Journal of Modern Mathematics and Statistics
MEDWELL PUBLICATIONS SINCE 2000

Automatic Continuity for Banach Algebras with Involution

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Key words: *-Algebra, *-simple algebra, *-semi-simple algebra, automatic continuity, separating ideal

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Page No.: 27-30
Volume: 14, Issue 2, 2020
ISSN: 1994-5388
Journal Modern Mathematic Statistic
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Abstract: Conditions are given for a Banach algebra A with involution which insure that every derivation on A is continuous. To do this, we define and study on a concept called *-semi-simplicity.

INTRODUCTION

In automatic continuity theory we are concerned with algebraic conditions on a linear map between Banach spaces which make this map automatically continuous. This theory has been mainly developed in the context of Banach algebras and there are excellent accounts on automatic continuity theory[1-4] in this associative context. In Singer and Wermer[5] proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical. They conjectured that the assumption of continuity is unnecessary. In Johnson[6] proved that if is a semi-simple Banach algebra, then every derivation on A is continuous and hence by the Singer-Wermer theorem it is zero.

In this research, we define and study on a unitary algebra provided with an involution *a notion which called *-semi-simplicity which generalizes the notion of semi-simplicity, it rests on the study of certain bilateral ideals called *-ideals. The interest, therefore is to restrict oneself to the level of a family of bilateral ideals instead of considering all the ideals on the left. This notion of *-semi-simplicity will also contribute to the study of the automatic continuity of linear operators on banach algebras in particular the continuity of derivations. We will show that on a *-semi-simple banach algebra, every derivation is continuous (Theorem 2).

PRELIMINARIES

In this study, the algebras considered are assumed complex, unitary, not necessarily commutative. An involution *on an algebra A is a mapping: satisfying the following:

Properties: $(x+y)^* = x^*+y^*$, $(x^*) = x$, $(xy)^* = y^*x^*$, $(\lambda x)^* = \bar{\lambda}x^*.\forall(x, y)\in A, \forall\lambda\in C.$

With involution *A is called *-algebra. An ideal of *-algebra is called *-ideal if $I^*\subseteq I$ (then $I^* = I$). Moreover, I is said to be a *-minimal (resp. * -maximal) ideal of A if is minimal (resp. maximal) in the set of nonzero (resp. proper) *-ideals of A. Observe that if I is an ideal of A, then $I+I^{*./}$, II^*, I^*I and $I\cap I^*$ are * -ideals of A. Moreover, if we denoted by $\overline{*}$ the map from A/I to A/I defined $\overline{*}$ $(a+I) = a^*+I$; then $\overline{*}$ a well-defined involution on A/I.

CHARATERIZATION OF *-SEMI-SIMPLE ALGEBRA

An algebra is called simple if it has no proper ideals. An *-algebra is called *-simple if it has no proper *-ideals. We observe that every simple algebra with involution (A, *) is a *-simple. The following counterexample shows the converse is not true.

Counterexample: Let A be a simple algebra, we denoted by A° the opposite al-gebra. Consider the algebra $B = A \oplus A^{\circ}$. Provided with the exchange involution defined by: $*(x, y) = (y, x)$. It is clear that B is not simple, since, the ideals of B are: $\{0\}$, B, $\{0\}\times A^{\circ}$, $A\times\{0\}$. But B is *-simple. Indeed, the only *-ideals of B are $\{0\}$ and B.

It is therefore, natural to ask under what conditions the converse is true. It is subject to the following proposition:

Proposition 1: Let (A, *) be *-simple algebra. If the involution * is anisotropic, then A is simple. Recall that involution is called anisotropic if; $\forall a\in A$; $a^*a = 0 \Rightarrow a = 0$.

Proof 1: Let I be an ideal of A, then $I\cap I^*$ is a *-ideal. It follows that, $I = \{0\}$ or A. Indeed, since, A is *-simple algebra, then $I\cap I^* = \{0\}$ or A. If $I\cap I^* = \{0\}$, then $x^*x = 0$ $\forall x\in I$. As that * is anisotropic, then $x = 0$, a result that $I = \{0\}$. if $I \cap I = A$, then $I = A$.

Proposition 2: Let A is a *-algebra. Then A is a *-simple if and only if, there exists a maximal ideal M such that, $M\cap M^* = \{0\}$.

Proof 2: $\Rightarrow$we assume A is *-simple. Let M be a maximal ideal of A, then $M\cap M^* = \{0\}$ or A. If $M\cap M^* = A$, then $M = A$ which contradicts the fact that M is proper ideal. Hence, $M\cap M^* = \{0\}$.

$\Leftarrow$Assume that, there exists a maximal ideal M such that $M\cap M^* = \{0\}$. Let I is a *-ideal of A. If $I\subseteq M$, then $I^* = I\subseteq M^*$ where $I\subseteq M\cap M^*) = \{0\}$. If $I\not\subseteq M$, then $A = M+I$ and we have: $M^* +I = (M^*+ I)\ A = (M^*+I)(M+I)\subseteq M^*M+I = I$. Which implies that $M^*\subseteq I$ as a result, $M\subset I$. Since, M is a maximal ideal of A, so, it follows that $A = I$.

Proposition 3: Tidli *et al*.[7] let A an *-simple algebra which is not simple. Then, there exists a sub-algebra simple unit I of A such that $A = I\oplus I^*$.

Proof 3: Let I a proper ideal of A. So, it follows that is $I\cap I^*$ is a *-ideal. Since, A is a *-simple algebra, then $I\cap I = \{0\}$ or A. If $I\cap I^* = A$, then $I = A$ which is absurd. from where $I\cap I^* = \{0\}$. There is also $I+I^*$ is a *-ideal, then $I+I^* = \{0\}$ or $= A$. If $I+I = \{0\}$, then $I = \{0\}$ which contradicts the fact that I is proper. Therefore, $A = I\oplus I^*$.

Let J an ideal of A such that $J\subseteq I$. According to what precedes, $A = J\oplus J^*$. Let $i\in I$, then there exists j, $j'\in J$ such that $i = j+(j')^*$. However, $i-j = (j')\in I\cap I^* = \{0\}$, from where $i = j$, therefore $I = J$. Consequently, I is a minimal ideal of A. Let J an ideal of I, then J is an ideal of A. Indeed, let $a \in A$ and $j\in J$, then it exists i, $(i')^*$ such that $a = i+(i')^*$. From where $aj = (i+(i')^*)j = ij+(i')^*j$. However, $(i')^*j\in I^*I$ and $I^*I\subseteq I\cap I = \{0\}$, it follows that $aj = ij\in J$. Since, I is a minimal ideal, $J = \{0\}$ or $I = J$. Thus, I is a simple sub algebra. On other hand, I a unital and if 1 indicates the unit of A, then there exists e, $e'\in I$ such that $1 = e+(e')^*$. Let $x\in I$, we are: $x = x1 = xe+x(e')^*$ but $x\text{-}xe = x(e^j)^*\in I\cap I^* = \{0\}$, from where $x = xe$. In the same way, we checked that $x = ex$. Consequently, I a unital of unit e.

Proposition 4: Let A be a *-algebra and M is a *-maximal ideal which not maximal. Then there exists a maximal ideal N of A such that $M = N\cap N^*$.

Proof 4: As M is not maximal, there is a maximal ideal N of A such that $M\subset N$. Since, $M = M\subseteq N^*$, it follows that $M\subseteq N\cap N^*$. As $N\cap N^*$ is a *-ideal of A, then $M = N\cap N^*$.

Definition 1: Let A be a *-algebra; We call *-radical of A, denoted Rad_*A, the inter-section of all *-maximals ideals of A. A is called *-semi-simple if $Rad_*A = \{0\}$.

Proposition 5: Let I be a *-ideal of a *-algebra A such that $I\subseteq Rad_*A$. So, $Rad_*(A/I) = Rad\ A/I$. In particular, A/Rad_*A is a *-semi-simple.

Proof 5: Let M is a *-maximal ideal of A. We put $\bar{A} = A/I$ and $\bar{M} = M/I$. We have: $I\subseteq Rad_*(A)\subseteq M$. So, from the following canonical isomorphism: $\bar{A}/\bar{M} \simeq A/M$ which is a *-simple; it follow that $\bar{A}/\bar{M}$ is a *-simple algebra. Consequently, M/I is a *-ideal *-maximal of A/I. From where:

$$Rad_*A/I = \bigcap\left\{\bar{M}\text{: M is }*\text{-maximal ideal of A}\right\} = \bigcap$$
$$\left\{\overline{\text{M: M is a }*\text{-maximal ideal of A}}\right\} = Rad_*\ A = Rad_*(A)/I$$

Now, we say that an algebra with involution (A, *) is a *-semi-simple if A is a sum of *-minimal ideals of A.

Lemma 1: Let A be a *-semi-simple algebra such that $A = \sum_{i\in S} = I_i$ where each I_i is a *-minimal ideal of A. If P is a *-minimal ideal of A, then there is a subset T of S such that: $A = P\oplus(\oplus_{j\in T}I_j)$.

Proof 6: Since, I_i are *-minimal and $P\neq A$, then there exists some $i\in S$ such that I_i+P is a direct sum. Indeed, otherwise $I_i\cap P = I_i$ for all $i\in S$ which implies that $P = A$. Applying Zorn' lemma, there is a subset T of S such that the collection $\{I_i: i\in T\}\cup\{P\}$ is a maximal with respect to independence: $(\oplus_{i\in T} I_j)+P = (\oplus_{i\in T} I_i\oplus P)$. Setting $B = \oplus_{i\in T} I_i+P$, the maximality of T implies that $I_i\cap B\neq(0)$ for all $i\in S$. Then, the *-minimality of I_i yields that $I_i\cap B = I_i$, hence, $I_i\subseteq B$ for all $i\in S$. Consequently, B = A.

Corollaire 6: For a algebra with involution (A, *), the following conditions are equivalent:

- A is *-semi-simple
- A is a direct sum of *-minimal *-ideals

Example: Let A_4 be the alternating group on 4 letters. Consider the group $\mathbb{R}[A_4]$ provided with its canonical involution *defined by $*(\sum_{g\in A4} r_g g) = \sum_{g\in A4} r_g g^{-1}$. From[1], the de-composition of the semi-simple algebra $\mathbb{R}[A_4]$ into a direct sum of simple components is as follows: $\mathbb{R}[A_4] = B_1\oplus B_2\oplus B_3$ where each B_i is invariant under*. More explicitly, $B_1\simeq\mathbb{R}$, $B_2\simeq C$ and $B_3\simeq M_3(\mathbb{R})$). In particular, each B_i is a *-minimal ideal of $\mathbb{R}[A_4]$. Consequently, $\mathbb{R}[A_4]$ is a *-semi-simple algebra.

Now, let A be a *-semi-simple algebra. Since, A is finitely generated (indeed, 1 generates A), then A has a finite length. Thus, $A = \oplus_{i=1}^{l} I_i$ where each I_i is a *-minimal ideal of A. It is easy to verify that each I_i is generated by a central symmetric idempotent element $e_i\in A$ (i.e.: $e^2_i = e_i$ and $^*(e_i) = e_i$) where $1 = \sum_{i=1}^{l} e_i$. Moreover, $e_ie_j = 0$ for all $i\neq j$. In what follows, we denoted by S the set of central symmetric orthogonal idempotents of A, i.e., $S = \{e_1, ..., e_l\}$ such that $I_i = Ae_i$.

Let $A = \oplus_{i=1}^{l} I_i$ be a *-semi-simple algebra, we have already seen that each I_i is gen-erated by a central symmetric idempotent e_i such that $1 =\sum_{i=1}^{l} e_i$. Hence, I_i is a sub algebra of A with unity e_i. Moreover, I_i is a *-simple algebra for all $1\leq i\leq l$. Consequently, every *-semi-simple algebra is a direct sum of *-simple algebras.

AUTOMATIC CONTINUITY

A derivation D on an algebra A is a linear mapping from A to itself satisfying $D(xy) = D(x)y+xD(y)$ for all x, $y\in A$. Let D a derivation of a banach algebra X. Then, the separating ideal $\delta(D)$ of X is the subset of X defined by: $\delta(D) = \{y\in X/\exists(x_n)_n\subseteq X: x_n\to 0 \text{ and } D(x_n)\to y\}$.

Lemma 2: Sinclair[4] let S be a linear operator from a Banach space X into a Banach space Y then:

- $\delta(S)$ is a closed linear space of Y
- S is continuous if only if $\delta(S) = \{0\}$
- If T and R are continuous linear operators on X and Y, respectively and if ST = RS, then $R(\delta(S))\subset\delta(S)$

Lemma 3: Sinclair[4] let S be a linear operator from a banach space X into a banach space Y and let R be a continuous operator from Y into a banach space Z. Then:

- RS is continuous if only if $R\delta(S) = \{0\}$
- $\overline{R\delta(S)} = \delta(RS)$
- There is a constant M (independent of R and Z) such that if RS is continuous then $\|RS\|\leq M\,\|R\|$

Proposition 7: Let A be a Banach *-algebra A. Then if M is a *-maximal ideal of A, then M is closed.

Proof: If M is a maximal ideal of A, then M is closed. Otherwise, if M not maximal ideal, then there exists a maximal ideal N of A such that $M = N\cap N^*$ (Proposition 4). Since, N (resp. N^*) is closed, it is deduced that M is closed in A.

Proposition 8: Let A be a Simple Banach Algebra. Then all derivations D on A is continuous.

Proof 7: Let $\delta(D)$ the separating ideal of D in A which is simple, so $\delta(D) = \{0\}$ or $\delta(D) = A$. If $\delta(D) = A$, that $e_A \in \delta(D)$, consequently $0 \in Sp(e_A)$ ([4]theorem 6-16). From where $\delta(D) = \{0\}$. And by Lemma (2) as a result D is continuous.

Theorem 1: Let A be a *-Simple Banach Algebra. Then all Derivation on A is continuous.

Proof 8: We have A is *-simple, there exists simple unital sub algebra I of A such that: $A = I\oplus I^*$ (Proposition 3); following algebraic isomorphism: $I\simeq A/I$, we deduce that I is a maximal ideal of A. From where I (resp. I^*) is closed in A.

Consequently, the algebra A/I (resp. A/I^*) is a simple banach algebra. Since, I is an ideal of A, then, so is D(I)+I; therefore, D(I)+I/I is an ideal of A/I. As A/I is a simple algebra, so, $D(I)+I/I = \{\bar{0}\}$ or D(I)+I/I = A/I. As I is a maximal ideal of A, then $D(I)+I/I = \{\bar{0}\}$, it followings that D(I)+I = I, so, $D(I)\subseteq I$. Consider the fonction $\tilde{D}$ on A/I defined by: $\tilde{D}(a+I) = D(a)+I$. it is clear that $\tilde{D}$ is well-defined, since, $D(I)\subseteq I$. Now, we show that $\tilde{D}$ is a derivation on A/I. Note that it is easy to show $\tilde{D}$ is a linear operator. Moreover, for a, $b\in A$, $\tilde{D}(a+I)(b+I) = \tilde{D}(ab+I) = D(ab)+I = aD(b)+D(a)b+I$. But then, $(a+I)\tilde{D}(b+I)+(\tilde{D}(a+I))(b+I) = (a+I)(D(b)+I)+(D(a)+I)(b+I) = aD(b)+I+D(a)b+I = aD(b)+D(a)b+I$. So, $\tilde{D}$ is a derivation on the simple banach algebra A/I, then by proposition (8), $\tilde{D}$ is continuous. To show that D is continuous, consider the canonical surjection π: $A\to A/I$; $a\to a+I$ which is continuous. In addition, we observe first that $\pi\circ D = \tilde{D}\circ\pi$ because for every $a\in A$, we have $\pi\circ D(a) = \pi(D(a)) = D(a)+I$ and $\tilde{D}\,\pi(a) = \tilde{D}(a+I) = D(a)+I$. $\tilde{D}\circ\pi$ is continuous, then we have $\delta(\tilde{D}\circ\pi) = \{0\}$ and $\pi\delta(D) = \delta(D\circ\pi) = \{0\}$ (Lemma 3) and this implied that $\delta(D)\subset I$. Following the same steps, we show that $\delta(D)\subset I^*$, then $\delta(D)\subset I\cap I^* = \{0\}$. Therefore, D is continuous (Lemma 2).

Theorem 2: Let A be a *-semi-simple Banach Algebra. Then all Derivation on A is continuous.

Proof 9: Since, A is a *-simple algebra, writing (by Lemma 1) $A = \oplus_{i=1}^{l} I_i$ where I_i isa *-minimal ideal of A and setting $L_i = \oplus_{i \neq j} I_j$, then $\forall 1 \leq i \leq l$, L_i is a *-maximal ideal of A. If L_i is a maximal ideal, then $D(L_i) \subseteq L_i$, if L_i is not maximal, then by Proposition (4) that exists a maximal ideal N_i such that $\forall 1 \leq i \leq l$, $L_i = N_i \cap N^*_i$. Consequently, $D(L_i) = D(N \cap N^*_i) \subseteq D(N_i) \cap D(N^*_i) \subseteq N_i \cap N^*_i = L_i$, $\forall 1 \leq i \leq l$. Now, consider the function $\tilde{D}$ on A/I defined by: $\forall 1 \leq i \leq l$, $\tilde{D}(a+L_i) = D(a)+L_i$.

Since, a *-maximal ideal is closed (Proposition 7) and as mentioned in theorem (1) wehave $\tilde{D}$ is a derivation on the *-simple Banach algebra A/L_i. Then by theorem (1), $\tilde{D}$ is continuous. To show that D is continuous, we observe first that $\pi \circ D = \tilde{D} \circ \pi$ (where π is the canonical surjection from A to A/I) because for every $a \in A$, we have $\pi \circ D(a) = \pi(D(a)) = D(a)+L$ and $\tilde{D}\,\pi(a) = \tilde{D}(a+L_i) = D(a)+L_i$. Since, $\tilde{D} \circ \pi$ is continuous, then we have $\delta(\tilde{D} \circ \pi) = \{0\}$ (Lemma 2) and as $\overline{\pi \delta D} = \delta(\tilde{D} \circ \pi) = \{0\}$ (Lemma 3); this implied that $\delta(D) \subset L_i$; $\forall 1 \leq i \leq l$. Tt follows that $l\, \delta(D) \subset \bigcap_{i=1}^{l} L_i = \{0\}$. Consequently, D is continuous.

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