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Some Applications of Skew Hermitian Linear Maps

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Abstract: If M be closed subspace of a Hilbert space H, then there exist a unique pair of Linear mappings P and Q such that P maps H onto M,Q maps H onto $M^{\perp}$, And x = Px + Qx for all x∈H. In this study we prove some similar results For any Hilbert space H, the identity map I and all skew Hermitian maps. Some applications also followed.

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INTRODUCTION

In view of the decomposition theorem on Hilbert spaces, any Hilbert space can be decomposed as a direct sum of any subspace M and its orthogonal complement. In this study it is showen that any z∈H has a unique representation z = x+Sx for all densely defined skew Hermitian maps and some x∈H. As a first step, we study some preliminary properties of the graph of a skew Hermitian map. Then, as a useful application, a decomposition theorem was proved.In the next, the theorem is generalized to densely defined skew hermitian maps.

MAXIMALLY ISOTROPIC SUBSPACE

Let $(H,\langle,\rangle)$ be a Hilbert space.The componentwise inner product on H⊕H

Is also denoted by $\langle,\rangle$. Let $\langle,\rangle_+$ be the pairing on H⊕H defined by,

$$\langle(x,y),(x',y')\rangle_+ = \langle x,y'\rangle + \langle y,x'\rangle, \forall x,x',y,y' \in H$$

With the above notations, a maximal isotropic subspace (m.i.s) on H, is a subspace L of H⊕H such that,

$$\langle(x,y),(x',y')\rangle_+ = 0, \forall (x,y),(x',y') \in L$$

furthermore, L has no proper extension with the above property.

Example: for a Hilbert space H the Set I of all isotropic subspaces of H⊕H is nonempty and partially ordered by inclusion relation in which every chain has an upper bound, so I possesses a maximal element by *Zorn's lemma.*

Example: let S: H→H be a skew Hermitian linear map and $L = \text{graph}(S) \subseteq H \oplus H$. L is a (m.i.s) on H (Courant, 1990).

Lemma: Let $M \subseteq H \oplus H$ be isotropic. Then $\overline{M}$ is also isotropic. PROOF. Assume that $(a,b) \in \overline{M}$ and (x_n, y_n) be a sequence of elements of M converging to (a,b), then,

$$\langle(a,b),(x,y)\rangle_+ = \langle a,y\rangle + \langle b,x\rangle = \langle \lim x_n, y\rangle + \langle \lim y_n, x\rangle = \lim\langle(x_n,y_n),(x,y)\rangle_+ = 0$$

Corollary: Every (m.i.s) on H, is a Hilbert subspace of H⊕H

Theorem: let L be a (m.i.s) on H and P_i: H→H (I = 1, 2) be the first and second projections.Then the map P_1+P_2: L→H is surjective.

Proof: Let $z \perp (P_1+P_2)(L)$, by the decomposition theorem on Hilbert spaces,

There are (x, y) (x', y') ∈ Lsuch that (z, z) = (x, y) + (y', x') Therefore,

$$2\|z\|^2 = \langle(z,z),(z,z)\rangle = \langle(z,z),(x,y)\rangle + \langle(z,z),(y',x')\rangle = \langle z, x+y\rangle + \langle z, x'+y'\rangle = 0$$

i.e., z = 0. Hence $(P_1+P_2)(L)$ is dense in H. P_1+P_2 is also norm preserving, because for (x, y) ∈ L,

$$\|(x,y)\|^2 = \langle(x,y),(x,y)\rangle = \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2 = \langle x+y, x+y\rangle^2 = \|x+y\|^2$$

consequently if $(P_1+P_2)(x_n)$ be a Cauchy sequence in H, then (x_n) is also a Cauchy sequence in L. Hence, $(P_1+P_2)(L) = H$.

Theorem: with the above notations, the map $P_1+P_2 : L \rightarrow H$ is an isomorphism of Hilbert spaces.

Proof: By theorem 2.5 and Banach,s theorem it is enough to show that the map is injective. Let z = (x, y) $\in$ ker (P_1+P_2) then x+y = 0 and (x, -y) $\in$ L.
Therefore,

$$0 = \langle (x,-x),(x,-x) \rangle = -2\|x\|^2$$

i.e., z = 0.

Theorem: Let S : H $\rightarrow$ H be a skew Hermitian linear map. Then every h$\in$H has a unique representation h = x + Sx for some x$\in$H such that, $\|h\|^2 = \|x\|^2 + \|Sx\|^2$

Proof: Let L = graph(S). By example 2.1 L is a (m.i.s) on H. Thus by theorem 2.6, P_1+P_2 is an isometry. Hence, for some x$\in$H we have h = x+S x with required identity on norms.

Some interesting examples will be appearing by the following lemma.

Lemma: Let H$\rightarrow$H is a densely defined Skew Hermitian linear map.Then graph(s)
Is closed.

Proof: Let (x_n, y_n) be a sequence in graph(S) converging to (x,y) and $\mu \in$ Dom(S), then,

$$\langle u, -Sx_n \rangle = \langle Su, x_n \rangle$$

Therefore,

$$\langle u, -y \rangle = \langle Su, x \rangle \text{and}\, x \in \text{Dom}(S) = \text{Dom}(S^*)$$

Thus $\langle u, -y \rangle = \langle u, -Sx \rangle$ and $u \perp y - Sx$ for all $u \in$ Dom(S). Since Dom(S) is dense in H, we have y-Sx = 0 and $(x,y) \in \text{graph}(S)$.

Theorem: Let S : H$\rightarrow$H be a densely defined skew Hermitian linear map. Then graph(S) is a (m.i.s) on H.

Proof: by lemma 2.8 graph(S) is a closed subspace of H$\oplus$H Let {(a, b)}$\cup$ graph(S) be an isotropic subset of H$\oplus$H Then we have,

$$\langle (a,b),(x,Sx) \rangle_+ = \langle a, Sx \rangle + \langle b, x \rangle = 0$$

for all x $\in$ Dom(S) Furtheremore without loss of generality we can assume,

$$\langle a, x \rangle + \langle b, Sx \rangle = 0$$

Consequently,

$$\langle a+b, x+Sx \rangle = \langle a, x \rangle + \langle b, Sx \rangle + \langle a, Sx \rangle + \langle b, x \rangle = 0$$

for all x$\in$Dom(S) Let u = a+b, then, and $0 = \langle x+Sx, u \rangle = \langle x, -u \rangle = \langle x, -Su \rangle\ \langle x, u \rangle + \langle Sx, u \rangle$, therefore and $\mu \in$ $\text{Dom}(S^*) = \text{Dom}(S)$ and $\langle x, -u \rangle = \langle Sx, u \rangle$ for all x$\in$ Dom(S). Consequently, Su = u. Now we have,

$$\|u\|^2 = \langle u, u \rangle = \langle Su, u \rangle = \langle u, -Su \rangle = -\|u\|^2$$

or u = 0 and a = -b. since $\text{graph}(S) \cup \{(a,b)\}$ is isotropic subspace, a = b = 0 and the graph(S) is maximally isotropic subspace of H$\oplus$H. i.e., graph(s) is a (m.i.s) on H.

Theorem: Let S : H $\rightarrow$ Hbe a densely defined skew Hermitian linear map.Then every h$\in$H has a unique representation h = x+Sx,for some h = x+Sx, for some x$\in$H such that,

$$\|h\|^2 = \|x\|^2 + \|Sx\|^2$$

Proof: let L = graph(S).by 2.9 L is a (m.i.s) on H.Thus P_1+P_2: L$\rightarrow$H is an isometry, by theorem 2.6. Hence, h = x+Sx for some x$\in$H and the above identity on norms.

Application: Let H = L^2 (R) and S = d/dx be differential operator defined on the space $C_0^1(R) = H$, the space of continuously differentiable real functions with compact support. Using integration by parts, it can be showen that S is skew Hermitian linear map. Thus by theorem 2.10, any H $\in$ L^2 (R) has a unique representation $h = f + \frac{df}{dx}$ for some $f \in C_0^1(R)$.

REFERENCES

Courant, T.J., 1990. Dirac Manifolds, Trans. Amer. Math. Soc., 319: 631-661.