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A Note on Efficiency and Proper Efficiency in Differentiable Multiobjective Programming

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Abstract: We give a suitable example to show that, for general nonlinear vector optimization problem, Kuhn-Tucker type theorem may be failed to hold true.

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Introduction

For the nonlinear vector maximization(NLVM) problem of the form

maximize $f(x) = (f_1(x), f_2(x), \ldots \ldots \ldots, f_k(x)), x \in X$ (1)

Where,

$X = \{x \in R^n : g_i(x) . 0 \text{ for } i \in M, h_j(x) = 0 \text{ for } j \in L\}$

$M = \{1, 2, \ldots \ldots \ldots, m\}, L = \{1, 2, \ldots \ldots \ldots, l\}$

The notion of optimal solution was first presented by Pareto (Paeto, V., 1896) in terms of efficient solution, which as the property that we can improve the value of some $f_r(x)$, $r \in K$, only at the cost of decreasing the value of at least one of the remaining component functions of the objective function f(x). More specifically, we have

Definition 1: A point $x^0 \in X$ is an efficient (or Pareto-optimal, or non-dominated, or non-inferior) solution of the Problem (NLVM) if and only if there exits no $x \in X$ such that

$f(x) . f(x^0)$ (and $f(x) = f(x^0)$)

(Where the inequality above is to be interpreted component-wise).

For any $x = (x_1, x_2, \ldots \ldots \ldots \ldots, x_n)$, $y = (y_1, y_2, \ldots \ldots \ldots \ldots, y_n) \in R^n$, we would write

1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots \ldots \ldots, n$

2) $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots \ldots \ldots, n$

3) $x \leq y$ if and only if $x \leq y$ and $x \neq y$.

Definition2: An efficient solution $x^0 \in x$ of the problem (NLVM) is said to properly efficient if there exists a scalar $N > 0$, such that for each $r = 1, 2, \ldots \ldots \ldots, k$; $f_r(x) < f_r(x^0)$ and $x \in X$; imply that $(f_r(\underline{x}^0) - f_r(\underline{x})) < N(f_j(x) - f_j(\underline{x}^0))$ for at least one j satisfying $f_j(x^p) < f_j(x)$

(Weir, T., 1988), obtained duality results with respect to properly efficient solution.

The following lemma, useful in checking whether $x^0 \in X$ is an efficient solution or not(of the Problem(NLVM)), may be proved in a similar fashion adopted by Chankong and Haimes (Chankong, V. and Y. Y. HAIMES, 1983) and Kanniappan (Kanniappan, V. and J. Optim, 1983).

Lemma1: $x^0 \in X$ is efficient solution of the Problem (NLVM) if and only if, it is optimal for each of the following k sub-problems: For each $r \in K$,

$$\underset{\underline{x} \in X}{\max imize}\ f_r(\underline{x})$$

subject to $f_p(x) > f_p(x^0)$ for all $p \in (K \setminus \{r\})$

The Problem(NVLM) was first treated by Singh (Singh, J. Optim, 1989), who have a set of necessary and sufficient conditions characterizing a Pareto optimal solution. The problem was later taken up by Gulati and Islam[6], who gave the Kun-Tucker and Fritz John types of sufficiency conditions under more generalized convexity assumptions. These are given below for future reference.

Theorem1(Kuhn-Tucker Type): Let f(x) be psoudoconcave and each of g(x) and h(x) be quasiconvex at x^0 ,X. Let there exist u^0 , R^k, v^0 , R^m, w^0 , R^l such that

$u^0 \nabla f(x^0) - v^0 \nabla g(x^0) - w^0 \nabla h(x^0) = 0$(2)

$v^0 g(x^0) = 0$(3)

$u^0 \geq 0$, $v^0 \geq 0$, $w^0 \geq 0$. (4)

Then, x^0 is an efficient solution of the Problem (NLVM).

For the definitions and concepts of different generalized convexity given in theorem1 above, we refer the readers to Mangasarian (Managasarian, O. L., 1969).

Theorem 2(Fritz John Type): Let f(x), g(x) and h(x) all satisfy the conditions of theorem1 and let u^0, v^0 and w^0 be as in theorem1, satisfying in addition to the conditions (2) – (4) , the condition that

$(u_j^0 , v_Q^0) \geq 0$ for $j \in K$

Q = {i , M : $g_i(x)$ is strictly pseudoconvex at x^0}.

Then, x^0 is an efficient solution of the problem(NLVM).

In this note, we show that, for the general nonlinear vector optimization problem, both the theorems may fail to hold true. This is illustrated in the example below:

Results and Discussion

Example

maximize $f(x) = (f_1(x) , f_2(x))$

subject to, $x \in X = \{ x \in \mathbf{R}^2 : g_i(x) \leq 0$ for i = 1, 2, 3 },

where, $f_1(x) = - x_1 - x_2$, $f_2(x) = x_1 - x_2$

$$g_1(x) = \begin{cases} (x_1-2)^2-1-x_2, & if\ x_1<3 \\ -x_2 & if\ x_1\geq 3 \end{cases}$$

$$g_2(x) = 2x_1+5x_2-10,$$

$$g_3(x) = 3x_1+4x_2-12$$

Note that

$$\max_{\underline{x} \in X} imize\ f_1(\underline{x}) = f_1(3/2,-3/4) = -3/4$$

$$\max_{\underline{x} \in X} imize\ f_2(\underline{x}) = f_2(4,0) = 4$$

Now, considering the sub-problem

$$\max imize\ f_1(\underline{x}) = -x_1 - x_2$$

$$subject\ to\ \underline{x} \in X,\ f_2(\underline{x}) = (x_1 - x_2) \geq f_2(13/4,0) = 13/4$$

We see that its(only) optimal solution is x* = (5/2 , - 3/4) and hence, by Lemma1, x^0= (13/4, 0) is not an efficient solution for the problem under consideration. With this x^0, the conditions (2) – (3) of Theorem1 read respectively as

$$(u_1{}^0, u_2{}^0)\begin{bmatrix} (-1, -1) \\ (1, -1) \end{bmatrix} - (v_1{}^0, v_2{}^0, v_3{}^0)\begin{bmatrix} (0, -1) \\ (2, 5) \\ (3, 4) \end{bmatrix} = \underline{0},$$

$$(v_1{}^0, v_2{}^0, v_3{}^0)\begin{pmatrix} -7/2 \\ -9/4 \end{pmatrix} = 0.$$

$$(v_1{}^0, v_2{}^0, v_3{}^0)\begin{bmatrix} 0 \\ -7/2 \\ -9/2 \end{bmatrix} = 0$$

with a solution

$\underline{u}^0 = (1, 1) > 0, \quad \underline{v}^0 = (2, 0, 0) \geq \underline{0}.$

These $\underline{u}^0$ and $\underline{v}^0$ satisfy all the conditions of Theorem1, yet $\underline{x}^0$ is not an efficient solution for the given problem.
In the first example all the functions excepting $g_1(x)$ are linear and hence all these functions satisfy the conditions of Theorem1. We note that $g_1(x)$ is not quasiconvex at $x = x^0 = (13/4, 0)$, since for example(with $x^* = (5/2, -3/4)$)

$$g_1(x^*) - g_1(\underline{x}^0) = 0, but \ \nabla g_1(\underline{x}^0)(\underline{x}^* - \underline{x}^0) = (0, -1)\begin{bmatrix} -3/4 \\ -3/4 \end{bmatrix} = 3/4 > 0.$$

This explains why the conditions (2) – (4) of Theorem1 are satisfied at $x = x^0$ though the point x^0 is not an efficient solution.
It may be mentioned here that $x = x^* = (5/2, -3/4)$ is an efficient solution for the problem given in example1 and the conditions of Theorem1 are satisfied with
$u^0 = (1, 1) > 0$ and $v^0 = (2, 0, 0) \geq 0$. The conditions (2) – (3) of Theorem1 are satisfied with $\underline{u}^0 = (0, 1) > \underline{0}$ and $v^0 = (1, 0, 0) \geq 0$ as well. This shows that the condition on u^0 in (3) of Theorem1 may be relaxed.

Conclusion

In the theoretical study of the multiobjective programming problems, one problem of interest is to relax the (generalized) convexity assumptions on different functions involved in the Problem (NLVM). Theorem1, due to Gulati and Islam (Gulati, T. R. and M. A. Islam, 1989), extends the convexity conditions treated by Singh (Singh, J. Optim, 1989) to the generalized convexity assumptions on different functions involved in the Problem(NLVM).

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