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Optimal Control for Oxygen Depletion in an Aquatic System

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Abstract: Oxygen is required to support aquatic life and maintain water quality, it is the most important dissolved gas in water. A small amount of oxygen, up to ten molecules of oxygen per million of water, is actually dissolved in water . Fish and zooplankton breath Dissolved Oxygen (DO) and without sufficient oxygen mortality will occur. In this study we give the mathematical model for the interaction between oxygen transport and oxygen use up by contaminants in polluted water bodies and derive the optimal control for the resulting coupled system. We also examine the existence and uniqueness of solution and then state the optimality conditions.

Key words: Optimal control, oxygen, transport, water pollution

INTRODUCTION

What's in a glass, a sink, a river full of water?... A refreshing drink... a cleansing wash... an invigorating swim... a home for plants, insects, fish, birds and mammals. It all depends on the water quality.

We tend to think of water in terms of a particular purpose: Is the quality of the water good enough for the use we want to make of it? Water fit for one use may be unfit for another. We may, for instance, trust the quality of lake water enough to swim in it, but not enough to drink it. Along the same line, drinking water can be used for irrigation, but water used for irrigation may not meet drinking water standards. It is the quality of the water which determines its uses.

Scientists, on the other hand, are interested in other aspects of water quality. To them quality is determined by the kinds and amounts of substances dissolved and suspended in the water and what those substances do to inhabitants of the ecosystem. It is the concentrations of these substances that determine the water quality and its suitability for particular purposes[1].

It is easy to dispose of waste by dumping it into a river or lake. In large or small amounts, dumped intentionally or accidentally, it may be carried away by the current, but will never disappear. It will reappear downstream, sometimes in changed form, or just diluted. Freshwater bodies have a great ability to break down some waste materials, but not in the quantities discarded by today's society. This overload that results, called pollution, eventually puts the ecosystem out of balance.

Sometimes nature itself can produce these imbalances. In some studys, the natural composition of the water makes it unfit for certain uses: e.g., water owing in the highly saline terrain of the prairies or gushing from highly mineralized springs in some parts of the country cannot sustain fish populations.

But most often our waterways are being polluted by municipal, agricultural and industrial wastes, including many toxic synthetic chemicals which cannot be broken down at all by natural processes. Even in tiny amounts, some of these substances can cause serious harm.

Many causes of pollution including sewage and fertilizers contain nutrients such as nitrates and phosphates. In excess levels, nutrients over stimulate the growth of aquatic plants and algae. Excessive growth of these types of organisms consequently clogs our waterways, use up dissolved oxygen as they decompose and block light to deeper waters. This, in turn, proves very harmful to aquatic organisms as it affects the respiration ability of fish and other invertebrates that reside in water[2].

Oxygen is required to support aquatic life and maintain water quality, it is the most important dissolved gas in water. Water in equilibrium with air at 25°C contains 8.3 mg L^{-1} of dissolved O_2. Although water molecules contain an oxygen atom, this oxygen is not what is needed by aquatic organisms living in natural waters. A small amount of oxygen, up to ten molecules of oxygen per million of water, is actually dissolved in water. Fish and zooplankton breath dissolved oxygen and without sufficient oxygen mortality will occur.

Dissolved Oxygen (DO) concentrations are affected by a number of factors. Higher DO is pro-duced by turbulent actions such as waves, which mix air and water. Lower water temperatures also allows for retention of

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higher DO concentrations. Low DO levels tend to occur more often in warmer, slow moving waters. In general, low DO levels occur during the warmest summer months and particularly during low ow periods. Water depth is also a factor. In deep slow moving waters DO concentrations may be high near the surface due to wind action and plant photosynthesis, but may be entirely depleted (anoxic) at the bottom[2].

Oxygen consuming wastes include decomposing or ganic matter or chemicals that reduce DO in the water. Raw domestic wastewater contains high concentrations of oxygen consuming wastes that need to be removed before it can be discharged into a waterway. Maintaining a sufficient level of DO in water is critical to most forms of aquatic life.

Microorganisms such as bacteria are responsible for decomposing organic waste. When organic matter such as dead plants, leaves, grass clippings, manure, sewage, or even food waste is present in a water supply, the bacteria will begin the process of breaking down this waste. When this happens, much of the available dissolved oxygen is consumed by aerobic bacteria, robbing other aquatic organisms of the oxygen they need to live.

The first part of this study considers the mathematical model which can be used to simulate the interaction of pollutants with oxygen in an aquatic media.

Interactive transport models: Water pollution is caused by wastewater discharges into rivers, lakes, estuaries, e.t.c. containing organic substances or excessive heat from both domestic or industrial origin. Apart from urban areas, chemical, food and paper industries are among the most important sources of pollution.

An organic pollutant, such as human and animal fecal wastes, is thoroughly mixed in the water of a river which is moving downstream at a constant velocity c. The concentration ρ of the pollutant in the river is homogeneous in all directions except that of the downstream ow, which we take to be from left to right along the x axis. The river is thereby modeled by an advective ow in one dimension. Diffusive effects due to river turbulence and irregularities in its contours as it meanders downstream are all ignored. However, the pollutant is allowed to decay in the water due to bacterial action, which gradually decomposes it[3].

Let k be the rate at which the pollutant density is degraded. We assume it to be proportional to the density itself:

$$k(x, t) = -\mu\rho(x, t),$$

where μ is a proportionality constant that measures the efficiency of bacterial action. Since an advective model is appropriate, the equation then is:

$$\frac{\partial}{\partial t}\rho(x,t) = -c\frac{\partial}{\partial x}\rho(x,t) - \mu\rho(x,t) \quad (1)$$

The bacterial decomposition of the pollutant requires an uptake of dissolved oxygen(DO) in the water. As the pollutant is degraded, oxygen is used up. Let δ (x, t) be the density of DO in the river. Its maximum value, which depends on temperature, is δ_m. We assume it to be a known fixed quantity.

The rate at which the oxygen dissipates is the same as that of the pollutant decay and is propor-tional to the pollutant concentration.

$$k_1(x, t) = -\mu\rho(x, t)$$

There is a source term due to the fact that the river surface, in contact with air above, draws oxygen in from the atmosphere by the process know as re-oxygenation. This happens at a rate proportional to the difference between the saturation level δ_m and the actual δ. Thus,

$$k_2(x, t) = \mu_1[\delta_m - \delta(x, t)]$$

Thus, the oxygen ow equation is given as:

$$\frac{\partial}{\partial t}\delta(x,t) = -c\frac{\partial}{\partial x}\delta(x,t) - \mu\rho(x,t) + \mu_1[\delta_m - \delta(x,t)] \quad (2)$$

Thus, our coupled system is:

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(x,t) = -c\frac{\partial}{\partial x}\rho(x,t) - \mu\rho(x,t) \\ &\frac{\partial}{\partial t}\delta(x,t) = -c\frac{\partial}{\partial x}\delta(x,t) - \mu\rho(x,t) + \mu_1[\delta_m - \delta(x,t)] \end{aligned} \quad (3)$$

Optimal control of the contaminant transport problem: Introducing the Neumann boundary control v1, v2 to the coupled equations fr om section 2 above we have the following Eqn

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(x,t) + c\frac{\partial}{\partial x}\rho(x,t) + \mu\rho(x,t) && = 0 \text{ in } Q \\ &\rho(x,0) && = \rho_0 \text{ in } \Omega \\ &\frac{\partial}{\partial n}\rho(x,t) && = \upsilon_1 \text{ on } \Sigma \\ &\frac{\partial}{\partial t}\delta(x,t) + c\frac{\partial}{\partial x}\delta(x,t) + \mu\rho(x,t) - \mu_1[\delta_m - \delta(x,t)] && = 0 \text{ in } Q \\ &\delta(x,0) && = \delta_0 \text{ in } \Omega \\ &\frac{\partial}{\partial n}\delta(x,t) && = \upsilon_2 \text{ on } \Sigma \end{aligned} \quad (4)$$

Let $\Omega \subset IR^1$ be a bounded open subset with the boundary $\partial\Omega$ and (0, T) an open interval. Q denotes the cylindrical domain $\Omega \times (0, T)$, while $\Sigma = \partial\Omega \times (0, T)$ which is the lateral boundary of Q.

Suppose $(\rho, \delta) \in H^1(\Omega) \times H^1(\Omega)$ satisfies problem (4). Then by Green's formula we easily obtain that (ρ, δ) is a solution of the following variational problem

$$(\rho', \varphi_1) + \alpha_1[\rho, \varphi_1] = (\upsilon_1, \varphi_1)\ \forall \varphi_1 \in H^1(\Omega),\ \upsilon_1 \in L^2(\Sigma) \tag{5}$$

$$(\delta', \varphi_1) + \alpha_2[\delta, \varphi_2] = (\upsilon_2, \varphi_2)\ \forall \varphi_2 \in H^1(\Omega),\ \upsilon_2 \in L^2(\Sigma) \tag{6}$$

(,) denoting inner product in $L^2(\Omega)$, where

$$\alpha_1[\rho, \varphi_1] = -\int_\Omega c \frac{\partial \varphi_1}{\partial x} \rho dx + \int_\Omega \mu \rho \varphi_1 dx$$

and

$$\alpha_2[\delta, \varphi_2] = -\int_\Omega c \frac{\partial \varphi_2}{\partial x} \delta dx + \int_\Omega \mu \rho \varphi_1 dx - \int_\Omega \mu_1 [\delta_m - \delta(x,t)] \varphi_2$$

$\alpha^1[\rho\ \varphi^1]$ is a continuous bilinear form such that

$$|\alpha_1[\rho, \varphi_1]| \le \alpha_1 \|\rho\|^2 H_{1(\Omega)} \|\varphi_1\|^2 H_{1(\Omega)} \quad \alpha_1 > 0$$

and

$$|\alpha_1[\rho, \varphi_1]| + \gamma_1 \|\rho\|^2 L_{2(\Omega)} \ge \beta_1 \|\rho\|^2 H_{1(\Omega)}, \quad \beta_1 > 0, \gamma_1 \ge 0 \tag{7}$$

Similarly $a_2[\delta, \varphi_2]$ is a continuous bilinear form such that

$$|\alpha_2[\delta, \varphi_1]| \le \alpha_1 \|\rho\|^2 H_{1(\Omega)} \|\varphi_1\|^2 H_{1(\Omega)} \quad \alpha_1 > 0$$

and

$$|\alpha_2[\delta, \varphi_2]| + \gamma_2 \|\delta\|^2 L_{2(\Omega)} \ge \beta_2 \|\rho\|^2 H_{1(\Omega)}, \quad \beta_2 > 0, \gamma_2 \ge 0 \tag{8}$$

Lemma 3.1: There exists a constants K_1, K_2, such that

$$\|\rho\|_{L^2(\Omega)} + \|\rho\|_{L^2(0,T,H_0^1(\Omega))} + \|\rho'\|_{L^2(0,T,H^{-1}(\Omega))} \le K_1\left(\|C_0\|_{L^2(\Omega)} + \|\upsilon_1\|_{L^2(\Sigma)}\right) \tag{9}$$

$$\|\delta\|_{L^2(\Omega)} + \|\delta\|_{L^2(0,T,H_0^1(\Omega))} + \|\delta'\|_{L^2(0,T,H^{-1}(\Omega))} \le K_2\left(\|C_0\|_{L^2(\Omega)} + \|\upsilon_2\|_{L^2(\Sigma)}\right) \tag{10}$$

Proof: See [4] pp 354: By the method of Galerkin, we can prove the existence and uniqueness of solution of problem (4) using the variational formulation Eq. 5 and 6 and the inequality of Eq. 7 and 8 and the estimates of lemma 4.

The optimal control problem: We state the optimal control problem. We look for a $(\rho, \delta, \upsilon) \in H^1(\Omega) \times H^1(\Omega) \times U$ such that the cost functional:

$$J(\rho, \upsilon) = \frac{1}{2}\int_Q |\rho - \rho_d|^2 dxdt + \frac{\xi}{2}\|\upsilon_1\|^2_{L^2(\Sigma)} + \frac{\zeta}{2}\|\upsilon_2\|^2_{L^2(\Sigma)} \tag{11}$$

is minimized subject to the constraints

$$\left.\begin{array}{ll}
(\rho', \varphi_1) + \alpha_1[\rho, \varphi_1] = (\upsilon_1, \varphi_1) & \forall \varphi_1 \in H^1(\Omega),\ \upsilon_1 \in L^2(\Sigma) \\
\rho(x,0) = \rho_0 & \text{in } \Omega \\
\frac{\partial}{\partial n}\rho(x,t) = \upsilon_1 & \text{on } \Sigma \\
(\delta', \varphi) + \alpha_2[\delta, \varphi_2] = (\upsilon_2, \varphi_2) & \forall \varphi_2 \in H^1(\Omega),\ \upsilon_2 \in L^2(\Sigma) \\
\delta(x,0) = \delta_0 & \text{in } \Omega \\
\frac{\partial}{\partial n}\delta(x,t) = \upsilon_2 & \text{on } \Sigma
\end{array}\right\} \tag{12}$$

The control space U is a closed convex subset of $L^2(\Sigma) \times L^2(\Sigma)$. $\upsilon = \{\upsilon_1, \upsilon_2\}$ is the control and the corresponding state $\rho = \rho(\upsilon_1)$ and $\delta = \delta(\upsilon^2)$ is the solution of (12) above. $\rho_d \in L^2(Q)$ is a target state that we would like to obtain by controlling ρ and $\xi, \zeta > 0$. Let U_{ad}, the admissible space of control, be defined as

$$U_{ad} = \{(\rho, \delta, \upsilon) \in H^1(\Omega) \times U : J(\rho, \upsilon) < \infty \text{ (12) are satisfied}\} \tag{13}$$

Then $(\hat{\rho}, \hat{\upsilon}) \in U_{ad}$ is called an optimal solution if there exists $\epsilon > 0$ such that

$$J(\hat{\rho}, \hat{\upsilon}) \le J(\rho, \upsilon) \qquad \forall (\rho, \upsilon) \in U_{ad} \tag{14}$$

satisfying

$$\|\hat{\rho} - \rho\|_{H^1(\Omega)} + \|\hat{\upsilon}_2 - \upsilon_2\|_{L^2(\Sigma)} + \|\hat{\upsilon}_2 - \upsilon_2\|_{L^2(\Sigma)} < \epsilon \tag{15}$$

If, for optimal solution $(\hat{\rho}, \hat{\upsilon}) \in U_{ad}$ inequalities (14) and (15) hold true with $\epsilon = +\infty$, then we say that $(\hat{\rho}, \hat{\upsilon})$ is the global minimum. The optimal control problem can now be formulated as a constrained minimization in a Hilbert space:

$$\min_{(\rho, \upsilon) \in U_{ad}} J(\rho, \upsilon) \tag{16}$$

The existence of an optimal solution: We now show the existence of an optimal solution. The existence of an optimal solution can be proved based on the a priori estimates and standard techniques

Theorem 3.1: There exists a unique optimal solution $(\hat{\rho},\hat{\upsilon})$ of (16).

Proof: The set U_{ad} is nonempty, thus we may choose a minimizing sequence $(\rho(\upsilon^n_1), \upsilon^n)$ in U_{ad} such that

$$\lim_{n\to\infty} J(\rho^n,\upsilon^n) = \inf_{(\rho,\upsilon)\in U_{ad}} J(\rho,\upsilon)$$

Set $\rho(\upsilon^n_1) = \rho^n$. By the definition of U_{ad}, we have

$$\left.\begin{array}{ll}(\rho^{n'},\varphi_1)+\alpha_1[\rho^n,\varphi_1]=(\upsilon^n_1,\varphi_1) & \forall\varphi_1\in H^1(\Omega)\\ (\delta^{n'},\varphi_2)+\alpha_2[\delta^n,\varphi_2]=(\upsilon^n_1,\varphi_2) & \forall\varphi_2\in H^1(\Omega)\\ \frac{\partial}{\partial n}\rho^n(x,t)=\upsilon_1 \quad \frac{\partial}{\partial n}\delta^n(x,t)=\upsilon_2 & \text{on }\Sigma\end{array}\right\} \quad (17)$$

By virtue of the term $\xi\,\|\upsilon_1\|^2 L^2(\Sigma)$ and $\zeta\,\|\upsilon_2\|^2 L^2(\Sigma)$ in (11) and from (13), we see that $\|\upsilon^n_1\| L^2(\Sigma)$ and $\|\upsilon^n_2\| L^2(\Sigma)$ are uniformly bounded. Also from (9) and (10) we have that the sequences $\|\rho^n\|\ H(\Omega)$ and $\|\delta\|^n H(\Omega)$ are uniformly bounded. since the embedding from $H^1(\Omega)\to\to L^2(\Omega)$ is compact and using a compactness lemma, it follows that we may extract a subsequence, denoted again by ρ^n, δ^n and υ^n such that

$$\begin{array}{llll}\upsilon^n_1\to\hat{\upsilon}_1 & \text{and} & \upsilon^n_2\to\hat{\upsilon}_2 & \text{in } L^2(\Sigma)\\ \rho^n\to\hat{\rho} & \text{and} & \delta^n\to\hat{\delta} & \text{in } L^2(0,T:H^1(\Omega))\\ \frac{dp^n}{dt}\to\frac{d\hat{\rho}}{dt} & \text{and} & \frac{d\delta^n}{dt}\to\frac{d\hat{\delta}}{dt} & \text{in } L^2(0,T:H^1(\Omega))\end{array}$$

and

$$\rho^n\to\hat{\rho} \quad \text{and} \quad \delta^n\to\hat{\delta} \quad \text{in } L^2(\Omega)$$

since $J(\rho,\upsilon)$ is lower semi continuous we conclude that $(\hat{\rho},\hat{\upsilon})$ is an optimal solution, i.e.,

$$J(\hat{\rho},\hat{\upsilon}) = \inf_{(\rho,\upsilon)\in U_{ad}} J(\rho,\upsilon)$$

Thus, we have shown that an optimal solution belonging to U^{ad} exists. Finally, the uniqueness of the optimal solution follows from the convexity of the functional and the linearity of the constraint equations.

Optimality system: Next we give the necessary and sufficient conditions for υ to be an optimal control. If υ is an optimal control then $J'(\upsilon).(\mu\,.\,\upsilon)\geq 0$ for all $\mu\in U_{ad}$.

Given

$$\left.\begin{array}{ll}\frac{\partial}{\partial t}\rho(x,t;\upsilon_1)+c\frac{\partial}{\partial x}\rho(x,t;\upsilon_1)+\mu\rho(x,t;\upsilon_1) & =0 \text{ in} Q\\ \frac{\partial}{\partial t}\delta(x,t;\upsilon_2)+c\frac{\partial}{\partial x}\delta(x,t;\upsilon_2)+\mu\rho(x,t;\upsilon_2)-\mu_1[\delta_m-\delta(x,t;\upsilon_2)] & =0 \text{ in} Q\\ \rho(x,t;\upsilon_1)=\rho_0 \quad \delta(x,t;\upsilon_2)=\delta_0 & \text{in}\Omega\\ \frac{\partial}{\partial t}\rho(x,t;\upsilon_1)=\upsilon_1 \quad \frac{\partial}{\partial x}\delta(x,t;\upsilon_2)=\upsilon_2 & \text{on}\Sigma\end{array}\right\}$$

(18)

For a control $\upsilon\in U$, the adjoint state $\rho^*(\upsilon_1)$, $\delta^*(\upsilon_2)\in L^2(0,T;L^2(Q))$ is defined by

$$-\frac{\partial}{\partial t}\rho^*(x,t;\upsilon_1)-c\frac{\partial}{\partial x}\rho^*(x,t;\upsilon_1)+\mu\rho^*(x,t;\upsilon_1)=\rho(\upsilon_1)-\rho_d \quad \text{in} Q \quad (19)$$

$$-\frac{\partial}{\partial t}\delta^*(x,t;\upsilon_2)-c\frac{\partial}{\partial x}\delta^*(x,t;\upsilon_2)+\mu\rho^*(x,t;\upsilon_2)-\mu_1[\delta_m-\delta^*(x,t;\upsilon_2)]=0 \text{ in} Q \quad (20)$$

$$\rho(x,T;\upsilon_1)=0 \quad \delta(x,T;\upsilon_2)=0 \text{ in } \Omega \quad (21)$$

$$\frac{\partial}{\partial t}\rho(x,T;\upsilon_1)=0 \quad \frac{\partial}{\partial n}\delta(x,T;\upsilon_2)=0 \text{ on}\Sigma \quad (22)$$

$$\rho^*(\upsilon_1),\delta^*(\upsilon_2)\in L^2(0,T;L^2(Q)) \quad (23)$$

Equations (19-23) admits a unique solution, a fact which follows from Theorem 4 Multiply-ing both sides of Eq. 14 by $\rho(\upsilon_1)\,.\,\rho(\mu_1)$ and both sides of Eq. 20 by $\delta(\upsilon_2)\,.\,\delta(\mu_2)$ with $(\upsilon_1=\mu_1)$ and $(\upsilon_2=\mu_2)$ noting that

$$\int_\Sigma\left(-c\frac{\partial}{\partial x}\rho^*(\mu_1),\rho(\upsilon_1)-\rho(\mu_1)\right)d\Sigma=\int_\Sigma\left(\begin{array}{l}\rho^*(\mu_1),\ c\frac{\partial}{\partial x}\rho(\upsilon_1)-\\ c\frac{\partial}{\partial x}\rho(\mu_1)\end{array}\right)d\Sigma$$

$$\int_\Sigma\left(-\frac{\partial}{\partial t}\rho^*(\mu_1),\rho(\upsilon_1)-\rho(\mu_1)\right)d\Sigma=\int_\Sigma\left(\begin{array}{l}\rho^*(\mu_1),\ \frac{\partial}{\partial t}\rho(\upsilon_1)-\\ \frac{\partial}{\partial t}\rho(\mu_1)\end{array}\right)d\Sigma$$

$$\int_\Sigma\left((\mu\rho^*(\mu_1),\rho(\upsilon_1)-\rho(\mu_1)\right)d\Sigma=\int_\Sigma\left((\rho^*(\mu_1),\mu\rho(\upsilon_1)-\mu\rho(\mu_1)\right)d\Sigma$$

So from (13), we have

$$\int_\Sigma (\rho(\upsilon_1-\rho_d,\rho(\upsilon_1)-\rho(\mu_1))d\Sigma = \int_\Sigma \left(\rho^*(\mu_1),\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial t}+\mu\right)(\rho(\upsilon_1)-\rho(\mu_1)\right)d\Sigma$$
$$=\int_\Sigma (\rho^*(\mu_1),\upsilon_1-\mu_1)d\Sigma$$

So:

$$\int_\Sigma (\rho^*(\mu_1)+\mu_1,\upsilon_1-\mu_1)d\Sigma \geq 0$$

Similarly

$$\int_\Sigma (\delta^*(\mu_2)+\mu_2,\upsilon_2-\mu_2)d\Sigma \geq 0$$

We have thus proved.

Theorem 4.1: Assume that, there exists a unique optimal solution (ρ, υ). The cost function being given by (8) a necessary and sufficient condition for v to be optimal control is that the following equations and inequalities be satisfied.

$$\frac{\partial}{\partial t}\rho(x,t;\mu_1)+c\frac{\partial}{\partial x}\rho(x,t;\mu_1)+\mu\rho(x,t;\mu_1) = 0 \quad \text{in } Q$$
$$\frac{\partial}{\partial t}\delta(x,t;\mu_2)+c\frac{\partial}{\partial x}\delta(x,t;\mu_2)+\mu\rho(x,t;\mu_2)-\mu_1[\delta_m-\delta(x,t;\mu_2)] = 0 \quad \text{in } Q$$
$$\rho(x,0,\mu_1)=\rho_0 \quad \delta(x,0,\mu_2)=\delta_0 \quad \text{in } \Omega$$
$$\frac{\partial}{\partial n}\rho(x,t;\mu_1)=\mu_1 \quad \frac{\partial}{\partial n}\delta(x,t;\mu_2)=\mu_2 \quad \text{in } \Sigma$$

$$-\frac{\partial}{\partial t}\rho^*(x,t;\mu_1)-c\frac{\partial}{\partial x}\rho^*(x,t;\mu_1)+\mu\rho^*(x,t;\mu_1) = \rho(\mu)-\rho_d \quad \text{in } Q$$
$$-\frac{\partial}{\partial t}\delta^*(x,t;\mu_2)-c\frac{\partial}{\partial x}\delta^*(x,t;\mu_2)+\mu\rho^*(x,t;\mu_2)-\mu_1[\delta_m-\delta^*(x,t;\mu_2)] = 0 \quad \text{in } Q$$
$$\rho(x,T;\mu_1)=0 \quad \delta(x,T;\mu_2)=0 \quad \text{in } \Omega$$
$$\frac{\partial}{\partial n}\rho(x,t;\mu_1)=0 \quad \frac{\partial}{\partial n}\delta(x,t;\mu_2)=0 \quad \text{in } \Sigma$$

$$\upsilon \in U_{ad},\ \int_\Sigma [\rho^*(\upsilon_1-\mu_1)+\delta^*(\upsilon_1-\mu_1)]d\Sigma + (\mu,\upsilon-\mu)\mu \geq 0,\ \mu \in U_{ad}$$

Since $U_{ad} = U$

$$\rho^*(\mu_1)+\mu_1 = 0 \quad \delta^*(\mu_2)+\mu_2 = 0$$

$$\text{so} \quad \mu_1 = \rho(\mu_1) \quad \mu_2 = -\delta(\mu_2)$$

together with

$$\rho(\mu_1) \in L^2(0,T;L^2(Q)) \quad \delta(\mu_2) \in L^2(0,T;L^2(Q))$$

$$\rho^*(\mu_1),\ \delta^*(\mu_2) \in L^2(0,T;L^2(Q))$$

CONCLUSION

We have established herein the uniqueness and existence of optimal solution for the coupled contaminant transport problem and have stated the optimality conditions of the problem. In a subsequent paper we shall discuss the decoupling and numerical solution for this system.

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