixed point of T then $d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*)<d(x , y)$, since, k<1. Which is contradiction. Hence, $||d(Tx^*, x^*)|| = 0$ and $x^* = y^*$. Therefore, the fixed point is unique.

Corollary 2.8: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose a mappings $T: X \to X$ satisfies for some positive integer n, $d(T^nx, T^ny) \leq M(x, y)$ where, $M(x, y) = \max \{\alpha_1 d(x, y), \alpha_2 d(x, Tx), \alpha_3 d(y, Ty), \alpha_4[d(x, Ty)+d(y, Tx)]\}$ for all $x, y \in X$ with $\alpha_1, \alpha_2, \alpha_3<1$, $2\alpha_4<1$. Then, T has a unique fixed point in X. Now, we can modified Theorem 2.7 by taking (X, d) be a sequentially com- pact cone metric space.

Corollary 2.9: Let (X, d) be a sequentially compact cone metric space and P be a normal cone with normal constant K. Suppose a mappings $T: X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq M(x, y)$ where, $M(x, y) = \max \{\alpha_1 d(x, y), \alpha_2 d(x, Tx), \alpha_3 d(y, Ty), \alpha_4[d(x, Ty)+d(y, Tx)]\}$ for all $x, y \in X$ with $\alpha_1, \alpha_2, \alpha_3<1$, $2\alpha_4<1$. Then T has a unique fixed point in X.

Proof: Choose $x_0 \in X$:

- $x_1 = Tx_0$
- $x_2 = Tx_1 = T^2x_0$
- $x_3 = Tx_2 = T^3x_0$
- $x_n = Tx_{n-1} = T^nx_0$
- $x_{n+1} = Tx_n = T^{n+1}x_0$

If for some n, $x_{n+1} = x_n$ then x_n has a fixed point of T, the proof is complete. Now, assumng that for all n, $x_{n+1} \neq x_n$. Set $d_n = d(x_n, x_{n+1})$ then $d_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq M(x_n, x_{n+1}) = M(x, y) = \max\{\alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x_{n+1}, x_{n+2}), \alpha_4[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})]\} = \max\{\alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x_{n+1}, x_{n+2}), \alpha_4 d(x_n, x_{n+2}) = \alpha_4[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\} = \alpha_4/1-\alpha_4\ d(xn, x_{n+1}) \leq d_n$.

We get decreasing sequence d_n and bounded by 0, since, P is regular, there is $d^* \in E$ such that $d_n \to d^*$ as $n \to \infty$. From the sequence compactness of X there are subsequence $\{x_{ni}\}$ of $\{x_n\}$ and $\{x^*\} \in X$ such that $\{x_{ni}\} \to \{x^*\}$ as $i \to \infty$. We have:

$$d(Tx_{ni}), Tx^*) \leq d(x_{ni}, x^*),\ i = 1, 2, ...$$
$$\text{So, } \|d(Tx_{ni}, Tx^*)\| K \|d(x_{ni}, x^*)\| \to 0$$

where, K is the normal constant of E. Hence, $Tx_{ni} \to Tx_*$ as $I \to \infty$. Similarly, $T^2x_{ni} \to T^2x^*$ as $i \to \infty$. By using Lemma 5, we have $d(Tx_{ni}; x_{ni}) \to d(Tx^*; x^*)$ as $i \to \infty$ and $d(T^2x_{ni}, Tx_{ni}) \to d(T^2x^*, Tx^*)$ as $i \to \infty$. It is obvious that $d(Tx_{ni}, x_{ni}) = d_{ni} \to d^* = d(Tx^*, x^*)$ as $i \to \infty$: Now, we shall prove that $Tx^* = x^*$. If $Tx^* \neq x^*$ then $d^* \neq 0$.

We have $d^* = d(Tx^*, x^*) > d(T^2x^*\ x^*) = \lim d(T^2x_{ni}, Tx_{ni}) = \lim d_{ni+1} = d^*$. We have a contradiction, so, $Tx^* = x^*$. That x^* is a fixed point of T. The uniqueness of fixed point is obvious.

Example 2.10: Let X = [0, 1/2), $E = R^2$ and P = (a, b): a and b are positive be a cone with metric d(x, y) = (|x-y|, |x-y|). Then (X, d) is a complete cone metric space with normal cone P. Define $T{:}X \to X$ by $Tx = x^2/2$. Then condition of Theorem 2:7 satisfying with fixed point 0 and $\{T^n(x)\}$ converges to fixed point, since $x \in X$.

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