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Lower Order Perturbations of Critical Fractional Laplacian Equations*

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Abstract: We give sufficient conditions for the existence of nontrivial solutions to a class of critical nonlocal problems of the Brezis-Nirenberg type. Our result extends some results in the literature for the local case to the nonlocal setting. It also complements the known results for the nonlocal case.

INTRODUCTION

Nonlinear elliptic equations involving critical Sobolev exponents have been extensively studied in the literature, beginning with the following celebrated result of Brezis and Nirenberg[1].

Theorem 1.1: Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$ and consider the problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where, $\lambda>0$ is a parameter and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Let $\lambda_1>0$ be the first dirichlet eigenvalue of $-\Delta$ in Ω.

- If $n \geq 4$, then problem (1.1) has a solution for all $\lambda \in (0, \lambda_1)$
- If n = 3, then there exists $\lambda_* \in [0, \lambda_1]$ such that problem Eq. 1 has a solution for all $\lambda \in (\lambda_*, \lambda_1)$
- If n = 3 and $\Omega = B_1(0)$ is the unit ball, then $\lambda_* = \lambda_1/4$ and problem Eq. 1 has no solution for $\lambda \leq \lambda_1/4$

Following[1], Gazzola and Ruf[2] considered the more general problem:

$$\begin{cases} -\Delta u = g(x, u) + |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

where, g is a Caratheodory function on $\Omega \times \mathbb{R}$ with sub critical growth:

$$\lim_{|t| \to +\infty} \frac{g(x,t)}{|t|^{2^*-1}} = 0$$

uniformly a.e., on Ω. Let $0 < \lambda_1 < \lambda_2 \leq, ..., \to +\infty$ be the sequence of Dirichet eigenvalues of $-\Delta$ in Ω, repeated according to multiplicity. The following extensions of Theorem 1.1 were obtained by Gazzola and Ruf[2].

Theorem 1.2: Assume the following conditions on g; for all $\epsilon>0$, there exists $a_\epsilon \in L^{2n/(n+2)}(\Omega)$ such that $|g(x, t)| \leq a_\epsilon(x) + \epsilon|t|^{2^*-1}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$. $G(x, t) := \int_0^t g\ (x,$

$\tau)d\tau\geq 0$ for a.a $x\in\Omega$ and all $t\in\mathbb{R}$; there exist $k\in\mathbb{N}$, δ, $\sigma>0$ and $\mu\in(\lambda_k, \lambda_{k+1})$ such that $1/2(\lambda_k+\sigma)t^2\leq G(x, t)\leq 1/2\ \mu\ t^2$ for a.a. $x\in\Omega$ and $|t|\leq\delta$; $G(x, t)\geq 1/2\ (\lambda_k+\sigma)t^2-\frac{1}{2^*}t^{2^*}$ for a.a $x\in\Omega$ and all $t\in\mathbb{R}$; if n = 3, there exists a nonempty open subest Ω_0 of Ω such that:

$$\lim_{t\to+\infty} e\ \frac{G(x,t)}{t^4}=+\infty$$

uniformly a.e. on Ω_0. Then problem (2) has a nontrivial solution.

Theorem 1.3: Assume conditions (1), (2) and there exists $\delta>0$, $k\in\mathbb{N}$ and $\mu\in(\lambda_k, \lambda_{k+1})$ such that $1/2\ \lambda_k\ t^2\leq 1/2\mu t^2$ for a.a. $x\in\Omega$ and $|t|\leq\delta$; there exists $\sigma\in(0, 1/2^*)$ such that $G(x, t)\geq 1/2\mu\ t^2-(1/2^*-\sigma)\ |t|^{2^*}$ for a.a. $x\in\Omega$ and all $t\in\mathbb{R}$; there exists a nonempty open subset Ω_0 of Ω such that:

$$\lim_{t\to+\infty} e\ \frac{G(x,t)}{t^{8n/(n^2-4)}}=+\infty$$

uniformly a.e. on Ω_0. Then, problem (1.2) has a nontrivial solution. Other extensions and generalizations can be found, e.g., by Capozzi *et al*.[3], Cerami *et al*.[4] and Tarantello[5]. More recently, Servadei and Valdinoci[6, 7] considered the nonlocal critical problem:

$$\begin{cases}(-\Delta)^s u=\lambda u+|u|^{2_s^*-2}u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\backslash\Omega\end{cases}\quad(3)$$

where, $s\in(0, 1)$, Ω is a bounded domain in $\mathbb{R}^n$, $n>2s$ with Lipschitz boundary, $\lambda>0$ is a parameter and $2^*_s = 2n/(n\ 2s)$ is the fractional critical Sobolev exponent. Here $(\Delta)^s$ is the fractional Laplacian operator, defined, up to a normalization factor, on smooth functions by:

$$(-\Delta)^s u(x)=2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n\backslash B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy,\quad x\in\mathbb{R}^n$$

Let us recall the definition of a weak solution of problem Eq. 3. Let:

$$H^s(\mathbb{R}^n)=\left\{u\in L^2(\mathbb{R}^n):\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}dx\ dy<+\infty\right\}$$

be the usual fractional Sobolev space endowed with the Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^n)}:=\left(\|u\|^2_{L^2(\mathbb{R}^n)}+\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}dx\ dy\right)^{1/2}$$

and let:

$$H_0^s(\Omega)=\left\{u\in H^s(\mathbb{R}^n):u=0\text{ a.e. in }\mathbb{R}^n\backslash\Omega\right\}$$

Then, $H_0^s(\Omega)$ is a closed linear subspace of $H^s(\mathbb{R}^n)$, equivalently renormed by the Gagliardo seminorm:

$$[u]_s:=\left(\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}dx\,dy\right)^{1/2}$$

and the imbedding $H_0^s(\Omega)\hookrightarrow L^r(\Omega)$ is continuous for $r\in[1, 2_s^*]$ and compact for $r\in[1, 2_s^*]$ [8]. A weak solution of problem Eq. 3 is a function $u\in H_0^s(\Omega)$ satisfying:

$$\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))(\upsilon(x)-\upsilon(y))}{|x-y|^{n+2s}}dx\ dy=\int_\Omega\left(\lambda u(x)+|u(x)|^{2_s^*-2}u(x)\right)\upsilon(x)dx\quad(4)$$

Let $0<\lambda_1<\lambda_2\leq, \ldots, \to+\infty$ denote the sequence of eigenvalues of the nonlocal eigenvalue problem:

$$\begin{cases}(-\Delta)^s u=\lambda u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\backslash\Omega\end{cases}$$

repeated according to multiplicity (Proposition)[9]. Servadei and Valdinoci[6, 7] obtained the following results.

Theorem 1.4: If $n\geq 4s$, then problem (3) has a nontrivial weak solution for each $\lambda>0$ that is not an eigenvalue of (4).

Theorem 1.5: If $2s<n<4s$, then there exists $\lambda s>0$ such that problem Eq. 3 has a nontrivial weak solution for each $\lambda>\lambda s$ that is not an eigenvalue of Eq. 4. By Servadei and Valdinoci[10], they also considered the more general problem:

$$\begin{cases}(-\Delta)^s u=\lambda u+|u|^{2_s^*-2}u+f(x, u) & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\backslash\Omega\end{cases}\quad(5)$$

where, f is a Caratheodory function on $\Omega\times R$ and obtained the following result.

Theorem 1.6: Assume the following conditions:

- For all $M>0$, $\sup\{|f(x, t)|: x\in\Omega, |t|\leq M\}<+\infty$
- $\lim_{|t|\to+\infty}\frac{f(x, t)}{t}=0$ uniformly a.e. on Ω
- $\lim_{|t|\to+\infty}\frac{f(x, t)}{|t|^{2_s^*-1}}=0$ uniformly a.e. on Ω

If $n \geq 4s$, then problem Eq. 5 has a nontrivial weak solution for all $\lambda \in (0, \lambda_1)$. In the present paper we consider the problem:

$$\begin{cases} (-\Delta)^s u = g(x, u) + |u|^{2_s^*-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \quad (6)$$

where $s \in (0, 1)$, Ω is a bounded domain in $\mathbb{R}^n$, $n>2s$ with Lipschitz boundary and g is a Caratheodory function on $\Omega \times \mathbb{R}$. Our main result is the following theorem.

Theorem 1.7: Assume the following conditions:

- H_1 there exist $p \in [1, 2_s^*)$ and $C>0$ such that $|g(x, t)| \leq C(|t|^{p-1}+1)$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$
- H_2 $G(x, t) \int_0^t g(x, \tau) d\tau \geq 0$ fora.a. $x \in \Omega$ and all $t \in \Omega$ and all $t \in \mathbb{R}$
- H_3 there exist $k \in \mathbb{N}$, $\delta, \sigma > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that $1/2\ (\lambda_k+\sigma)t^2 \leq G(x, t) \leq /2\mu\ t^2$ for a.a. $x \in \Omega$ and $|t| \leq \delta$
- H_4 $G(x, t) \geq 1/2\ (\lambda_k+\sigma)\ t^2 - \frac{1}{2_s^*}|t|^{2_s^*}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$
- H_5 there exists a nonempty open subset Ω_0 of Ω such that $\lim_{|t| \to +\infty} \frac{G(x, t)}{t^{(n+2s)/(n-2s)}} = +\infty$ uniformly a.e. on Ω_0

Then problem Eq. 6 has a nontrivial weak solution. Theorem 1.7 extends the results of Gazzola and Ruf[2] to the nonlocal case and complements the results of Servadei and Valdinoci[6, 7, 10]. This theorem will be proved after some preliminaries in the next section.

PRELIMINARIES

A function $u \in H_0^s(\Omega)$ is a weak solution of problem Eq. 6 if:

$$\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)\left(u(x)-u\right)(y)\right)}{|x-y|^{n+2s}} dx\ dy = \int_\Omega \left(g(x, u) + |u(x)|^{2_s^*-2} u(x)\right) \upsilon(x) dx$$

for all $u \in H_0^s(\Omega)$. Weak solutions coincide with critical points of the C^1-functional:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(G(x, u) + \frac{1}{2_s^*}|u|^{2_s^*}\right) dx, \quad u \in H_0^s(\Omega)$$

Recall that E satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$ or the $(PS)_c$ condition for short, if every sequence $(u_j) \subset H_0^s(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$, called a $(PS)_c$ sequence has a convergent subsequence. Let:

$$S = \inf_{u \in H_0^s(\Omega) \backslash \{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)\right)^2}{|x-y|^{n+2s}} dx\ dy}{\left(\int_\Omega |u|^{2_s^*} dx\right)^{2/2_s^*}} \quad (7)$$

be the best constant for the fractional Sobolev imbedding $H_0^s(\Omega) \to L^{2_s^*}(\Omega)$. Proof of theorem 1.7 will be based on the following proposition.

Proposition 2.1: If $0 < c < s/n\ S^{n/2s}$, then every $(PS)_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of E.

Proof: Let (u_j) be a $(PS)_c$ sequence. Then:

$$E(u_j) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(G(x, u_j) + \frac{1}{2_s^*}|u_j|^{2_s^*}\right) dx = c + o(1) \quad (8)$$

and

$$E(u_j)u_j = \int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(u_j g(x, u_j) + |u_j|^{2_s^*}\right) dx = o(1)\|u_j\| \quad (9)$$

Dividing Eq. 9 by 2 and subtracting from Eq. 8 gives:

$$\int_\Omega \left[\frac{1}{2} u_j g(x, u_j) - G(x, u_j) + \frac{s}{n}|u_j|^{2_s^*}\right] dx = o(1)\|u_j\| + O(1)$$

which together with (H_1) and the Holder and Young's inequalities gives:

$$\int_\Omega |u_j|^{2_s^*} dx \leq o(1)\|u_j\| + O(1)$$

This together with (H_1) and Eq. 8 implies that (u_j) is bounded in $H_0^s(\Omega)$. So, a renamed subsequence converges to some u weakly in $H_0^s(\Omega)$ strongly in $L^q(\Omega)$ for all $q \in [1, 2_s^*]$ and a.e. in Ω. Then, u is a critical point of E by the weak continuity of E'. Suppose u = 0. Since, (u_j) is bounded in $H_0^s(\Omega)$ and converges to 0 in $L^p(\Omega)$, Eq. 9, (H_1), and Eq. 7 give:

$$O(1) = \int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega |u_j|^{2_s^*} dx \geq \|u_j\|^2 \left(1 - \frac{\|u_j\|^{2_s^*-2}}{S^{2_s^*/2}}\right)$$

If $\|u_j\| \to 0$, then $E(u_j) \to 0$, contradicting $c > 0$, so, this implies:

$$\|u_j\|^2 \geq S^{n/2s} + o(1)$$

for a renamed subsequence. Dividing Eq. 9 by 2*s and subtracting from Eq. 8 then gives:

$$c = \frac{s}{n}\int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}}\,dx\,dy + o(1) \geq \frac{s}{n} S^{n/2s} + o(1)$$

contradicting $c < \frac{s}{n} S^{n/2s}$. To produce $(PS)_c$ sequences with $0<x<s/n\ S^{n/2s}$, we will use the following linking theorem of Rabinowitz[11, 12].

Theorem 2.2: Let E be a C^1 functional on a Banach space V and let $V = V^- \oplus V^+$ be a direct sum decomposition with dim $V^- < \infty$. Assume that there exist $R>\rho>0$ and $w_0 \in V^+$ with $\|w_0\| = 1$ such that:

$$\max_{u\in\partial Q} E(u) < \inf_{u\in\partial B_\rho \cap V^+} E(u)$$

where:

$$Q = \left\{u + two: \upsilon \in V^-, \|\upsilon\| \leq R,\ t \in [0, R]\right\}$$

Let $\Gamma = \{h \in C(Q, V): h|_{\partial Q} = id\}$ and set:

$$c := \inf_{h\in\Gamma} \max_{u\in h(Q)} E(u)$$

Then:

$$\inf_{u\in\partial B_\rho \cap V^+} E(u) \leq c \leq \max_{u\in Q} E(u)$$

and E has a $(PS)_c$ sequence.

Proof of Theorem 1.7: In this section we prove Theorem 1.7. Let $e_1, ..., e_k$ be L^2-orthonormal eigenfunc-tions for $\lambda_1, ..., \lambda_k$, let $H^- = \text{span}\{e_1, ..., e_k\}$ and let $H^+ = (H^-)^\perp$. Without loss of generality we may assume that $0 \in \Omega_0$. For $m \in \mathbb{N}$, so, large that $B4/m := \{x \in \mathbb{R}^n: |x| < 4/m\} \subset \Omega_0$, let:

$$\zeta_m(x) = \begin{cases} 0, & x \in B_{1/m} \\ m|x|-1, & x \in A_m = B_{2/m} \backslash B_{1/m} \\ 1, & x \in \Omega \setminus B_{2/m} \end{cases}$$

It is easily seen that:

$$\left|\zeta_m(x) - \zeta_m(y)\right| \leq m|x-y| \quad \forall x, y \in \Omega \tag{10}$$

Let $e_j^m = \zeta_m e_j,\ j = 1, ..., k$ and let $H_m^- = \text{span}\left\{e_1^m, ..., e_k^m\right\}$

Lemma 3.1: Let $f \in L^\infty(\Omega)$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then:

$$\|\zeta_m u\|^2 \leq \|u\|^2 + \frac{C|f|_\infty^2}{m^{n-2s}}$$

where, $C = C(n, \Omega, s) > 0$. To prove this lemma we will need the following estimates from[13].

Lemma 3.2; ([6], Lemma 2.3): Let $f \in L^q(\Omega)$, $1<q\leq\infty$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then $|u|_r \leq C|f|_q$ where:

$$r = \begin{cases} nq/(n-2sq), & 1<q<n/2s \\ \infty, & n/2s<q\leq\infty \end{cases}$$

and $C = C(n, \Omega, s, q) > 0$. In particular, if $f \in L^\infty(\Omega)$, then $|u|_\infty = C\,|f|_\infty$.

Lemma 3.3 (Lemma 2.5)[13]: Let $f \in L^q(\Omega)$, $n/2s<q\leq\infty$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then:

$$\|\varphi u\|^2 \leq C|f|_q^2 \left(|\varphi|_{2q'}^2 + \|\varphi\|^2\right) \quad \forall \varphi \in L^{2q'}(\Omega) \cap H_0^s(\Omega)$$

where, $C = C(n, \Omega, s, q) > 0$ and $q' = q/(q-1)$.

Proof of Lemma 3.1: We have:

$$\begin{aligned} \|\zeta_m u\|^2 \leq & \int_{A1} \frac{\left(u(x)-u(y)\right)^2}{|x-y|^{n+2s}}\,dx\,dy + \\ & \int_{A2} \frac{\left|\zeta_m(x)u(x) - \zeta_m(y)u(y)\right|^2}{|x-y|^{n+2s}}\,dx\,dy + \\ & 2\int_{A3} \frac{\left(\zeta_m(x)u(x) - u(y)\right)^2}{|x-y|^{n+2s}}\,dx\,dy =: I_1 + I_2 + I_3 \end{aligned}$$

where, $A_1 = B_{2/m}^c \times B_{2/m}^c$, $A_2 = B_{3/m} \times B_{3/m}$ and $A_3 = B_{2/m} \times B_{3/m}^c$ we have $I_1 \leq \|u\|^2$. To estimate I_2, let:

$$\varphi_m(x) = \begin{cases} \zeta_m(x), & x \in B_{3/m} \\ 4-m|x|, & x \in B_{4/m} \backslash B_{3/m} \\ 0, & x \in B_{4/m}^c \end{cases}$$

Applying Lemma 3.3 to φ_m with $q = \infty$:

$$I_2 \leq \|\varphi_m u\|^2 \leq C|f|_\infty^2 \left(|\varphi_m|_2^2 + \|\varphi_m\|^2\right)$$

where, $C = C(n, \Omega, s) > 0$. Since, $\varphi_m(x) = \varphi_1(mx)$:

$$|\varphi_m|_2^2 = \int_{\mathbb{R}^n} \varphi_m(x)^2\,dx = \int_{\mathbb{R}^n} \varphi_1(mx)^2\,dx = \frac{|\varphi_1|_2^2}{m^2}$$

and:

$$\|\varphi_m\|^2 = \int_{\mathbb{R}^{2n}} \frac{|\varphi_m(x)-\varphi_m(y)|^2}{|x-y|^{n+2s}} dx\, dy = \int_{\mathbb{R}^{2n}} \frac{|\varphi_1(mx)-\varphi_1(my)|^2}{|x-y|^{n+2s}} dx\, dy = \frac{\|\varphi_1\|^2}{m^{n-2s}}$$

So:

$$I_2 \le \frac{C|f|_\infty^2}{m^{n-2s}}$$

For $(x, y) \in A_3$, $|x-y| \ge |y|-|x| > |y|-2/m \ge |y|-(2/3)|y| = |y|/3$, so:

$$I_3 \le C|u|_\infty^2 \int_{A_3} \frac{1}{|y|^{n+2s}} dx\, dy \le \frac{C|f|_\infty^2}{m^{n-2s}}$$

by Lemma 3.2. The desired conclusion follows.

Lemma 3.4: We have $e_j^m \to e_j$ in $H_0^s(\Omega)$ as $m \to \infty$ and:

$$\max_{\left\{u \in H_m^-: \int_\Omega u^2 dx = 1\right\}} \|u\|^2 \le \lambda_k + \frac{C}{m^{n-2s}} \quad (11)$$

for some constant C>0.

Proof: We have:

$$\|e_j^m - e_j\|^2 = \int_{\mathbb{R}^{2n}} \frac{\left[\begin{matrix}(\zeta_m(x)e_j(x)-e_j(x))- \\ (\zeta_m(y)e_j(y)-e_j(y))\end{matrix}\right]^2}{|x-y|^{n+2s}} dx\, dy = \int_{\mathbb{R}^{2n}} \frac{\left|\begin{matrix}e_j(x)[\zeta_m(x)-\zeta_m(y)]+ \\ [\zeta_m(y)-1][e_j(x)e_j(y)]\end{matrix}\right|^2}{|x-y|^{n+2s}} dx\, dy \le \quad (12)$$

$$2\int_{\mathbb{R}^{2n}} \frac{e_j(x)^2[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\, dy + \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y)-1]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\, dy \le 2\left(|e_j|_\infty^2 I_1 + I_2\right)$$

Where:

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\, dy,$$

$$I_2 \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y)-1]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\, dy$$

We will show that I_1 and I_2 go to 0 as $m \to \infty$. Since, $\zeta_m = 1$ in $B_{2/m}^c$:

$$I_1 = \int_{B_{2/m} \times B_{2/m}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\, dy + 2I_1 = \int_{B_{2/m} \times B_{2/m}^c} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\, dy =: I_3 + 2I_4$$

Write:

$$\int_{B_{2/m} \times B_{2/m}^c} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\, dy + \int_{B_{2/m} \times (B3_{/m} \setminus B_{2/m})} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\, dy =: I_5 + I_6$$

Clearly, I_3 and I_6 are less than or equal to:

$$\int_{B_{2/m} \times B3_{/m}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\, dy =: I_7$$

so, $I_1 = 2I_5 + 3I_7$. To estimate I_5 and I_7, we change variables from (x, y) to (x, ζ) where, ζ = x-y. For $(x, y) \in B_{2/m} \times B_{3/m}^c$, $|\xi| \ge |y|-|x| > 1/m$ and hence:

$$I_5 \le \int_{B_{2/m} \times B_{2/m}^c} \frac{dx\, dy}{|x-y|^{n+2s}} \le \int_{B_{2/m} \times B_{1/m}^c} \frac{dx\, dy}{|\xi|^{n+2s}} \le \frac{C}{m^{n-2s}} \quad (13)$$

For $(x, y) \in B_{2/m} \times B_{3/m}$, $|\xi| \le |x|+|y| < 5/m$ and hence (11) gives:

$$I_7 \le m^2 \int_{B_{2/m} \times B_{3/m}} \frac{dx\, dy}{|x-y|^{n-2(1-s)}} \le m^2 \int_{B_{2/m} \times B_{5/m}} \frac{dx\, dy}{|\xi|^{n-2(1-s)}} \le \frac{C}{m^{n-2s}}$$

Thus, $I_1 \le C/m^{n-2s}$. Now we estimate I_2. We have:

$$I_2 = \int_{\mathbb{R}^n \times B_{2/m}} \frac{[1-\zeta_m(y)]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\, dy \le I_8 + 4|e_j|_\infty^2 I_9$$

Where:

$$I_8 \int_{B_{2/m} \times B_{2/m}} \frac{[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\, dy, \quad = I_9 \int_{B_{3/m}^c \times B_{2/m}} \frac{dx\, dy}{|x-y|^{n+2s}}$$

Since, $e_j \in H_0^s(\Omega)$ and $|B_{3/m} \times B_{2/m}| \to 0$, $I_8 \to 0$. As in Eq. 13, $I_9 \le C/m^{n-2s}$. Thus, $I_2 \le C/m^{n-2s} + o(1)$. To prove Eq. 11, let $\upsilon = \sum_{j=1}^k \alpha_j e_j \in H^-$. By Lemma 3.1:

$$\|\zeta_m \upsilon\|^2 \le \|\upsilon\|^2 + \frac{C|f|_\infty^2}{m^{n-2s}} \quad (14)$$

Where:

$$f = (-\Delta)^s \upsilon = \sum_{j=1}^k \lambda_j \alpha_j e_j \in H^-$$

Since, dim $H^- < \infty$:

$$|f|_\infty^2 \le c_1|f|_2^2 = c_1\sum_{j=1}^{k}\lambda_j^2\alpha_j^2 \le c_1\lambda_k^2\sum_{j=1}^{k}\alpha_j^2 = c_2|\upsilon|_2^2$$

for some constants c_1, $c_2>0$. Since, $\|\upsilon\|^2\le\lambda_k|\upsilon|_2^2$, this together with Eq. 14 gives:

$$\|\zeta_m\upsilon\|_2^2 \le \left(\lambda_k \frac{C}{m^{n-2s}}\right)|\upsilon|_2^2 \tag{15}$$

On the other hand:

$$\|\zeta_m\upsilon\|_2^2 = \int_{\Omega\setminus B_2/m}\upsilon^2\,dx + \int_{B_2/m}(\zeta_m\upsilon)^2\,dx \ge \int_\Omega \upsilon^2 dx - \int_{B_2/m}\upsilon^2 dx$$

and:

$$\int_{B_2/m}\upsilon^2\,dx \ge c_3\frac{|\upsilon|_\infty^2}{m^n} \le c_4\frac{|\upsilon|_2^2}{m^n}$$

for some constants c_3, $c_4>0$, so:

$$\|\zeta_m\upsilon\|_2^2 \ge \left(1-\frac{c_4}{m^n}\right)|\upsilon|_2^2 \tag{16}$$

Combining Eq. 15 and 16 gives:

$$\|\zeta_m\upsilon\|^2 \le \left(\lambda_k + \frac{C}{m^{n-2s}}\right)|\zeta_m\upsilon|_2^2$$

Since, Eq. 11 follows from this.

Lemma 3.5: For all sciently large m, $H_0^s(\Omega) = H_m^- \oplus H^+$.

Proof: Let $P: H_0^s(\Omega) \to H^-$ be the orthogonal projection. First we show that $PH_m^- = H^-$ for all sufficiently large m. Since, $PH_m^- \subset H^-$ and dim H^- = k, it suffices to show that $Pe_1^m, ..., Pe_k^m$ are linearly independent. Suppose not. Then there exists $\alpha^m = (\alpha_1^m, ..., \alpha_k^m) \in S^{n-1}$ such that:

$$\sum_{j=1}^{k}\alpha_j^2 Pe_j^m = 0 \tag{17}$$

where, S^{n-1} is the unit sphere in $\mathbb{R}^n$. Passing to a subsequence, we may assume that $\alpha^m \to \alpha = (\alpha_1, ..., \alpha_n) \in S^{n-1}$. Since, $Pe_j^m \to Pe_j = e_j$ by Lemma 3.4, then passing to the limit is Eq. 17 gives:

$$\sum_{j=1}^{k}\alpha_j e_j = 0$$

Since, e_1, ..., e_k are linearly independent, then $\alpha_1 = \cdots = \alpha_k = 0$, contradicting $\alpha \in S^{n-1}$.Given $u \in H_0^s(\Omega)$, write $u = \upsilon + w$ with $\upsilon \in H^-$, $w \in H^+$. Since, $PH_m^- = H^-$, there exists $z \in H_m^-$ such that $Pz = \upsilon$. Then $u = z+(\upsilon-z+w)$ and $\upsilon-z+w \in H^+$ since, $P(\upsilon-z+w) = 0$. Finally, suppose $u \in H_m^- \cap H^+$. Since, $u \in H_m^-$:

$$u = \sum_{j=1}^{k}\alpha_j e_j^m$$

for some α_1, ..., $a_k \in \mathbb{R}$. Since, $u \in H^+$:

$$P_u = \sum_{j=1}^{k}\alpha_j Pe_j^m = 0$$

Since, $Pe_1^m, ..., Pe_k^m$ are linearly independent for sufficiently large m, then $\alpha_1 = \cdots = \alpha_k = 0$ and hence, u = 0. As by Rabinowitz[11], set:

$$U_\varepsilon(x) = \frac{c(n,s)\,\varepsilon^{(n-2s)/2}}{\left(\varepsilon^2+|x|^2\right)^{(n-2s)/2}},\ \varepsilon>0$$

where, c(n, s)>0 is such that:

$$\|U_\varepsilon\|^2 = |U_\varepsilon|_{2_s^*}^{2_s^*} = S^{n/2s}$$

Then take a smooth function $\eta_m:\mathbb{R}^n\to[0, 1]$ such that $\eta_m = 1$ in $B_1/_{4m}$ and $\eta = 0$ outside $B_1/_{2m}$ and set $u_\varepsilon^m = \eta_m U_\varepsilon$. The following estimates were obtained Rabinowitz[11]:

$$\|u_\varepsilon^m\|^2 = S^{n/2s} + O(\varepsilon^{n-2s}), \quad |u_\varepsilon^m|_{2_s^*}^{2_s^*} = S^{n/2s} + O(\varepsilon^n) \tag{18}$$

as $\varepsilon\to 0$. We prove Theorem 1.7 by applying Theorem 2.2 using the direct sum decomposition $H_0^s(\Omega) = H_m^- \oplus H^+$ and taking $w_0 = u_\varepsilon^m$. We will show that:

$$\max_{u\in\partial Q_\varepsilon^m} E(u) \le 0 < \inf_{u\in\partial B_\rho\cap H^+} E(u)$$

if $\rho,\varepsilon>0$ are sufficiently small and m, $R>\rho$ are sufficiently large where:

$$Q_\varepsilon^m = \left\{\upsilon + tu_\varepsilon^m : \upsilon \in H_m^-, \|\upsilon\| \le R,\ t\in[0, R]\right\}$$

Let $\Gamma = \left\{h \in C\left(Q_\varepsilon^m, H_0^s(\Omega)\right) : h|_{\partial Q_\varepsilon^m} = id\right\}$ and set:

$$c := \inf_{h\in\Gamma}\max_{u\in h(Q_\varepsilon^m)} E(u)$$

Then Theorem 2.2 gives a $(PS)_c$ sequence with:

$$\inf_{u\in\partial B_\rho\cap H^+} E(u) \le c \le \max_{u\in Q_\varepsilon^m} E(u)$$

We will show that:

$$\max_{u\in Q_\varepsilon^m} E(u) < \frac{S}{n} S^{n/2s} \tag{19}$$

if ε is sufficiently small and apply Proposition 2.1 to obtain a nontrivial critical point of E.

Lemma 3.6: If ρ>0 is sufficiently small, then:

$$\inf_{u\in\partial B_\rho \cap H^+} E(u) > 0$$

Proof: By (H_1) and (H_3), $G(x, t) \le 1/2\mu t^2 + c_5|t|^p$ for a.a.$x\in\Omega$ and all $t\in\mathbb{R}$ for some constant $c_5>0$.

For $u\in H^+$, this together with the fact that $\frac{\|u\|^2}{|u|_2^2} \ge \lambda_{k+1}$ and the fractional Sobolev embedding theorem gives:

$$E(u) \ge \frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{1}{2}\mu u^2 + c_5|u|^p + \frac{1}{2_s^*}|u|^{2_s^*}\right)dx \ge \frac{1}{2}\left(1-\frac{\mu}{\lambda_{k+1}}\right)\|u\|^2 - c_6\left(\|u\|^p + \|u\|^{2_s^*}\right)$$

for some constant $c_6>0$. Since, $\mu<\lambda_{k+1}$ and $2<p<2^*_s$, the desired conclusion follows from this for sufficiently small ρ.

Lemma 3.7: If m and R>ρ are sufficiently large and ε>0 is sufficiently small, then:

$$\max_{u\in\partial Q_\varepsilon^m} E(u) \le 0 \tag{20}$$

Proof: For $u \in H_m^-$ with $\|\upsilon\|\le R$ and $t\in[0, R]$:

$$E(\upsilon + tu_\varepsilon^m) = E(\upsilon) + E(tu_\varepsilon^m) - 4t\int_{B_{1/m}^c \times B_{1/2m}} \frac{\upsilon(x)u_\varepsilon^m(y)}{|x-y|^{n+2s}}dx\,dy \tag{21}$$

since, υ = 0 in $B_{1/m}$ and $u_\varepsilon^m = 0$ outside $B_{1/2m}$. By Lemma 3.4 and (H_4):

$$E(\upsilon) \le \frac{1}{2}\left(\lambda_k + \frac{C}{m^{n-2s}}\right)\int_\Omega \upsilon^2 dx - \frac{1}{2}(\lambda_k + \sigma)\int_\Omega \upsilon^2 dx = -\frac{1}{2}\left(\sigma - \frac{C}{m^{n-2s}}\right)\int_\Omega \upsilon^2 dx \le -\frac{\sigma}{4}\int_\Omega \upsilon^2 dx$$

for sufficiently large m. Since, H_m^- is finite dimensional, it follows from this that:

$$E(\upsilon) \le -c_7\|\upsilon\|^2 \tag{22}$$

for some constant $c_7>0$ in particular, $E(\upsilon)\le 0$. By (H_2) and Eq. 18:

$$E(tu_\varepsilon^m) \le \frac{t^2}{2}\|u_\varepsilon^m\|^2 - \frac{t^{2_s^*}}{2_s^*}|u_\varepsilon^m|_{2_s^*}^{2_s^*} \ge \left(\frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_8 R^{2_s^*}\varepsilon^{n-2s} \tag{23}$$

for some constant $c_8>0$. The last integral in Eq. 21 is bounded by:

$$c(n, s)|\upsilon|_\infty\,\varepsilon^{(n-2s)/2}\int_{B_{1/m}^c\times B_{1/2m}} \frac{dx\,dy}{|x-y|^{n+2s}\left(\varepsilon^2 + |y|^2\right)^{(n-2s)/2}}$$

Changing variables from (x, y)-(ζ, y) where ζ = x-y, $|\zeta|\ge|x|-|y|>1/2m$ and hence, the integral on the right is bounded by:

$$\int_{B_{1/2m}^c\times B_{1/2m}} \frac{d\zeta\,dy}{|\zeta|^{n+2s}|y|^{n-2s}}$$

and the scaling $(\zeta, y)\mapsto(m\zeta, my)$ shows that this integral is independent of m. Since, $|\upsilon|\le R$, it now follows that:

$$\left|\int_{B_{1/2m}^c\times B_{1/2m}} \frac{\upsilon(x)u_\varepsilon^m(y)}{|x-y|^{n+2s}}dx\,dy\right| \le c_9 R\,\varepsilon^{(n-2s)/2} \tag{24}$$

for some constant $c_9>0$. Combining Eq. 21-24 gives:

$$E(\upsilon + tu_\varepsilon^m) \le -c_7\|\upsilon\|^2 + \left(\frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_8 R^{2_s^*}\varepsilon^{n-2s} + c_{10}R^2\varepsilon^{(n-2s)/2}$$

where $c_{10} = 4c_9$. For $\upsilon + tu_\varepsilon^m \in \partial Q_\varepsilon^m \backslash H_m^-$, either $\|\upsilon\| = R$ or t = R, so, it follows from this that there exists R>ρ such that Eq. 20 holds for all sufficiently small ε. Turning to Eq.19 by contradiction, suppose:

$$\max_{u\in Q_{\varepsilon_j}^m} E(u) \ge \frac{s}{n}S^{n/2s}$$

for some sequence $\varepsilon_j\searrow 0$. Since, H_m^- is finite dimensional, $Q_{\varepsilon_j}^m$ is compact and hence, the above maximum is attained at some point $u_j = \upsilon_j + t_j u_{\varepsilon_j}^m \in Q_{\varepsilon_j}^m$. Then:

$$\frac{s}{n}S^{n/2s} \le E(u_j) = E(\upsilon_j) + E(t_j u_{\varepsilon_j}^m) - 4t_j\int_{B_{1/2m}^c\times B_{1/2m}} \frac{\upsilon(x)u_{\varepsilon_j}^m(y)}{|x-y|^{n+2s}}dx\,dy \le \frac{t_j^2}{2}\|u_{\varepsilon_j}^m\|^2 - \frac{t_j^{2_s^*}}{2_s^*}|u_{\varepsilon_j}^m|_{2_s^*}^{2_s^*} - \int_\Omega G\left(x, t_j u_{\varepsilon_j}^m\right)dx + c_{11}\varepsilon_j^{(n-2s)/2} \tag{25}$$

for some constant $c_{11}>0$ as in the proof of Lemma 3.7. The estimates in Eq. 18 give:

$$\frac{t_j^2}{2}\|u_{\varepsilon_j}^m\|^2 - \frac{t_j^{2_s^*}}{2_s^*}|u_{\varepsilon_j}^m|_{2_s^*}^{2_s^*} \le \left(\frac{t_j^2}{2} - \frac{t_j^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_{12}\varepsilon_j^{n-2s} \tag{26}$$

$$\leq \max_{t\in[0,\infty)}\left(\frac{t^2}{2}-\frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s}+c_{12}\,\varepsilon_j^{n-2s}=\frac{s}{n}S^{n/2s}+c_{12}\,\varepsilon_j^{n-2s} \qquad (27)$$

for some constant $c_{12}>0$, so, Eq. 25 gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \leq c_{13}\,\varepsilon_j^{(n-2s)/2} \qquad (28)$$

for some constant $c_{13}>0$. Since, $t_j\in[0, R]$, t_j converges to some $t_0\in[0, R]$ for a renamed subsequence. In Eq. 25 and 26 (H_2):

$$\frac{s}{n}S^{n/2s}\leq\left(\frac{t_j^2}{2}-\frac{t_j^{2_s^*}}{2_s^*}\right)S^{n/2s}+c_{14}\varepsilon_j^{(n-2s)/2}$$

for some constant $c_{14}>0$ and passing to the limit gives:

$$\frac{t_0^2}{2}-\frac{t_0^{2_s^*}}{2_s^*}\geq\frac{s}{n}$$

Since, the function $[0, \infty)\to\mathbb{R}$, $t\mapsto\frac{t^2}{2}-\frac{t^{2_s^*}}{2_s^*}$ attains its maximum value of s/n only at t = 1, it follows that $t_0 = 1$. We now show that (28) together with (H_2) and (H_5) leads to a contradiction. For j, so, large that $B_{\varepsilon j}\subset B_{4/m}$, ($H_2$) gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq \int_{B_{\varepsilon_j}} G\left(x,\, t_j U_{\varepsilon_j}\right)dx \qquad (29)$$

since, $\eta_m = 1$ in $B_{1/4m}$. Set:

$$\varphi(t)=\inf_{x\in\Omega_0,\tau\geq t}\frac{G(x,\tau)}{\tau^{(n+2s)/(n-2s)}},\ t\geq 0$$

Then φf is nondecreasing:

$$\lim_{t\to+\infty}\varphi(t)=+\infty \qquad (30)$$

by (H_5) and $G(x, t)\geq\varphi(t)t^{(n+2s)/(n-2s)}$ for a.a. $x\in\in\Omega_0$ and$t\geq 0$. Since, $B_{\varepsilon j}\subset B_{4/m}\subset\Omega_0$, this together with (29) gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq \int_{B_{\varepsilon_j}} G\left(t_j U_{\varepsilon_j}\right)dx\left(t_j U_{\varepsilon_j}\right)^{(n+2s)/(n-2s)}dx \qquad (31)$$

For $x\in B_{\varepsilon j}$:

$$U_{\varepsilon_j}(x)=U_{\varepsilon_j}\left(\left|\varepsilon_j\right|\right)\geq U_{\varepsilon_j}\left(\varepsilon_j\right)=c_{15}\varepsilon_j^{-(n-2s)/2}$$

for some constant $c_{15}>0$. Since, $t_j\to 1$ and φ is nondecreasing, this together with Eq. 31 gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq c_{16}\int_{B_{\varepsilon_j}}\varphi\left(c_{17}\,\varepsilon_j^{-(n-2s)/2}\right)\varepsilon_j^{-(n+2s)/2}dx =$$
$$c_{18}\varphi\left(c_{17}\varepsilon_j^{-(n-2s)/2}\right)\varepsilon_j^{(n-2s)/2}$$

for some constants c_{16}, c_{17}, $c_{18}>0$ and all sufficiently large j. This together with (28) implies that $f(c_{17}\varepsilon_j^{-(n-2s)/2})$ is bounded, contradicting (30). This completes the proof of Theorem 1.7.

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