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An Exact Analytical Solution of K-DV Equation by Extended Direct Method

Rezaul Karim
Department of Accounting and Information Systems, University of Chittagong, Bangaldesh

Abstract: We solve the K-dV equation by an extended version of the direct method due to Clarkson and Kruskal.

Key words: Analytical, normalize, reduction

INTRODUCTION

The remarkable form of the Korteweg- de Vries nonlinear partial differential equation (Tyn Myint-U with Lokenath, Third Edition)

$$U_t + C_0 (1 + \quad)U_x + \frac{3U_x}{2h_0} U_{xxx} = \delta \qquad (1)$$

was introduced by Korteweg- deσVries in 1895, to describe long water waves in a channel of depth h_0, where $\sigma = \frac{1}{6} C_0 h_0^2$ is a constant for fairly long waves, $C_0 = (gh_0)^{1/2}$, U is displacement of wave and g is the acceleration due to gravity. Several methods have been applied to solve this equation with considerable degree of success [1-3]. In this paper, our purpose is to apply the extension of Clarkson and Kruskal direct method to obtain new solution of (1).

An Extension To The Direct Method of Clarkson and Kruskal: We can easily transform from the remarkable form of the K-dV nonlinear partial differential equation (1) to the simple form of K-dV equation

$$U_t + UU_x + U_{xxx} = 0 \qquad (2)$$

This new method is similar to the original method but begins with a more general concept: the usual idea is to seek reduction to a single ODE, instead, we seek a transformation which "reduces" the given PDE to a system of ODEs in p (ξ) and q (η) by means of the Ansatz

$$u(x,t) = \alpha(x,t) + \beta(x,t)p(\xi(x,t)) + \gamma(x,t)q(\eta(x,t)) \qquad (3)$$

Now we substitute equation (3) into our given K-dV equation and impose that the result is a pair of ODEs in.

We consider the general 3rd order PDE in two independent variables x and t: $p(\xi)$ and $q(\eta)$

$$\Delta(u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}, u_{xx}, u_{xt}, u_{tt}, u_x, u_t, u; x, t) = 0 \qquad (4)$$

Substituting equation (3) into (4) we get

$$\sum(p_{\xi\xi\xi}, q_{\eta\eta\eta}, p_{\xi\xi}, q_{\eta\eta}, p_\xi, q_\eta, p, q, \alpha, \beta, \gamma, \xi, \eta; x, t) = 0 \qquad (5)$$

where Σ is a known function which depends on the x and t derivatives of arguments and which are functions of these variables. Next we divide equation (5) into two separable equations, which are ODE and normalize the coefficients of powers and derivatives of p and q in each equation, in exactly the same way as in the direct method. There are two cases to consider:

- When $\xi = \eta$: If $\xi(x,t) \equiv \eta$ then we divide equation (5) into a pair of coupled ODEs for $p(\xi)$ and $q(\xi)$. There are of course many ways in which we may partition (5) in this manner, each distinct partition, perhaps yielding different solutions/ reductions.
- When $\xi \neq \eta$: In this case we must divide equation (5) into a pair of ODEs, which are not coupled one for $p(\xi)$ and the other for q (η).

We now apply this method to a specific example.

Extended Direct Method For K-dV Equation: We consider the K-dV equation in the form

$$u_t + uu_x + u_{xxx} = 0 \qquad (6)$$

Let $u(x,t) = \alpha(x,t) + \beta(x,t)p(\xi(x,t)) + \gamma(x,t)q(\eta(x,t))$

On substituting the value of u_t, u_x & u_{xxx} in equation (6) we get

$$\begin{aligned} &p_{\xi\xi\xi}\beta(\xi_x)^3 + q_{\eta\eta\eta}\gamma(\eta_x)^3 + p_{\xi\xi}[3\beta_x(\xi_x)^2 + 3\beta\xi_{xx}\xi_x] + q_{\eta\eta}[3\gamma_x(\eta_x)^2 + \\ &3\gamma\eta_{xx}\eta_x] + pp_\xi\beta^2\xi_x + qq_\eta\gamma^2\eta_x + p^2\beta\beta_x + q^2\gamma\gamma_x + p_\xi[\beta\xi_t + \alpha\beta\xi_x + \\ &\beta\gamma q\xi_x + \beta\xi_{xxx} + 3\beta_x\xi_{xx} + 3\beta_{xx}\xi_x] + q_\eta[\gamma\eta_t + \beta\gamma p\eta_x + \alpha\gamma\eta_x + \gamma\eta_{xxx} \\ &+3\gamma_x\eta_{xx} + 3\beta_{xx}\xi_x] + p[\beta_t + \alpha\beta_x + \alpha_x\beta + \beta\gamma_x q + \gamma\beta_x q + \beta_{xxx}] + \\ &q[\gamma_t + \alpha\gamma_x + \alpha_x\gamma + \gamma_{xxx}] + \alpha_t + \alpha\alpha_x + \alpha_{xxx} = 0 \end{aligned} \qquad (7)$$

We see from the coefficients of the 3^{rd} order derivatives of p and q in equation (7), that there are four cases to consider (I) $\xi_x\eta_x \neq 0$ (ii), $\xi_x = \eta_x = 0$ (iii) $\xi_x = 0$, with $\eta_x \neq 0$ and (iv) $\xi_x \neq 0$ with $\eta_x = 0$.

Notation: Unless otherwise stated, we use the following notation: T_i, $i = 1, 2, \ldots$ are functions introduced when normalizing coefficients of powers and derivatives of $p(\xi)$ and $q(\eta)$ are to determined; the corresponding symbols γ_{ij}, j = 1, 2,... ... are introduced if the explicit (rational) form for Γ_i is known; in special casesis Γ_i necessarily constant and we write γ_i; λ_1, λ_2, ... are miscellaneous constant introduced during the computation, for example, constants of integration; c_1, c_2, ... are constants introduced upon integration of the reduced equations (ODEs): we reserve the prime, ' , to represent derivatives with respect to t that is ' . $\equiv \frac{d}{dt}$

The case $\xi_x\eta_x \neq 0$: As stated above, we must divide equation (7) into a pair of ODEs.

$$P_{\xi\xi\xi}\beta(\xi x)^3 + P_{\xi\xi}[3\beta_x(\xi x)^2 + 3\beta\xi_{xx}\xi_x] + PP_\xi(\beta)^2\xi_x + P^2\beta\beta_x + P_\xi[\beta\xi_t + \alpha\beta\xi_x + \beta\xi_{xxx} + 3\beta_x\xi_{xx} + 3\xi_{xx}\xi_x] + P[\beta_t + \alpha\beta_x + \alpha_x\beta + \beta\gamma_x q + \gamma\beta_x q + \beta_{xxx}] + P_\xi q\beta\gamma\xi_x + (1-\delta)[\alpha_t + \alpha\alpha_x + \alpha_{xxx}] = 0 \quad (8)$$

and

$$q_{\eta\eta\eta}\gamma(\eta_x)^3 + q_{\eta\eta}[3\gamma_x(\eta_x)^2 + 3\gamma\eta_{xx}\eta_x] + qq_\eta(\gamma)^2\eta_x + q^2\gamma\gamma_x + q_\eta[\gamma\eta_t + \beta\gamma p\eta_x + \alpha\gamma\eta_x + \gamma\eta_{xxx} + 3\gamma_x\eta_{xx} + 3\gamma_{xx}\eta_x] + q[\gamma_t + \alpha\gamma_x + \alpha_x\gamma + \gamma_{xxx}] + \delta[\alpha_t + \alpha\alpha_x + \alpha_{xxx}] = 0 \quad (9)$$

Where we have introduced δ, $0 \leq \delta \leq 1$. We now normalize the coefficients of pp_ξ, p^2, and p_ξ against that of $p_{\xi\xi\xi}$ in equation (8) and use freedom of scaling and translation of and freedom to redefine $p(\xi)$. Finally we get

$$\alpha = -\frac{x\phi'(t) + \theta'(t)}{\phi(t)} \; ; \; \beta = \phi^2(t) \; ; \; \xi = x\phi(t) + \theta(t) \quad (10)$$

where $\phi(t)$ are & θ (t) function of integration, to be determined and consequently the coefficients of p is found to be identically zero. Similarly we normalize the coefficients of qq_η and q^2 against that of $q_{\eta\eta\eta}$ in equation (9), then use freedom of rescaling of $q(\eta)$ and redefine η (x,t). We get

$$\eta = x\mu(t) + \gamma(t); \quad \gamma = \mu^2(t) \quad (11)$$

where μ (t) and γ (t) are functions of integration to be determined.

Substituting results (10) and (11) into equation (8) and (9) and normalizing the remaining coefficients of equation (8) and (9) appropriately, we obtain the determining system:

$$\mu^3\Gamma_1(\eta) = x\mu' + \upsilon' - \frac{\mu(x\phi' + \theta')}{\phi} \quad (12)$$

$$\mu^4\Gamma_2(\eta) = 2\mu' - \frac{\mu\phi'}{\phi} \quad (13)$$

$$\phi^7\Gamma_{3i}(\xi) = (1-\delta)[x(2\phi'^2 - \phi\phi'') + 2\theta'\phi' - \phi\theta''] \quad (14)$$

$$\mu^5\phi^2\Gamma_{3ii}(\eta) = \delta[x(2\phi'^2 - \phi\phi'') + 2\theta'\phi' - \phi\theta''] \quad (15)$$

and $p(\xi)$ & $q(\eta)$ satisfy

$$p_{\xi\xi\xi} + pp_\xi + \Gamma_{3i}(\xi) = 0$$

and $q_{\eta\eta\eta} + qq_\eta + \Gamma_1(\eta)q_\eta + \Gamma_2(\eta)q + \Gamma_{3ii}(\eta) = 0$

Notice that (13) is a differential consequence of (12) i.e. taking the x derivative of (12) yields (13) so we may neglect (13) as a constraint and use it only to compute . $\Gamma_2(\eta)$ Since ξ is linear in x (cf. eq. (10)) then it is clear from (14) that

$$\Gamma_{3i}(\xi) = \gamma_{3i}\xi + \gamma_{3i_0}$$

Therefore,

$$p_{\xi\xi\xi} + pp_\xi + \gamma_{3i_1}\xi + \gamma_{3i_0} = 0$$

Integrating with respect to ξ we get,

$$p_{\xi\xi} + \frac{1}{2}p^2 + \frac{1}{2}\gamma_{3i_1}\xi^2 + \gamma_{3i_0}\xi + c_1 = 0$$

This is an ordinary differential equation of $p(\xi)$, i.e. the reduction from our given PDE to ODE.
Now from the second equation we have

$$q_{\eta\eta\eta} + qq_\eta + \Gamma_1(\eta)q_\eta + \Gamma_2(\eta)q + \Gamma_{3ii}(\eta) = 0$$

This is also an ordinary differential equation of $q(\eta)$.

The case $\xi_x = 0$, $\eta_x = 0$: We set $\xi = \eta = t$ in this case, without loss of generality. Hence equation (8) and (10) are simplified considerably. From (8) and (9) we get finally:

$$\beta p_\xi + \beta\beta_x p^2 + [\beta_t + (\alpha\beta)_x + \beta_{xxx}]p + (\beta\gamma)_x pq + (1-\delta)(\alpha_t + \alpha\alpha_x + \alpha_{xxx}) = 0 \quad (16)$$

and, $\gamma q_\eta + \gamma\gamma_x q^2 + [\gamma_t + (\alpha\gamma)_x + \gamma_{xxx}]q + \delta[\alpha_t + \alpha\alpha_x + \alpha_{xxx}] = 0 \quad (17)$

Normalizing the coefficient of p^2 and p against that of

p_ξ and the coefficient of q^2 against that of q^2 and using the freedom of scaling in both p and q and translation in p we get:

$$\beta(x,t)=x+\psi(t),\ \ \gamma(x,t)=x+\sigma(t),\ \ \alpha(x,t)=\frac{\alpha_0(t)-x\psi'(t)}{x+\psi(t)} \tag{18}$$

where $\psi(t)$, $\sigma(t)$, and $\alpha_0(t)$ are functions of integration, to be determined.

Normalizing the remaining coefficients in equation (16) and (17) we obtain the determining system:

$$(x+\sigma)\Gamma(t)=\alpha+(x+\sigma)\alpha_x+\sigma' \tag{19}$$

$$(x+\psi)\Gamma_{2i}(t)=(1-\delta)(\alpha_t+\alpha\alpha_x+\alpha_{xxx}) \tag{20}$$

$$(x+\psi)\Gamma_{2i_1}(t)=2x+\psi(t)+\sigma(t) \tag{21}$$

and, $$(x+\sigma)\Gamma_{2ii}(t)=\delta(\alpha_t+\alpha\alpha_x+\alpha_{xxx}) \tag{22}$$

Substituting equation (18) into (20), (21) and (22) and after some calculation we get:

$$\Gamma_{2i}=\Gamma_{2ii}=0,\quad \Gamma_{2i1}=2 \tag{23}$$

$$\psi''=0 \tag{24}$$

$$\alpha'_0-3\psi\psi''+\psi'^2=0 \tag{25}$$

$$\alpha'_0\psi+\psi\psi'^2-\psi^2\psi''=0 \tag{26}$$

$$3\alpha'_0\psi^2-\psi''\psi^3+2\psi'\psi^2-2\alpha_0\psi\psi'-\alpha_0{}^2=0 \tag{27}$$

and, $$\alpha'_0\psi^3-2\alpha_0\psi^2\psi'-\alpha_0{}^2\psi-6\alpha_0-6\psi\psi'=0 \tag{28}$$

Integrating equations (24) and (25) we obtain:

$$\psi(t)=\lambda_1 t+\lambda_2 \ \text{ and } \ \alpha_0(t)=\lambda_3-\lambda_1{}^2 t \tag{29}$$

Equation (26) is identically satisfied using after (24) and (25). Substituting equation (29) into equation (27), we find that all terms involving 't' vanish to leave simply:

$$\lambda_3{}^2+2\lambda_1\lambda_2\lambda_3+\lambda_1{}^2\lambda_2{}^2=0 \tag{30}$$

which gives λ_3 in terms of $\lambda_1$and λ_2.

Again on substituting equation (29) into equation (28) we get:

$$-\lambda_1{}^2\lambda_2{}^3-2\lambda_1\lambda_2{}^2\lambda_3-\lambda_2\lambda_3{}^2+6\lambda_3+6\lambda_1\lambda_2=0 \tag{31}$$

which also gives λ_3 in terms of $\lambda_1$and λ_2.

From (18) and (19) we get:

$$\Gamma_1(t)=0 \tag{32}$$

$$\sigma'-\psi'=0 \tag{33}$$

$$2\psi(\sigma'-\psi')=0 \tag{34}$$

and, $$\alpha_0\psi-\sigma\alpha_0-\sigma\psi\psi'+\sigma'\psi^2=0 \tag{35}$$

The new dependent variables p (t) and q (t) now satisfy:

$$p'(t)+p^2(t)+p(t)\,q(t)=0 \tag{36}$$

and $$q'(t)+q^2(t)=0 \tag{37}$$

Integrating (37) with respect to t we get, $q(t)=\frac{1}{t}$ and equation (36) now becomes,

$$p'(t)+p^2(t)+\frac{p(t)}{t}=0$$

On integration this equation we get: $p(t)=\frac{1}{t\ln t}$

Therefore, $$p(t)=\frac{1}{t\ln t},\ \ \Gamma_1(t)=0 \text{ and } q(t)=\frac{1}{t} \tag{38}$$

integrating (33) we get (cf. 29):

$$\sigma(t)=\lambda_1 t+\lambda_2 \tag{39}$$

We note that equations (34) and (35) vanish identically.

Reduction: We are free to translate $x \rightarrow x-\lambda_2$ and therefore set$\lambda_2=0$ without loss of generality. Hence we obtain the exact solution:

$$u(x,t) \ = \ \alpha(x,t)+\beta(x,t)p(\xi(x,t))+\gamma(x,t)q(\eta(x,t))$$

becomes,

$$u(x,t)=\frac{\lambda_3-\lambda_1(x+\lambda_1 t)}{x+\lambda_1 t}+\frac{x+\lambda_1 t}{t\ln t}+\frac{x+\lambda_1 t}{t}$$

That is,

$$u(x,t)=\frac{\lambda_3}{x+\lambda_1 t}+\frac{x}{t}+\frac{x+\lambda_1 t}{t\ln t} \tag{40}$$

This is the complete reduction i.e. solution of K-dV equation.

Similar analysis for the cases of .

$$\xi_x\neq 0\ \ \&\ \ \eta_x=0 \ \text{ and } \ \xi_x=0\ \ \&\ \ \eta_x\neq 0$$

CONCLUSIONS

Currently, there is much mathematical interest in the determination of similarity reduction of a given PDE. To find some similarity solutions of a nonlinear physical problem, one may use the classical Lie group approach, the non-classical Lie group approach, the direct method and the multiple singular manifold method. Though the direct method has been widely used to find the similarity solutions for many real physical models, but Extended Direct Method is more general concept to get the solution of non-linear PDE like K-dV equation. However, it is just when the Clarkson and Kruskal direct method was developed, much more similarity reductions for nonlinear system were found. By using this method we have computed new classes of solutions of K-dV equation. It appears likely that the application of the Extended Direct Method to a Partial Differential Equation of higher order (other than K-dV equation) will generate an equation equivalent to (7) of correspondingly higher order which might lead to reductions to two or more different systems of Ordinary Differential Equations. This characteristic of the Extended Direct Method adds complexity, but each case may be treated separately. In fact it should be considered a positive feature as all possible partitions may be obtained algorithmically and those leading to trivial reductions eliminated in a straightforward manner. Finally the solution found by our method can also be used as models for numerical experiments differing from known exact solutions of K-dV equation.

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