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An Algoritm for Solving Electromagnetic Field Equations by Finite Element Method

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Abstract: Describing the behaviour of electromagnetic frequency responses from vertically inhomogenous and anisotropic earth of 2-Dimensional structures energized finite sources is computationally laborious. Differential equations were derived and their numerical solutions also sought for the desired components of electric and magnetic fields. Also expressions for the impedance and apparent conductivity were stated. An algorithm based on the finite element method for computing approximate numerical solutions for these problems were dealinated.

Key words: Electromagnetic frequency response, inhomogenous and anisotropic earth, algorithm, finite element method

INTRODUCTION

The study of electrical resistivity from vertically inhomogenous earth wave conductivity changes continually with depth and has been investigated by many workers. The study of the d.c resistivity soundings on a model earth with transition layer was also made by Mallick and Roy (1968), Mallick and Jain (1979), Patella (1971), Koefold (1979) and Benerjee *et al*. (1980). Models of conductivity and resistivity varying linearly for Magnetotelluric (MT) soundings was also investigated by Mallick (1970), Abramovic (1974), Rankin and Reddy (1975), Rankin and Kao (1980) and Kao (1982).

Berdichevskiy *et al*. (1974) investigated models of resistivity decreasing exponentially with depth. Kao (1982) studied the models with an inhomogenous layer where the resistivity or conductivity increases and decreases exponentially with depth. Athough Kao's procedure seems to be more reliable, it is desirable to consider a model of 2-Dimensional vertically inhomogenous earth which may find application in the study of deep interior of the earth.

A numerical finite element approach has been studied by many workers in solving magnotelluric problems of any type in various dimensions. Kaikkonen (1984) investigates the finite element modeling in geophysical applications of magnetolluric fields. Kaikkonene and Sharma (1998) gave a precise explanation of an automated finite mesh generation and element coding in a 2-Dimensional magnetotelluric inversion. Kenneth and Whittal (1986), Kaikkonene (1992) studied a 2-dimensional inversion of magnetotelluric data with a variable model geometry. Eric *et al*. (1981), Rannacher (1995) explained the finite element solutions of diffusion problems. Kaikkonene (1996) investigated the boundary integral solution of a d.c geoelctric problem for a 2-dimensional body embedded in a two-layered earth.

The computation of magnetotelluric impedances from surface measurements of the magnetotelluric fields exhibit anisotropy, which may be due to vertically inhomogenous structures with different electrical properties. It is assumed here that the primary, natural electromagnetic fields are plane waves and the distance between the measuring electrodes is small relative to the dimensions of the structure.

The finite element method makes provision for the field as it involves the consideration of the field into smaller elements provide a very good approximation in the field. Thus, field characteristics as well as conductivity which is the most important property in studying the electric current flow in the earth medium can be considered.

MATERIALS AND METHODS

Solutions of the electromagnetic field equations have been considered. Differential equations have been solved for different components of the electromagnetic waves in 2-dimension for vertically inhomogenous earth medium and an algorithm for numerical solutions by finite element method has been considered, examining the Galerkin process.

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Expressions were also stated for the wave impedance and conductivity in anisotropic vertically inhomogenous earth as vital components of the electrical properties in magnetotelluric survey and analysis of current flow in erath medium.

RESULTS

The impedance at any reference point can be determined independently by numerical approach, so that the impedance at any other point could be determined provided the vertical distance h between them is known.

$$\partial E_x / \partial z = -iw\mu H_y \quad (1)$$

$$\partial E_y / \partial z = (-\sigma + iw\varepsilon) E_x \quad (2)$$

Obviously, these two equations contain two variables (E_x and H_y).

Because it is difficult to attempt to solve any of the 2 Eq numerically, we shall therefore combine the equations to give another equation with one variable.

Hence, differentiating (1) with respect to z yields:

$$\frac{\partial^2 E}{\partial z^2} = -iw\mu \frac{\partial H_y}{\partial z} \quad (3)$$

Substituting from (3) in (1) gives;

$$\frac{\partial^2 E_x}{\partial z^2} = -i\mu w\left(-(\sigma + iw\varepsilon)E_x\right) \quad (4)$$

To compute the magnitudes and phases of E_x and H_y, their real and imaginary parts have to be known. Splitting 4 yields,

Real:

$$\frac{\partial^2 E_x}{\partial z^2} = -w^2 \mu\varepsilon E_x \quad (5)$$

Imaginary:

$$\frac{\partial^2 E_x}{\partial z^2} = \sigma w \mu E_x \quad (6)$$

Also from (3) and (4), we found for H_y, that:

$$-iw\mu \frac{\partial H_y}{\partial z} = -i\mu w\left(-(\sigma + iw\varepsilon)E_x\right)$$

Thus;

Real part:

$$\partial H_y / \partial z = -\frac{E_x}{\rho} = -\sigma E_x \quad (7)$$

Imaginary part:

$$\partial H_y / \partial z = -w\varepsilon E_x \quad (8)$$

R.H.S of Eq. 7 and 8 become known constants, having computed numerical values for the components of E from Eq. 5 and 6. The electric vector oscillates along the x- direction and the magnetic vector ,in the y-direction. Since we have an electromagnetic field traveling in the z-direction, both E_x and H_y are being calculated at various points along the z-direction in the medium.

Choice of initial conditions for the differential equations

Electric field component:

Real part:

$$\frac{\partial^2 E_x}{\partial z^2} = -w^2 \mu \varepsilon E_x$$

Imaginary part:

$$\frac{\partial^2 E_x}{\partial z^2} = w \mu \sigma E_x$$

Magnetic field component:

Real part:

$$\frac{\partial H_y}{\partial z} = -\sigma E_x$$

Imaginary part:

$$\frac{\partial H_y}{\partial z} = -w \varepsilon E_x$$

For this method, we require some defined initial conditions for the 2 variables $E_x(0)$ and $H_y(0)$ having decided to apply a suitable finite element method to solve derived differential equations for E_x and H_y.

DISCUSSION

We sought for numerical solution for E_x and H_y by applying the Galerkin process, which is an aspect of the weighted residual method. We find an interpolation function for the electric field component (and the application of the weighted residual function is discussed).

For the numerical solution of E_x:

$$\frac{\partial^2 E_x}{\partial z^2} + CE_x = 0$$

Where $C = w^2 \mu E$

And by Galerkin process,

$$\int N_i \left(\frac{\partial^2 E_x}{\partial z^2} + CE_x \right) dz$$

Where N is known as the shape function:

$$\int N_i \frac{\partial^2 E_x}{\partial z^2} dz + \int N_i \ CE_x \ dz = 0$$
$$E_x = [N] \ \{E_x\}$$

Therefore,

$$\int N_i \frac{\partial^2 [N]}{\partial z^2} \{E_x\} + \int CN_i \ [N] \ \{E_x\} \ dz = 0$$

Integrating, we obtain

$$N_i \int \frac{\partial^2 [N]}{\partial z^2} \{E_x\} dz - \int \frac{\partial N_i}{\partial z} \int \frac{\partial^2 [N]}{\partial z^2}$$
$$\{E_x\} dz + \int CN_i [N] \ \{E_x\} dz = 0$$
$$N_i \frac{\partial [N]}{\partial z} \{E_x\} - \int \frac{\partial N_i}{\partial z} \cdot \frac{\partial [N]}{\partial z}$$
$$\{E_x\} + \int CN_i [N] \ \{E_x\} dz = 0$$

Where the first term i.e

$$N_i \frac{\partial [N]}{\partial z} \{E_x\}$$

Accounts for the boundary condition and in matrix notation; we can rewrite the last expression as:

$$[K_{ij}] \ \{E_x\} - C[M_{ij}] \ \{E_x\} = 0$$
$$[K_{ij}] = \int \frac{\partial N_i}{\partial z} \cdot \frac{\partial N_j}{\partial z} dz \ \text{ and } \ [M_{ij}] = C \int N_i N_j dz$$

Next we move further to choose an interpolation function for our equation i.e.

$$\frac{\partial^2 E_x}{\partial z^2} + CE_x = 0$$

Thus,

$$E_x = \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \alpha_4 z^3$$

The minimum order of the polynomial is quadratic for this type of equation as we need to select a displacement model for E_x and axial displacements dE_x/∂_z are directly related to the transverse displacement, E_x. The joint degrees of freedom at the end nodes, designated E_{x1} and E_{x2}. An additional (internal) degree of freedom is necessary in order to employ the suggested model. Instead , let us employ two additional joint degrees of freedom in order to preserve interclement compartbility for the slope.

The slope is defined as;

$$\theta = \frac{dE_x}{\partial z}$$

The two additional degrees of freedom are θ_i and θ_j. Thus,

$$\theta = \frac{dE_x}{d z} = \alpha_2 + 2\alpha_3 z + 3\alpha_4 z^2$$

The nodal values of of E_x and dE_x/∂_z are obtained by evaluating at each node, thus:

$$E_{x_i} = \alpha_1 + \alpha_2 z_i + \alpha_3 z_i^2 + \alpha_4 z_i^3$$

$$\frac{dE_x}{\partial z} = \alpha_2 + 2\alpha_3 z_i + 3\alpha_4 z_i^2$$

$$E_{x_j} = \alpha_1 + \alpha_2 z_j + \alpha_3 z_j^2 + \alpha_4 z_i^3$$

$$\frac{dE_{x_j}}{dz} = \alpha_2 + 2\alpha_3 z_j + 3\alpha_4 z_i^2$$

These are chosen in order for the boundedness to be assured, that is;

- The displacement models must be continous within the elements and the displacements must be compartible between adjacent elements.
- The displacement models must include the rigid body displacements of the elements.
- The displacement models must include the constant strain states of the element.

For convenience, transforming our coordinate system as follows:

$$z = z - z_1$$

Where $z_1 = 0$ and $z_2 = \ell$ and our displacement model becomes;

$$E_x = \{\phi\}^T \{\alpha\}$$

$$\underset{1\times 4}{\{\phi\}^T} = \underset{4\times 1}{\begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix}} \qquad (*)$$

Also,

$$\theta(2) = \begin{bmatrix} 0 & 1 & 2z & 3z^2 \end{bmatrix} \{\alpha\}$$

In matrix notation, we now express the nodal displacement in terms of the generalized coordinates:

$$\{q\} = \begin{Bmatrix} E_x(z=0) \\ \theta(z=0) \\ E_x(z=\ell) \\ \theta(z=\ell) \end{Bmatrix} = \begin{Bmatrix} E_{x_1} \\ \theta_1 \\ E_{x_2} \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^2 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \{\alpha\}$$

$$= [A]\ \{\alpha\}$$

Inverting the equations, we obtain

$$\alpha = [A]^{-1}\{q\}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/\ell^2 & -2/\ell & 3/\ell^2 & -1/\ell \\ 2/\ell^3 & 1/\ell^2 & -2/\ell^3 & 1/\ell^2 \end{bmatrix} \{q\}$$

Substituting (**) in (*) gives the formulation of the equation.

$$\{q\} = \{\phi\}\ \left[A^{-1}\right]\ \{q\}$$

$$= [N]\{q\}$$

Where:

$$N = \left[\left(1 - \frac{3z^2}{\ell^2} + \frac{2z^3}{\ell^3}\right), \left(z - \frac{2z^2}{\ell} + \frac{z^3}{\ell^2}\right), \left(\frac{3z^2}{\ell^2} - \frac{2z^3}{\ell^3}\right), \left(\frac{z^3}{\ell^2} - \frac{z^2}{\ell}\right)\right]$$

$$\frac{\partial N}{\partial z} = \left[\left(\frac{6z^2}{\ell^3} - \frac{6z}{\ell^3}\right), \left(1 - \frac{4z}{\ell} + \frac{3z^2}{\ell^2}\right), \left(\frac{6z}{\ell^2} - \frac{6z^2}{\ell^3}\right), \left(\frac{3z^2}{\ell^2} - \frac{2z}{\ell}\right)\right]$$

Recall that

$$\left[K_{ij}\right] - C\left[M_{ij}\right] = 0$$

When this terms are substituted for a set of algebraic linear equations that can be solved simultaneously are generated, this is achieved using the Gaussian Elimination Method.

The same procedure can be carried out for $\partial^2 E_x/\partial z^2 = w\mu\sigma E_x$, the imaginary part of the field component.

Numerical solution for H:

Real part:

$$\frac{\partial H_y}{\partial z} = -\sigma E_x$$

Imaginary part:

$$\frac{\partial H_y}{\partial z} = -w\varepsilon E_x$$

The Galerkin processes are also applied to these equations.

$$\frac{\partial H_y}{\partial z} = -\sigma E_x$$

$$\frac{\partial H_y}{\partial z} + C = 0 \text{ where } C = \sigma E_x$$

$$\int N_i \left(\frac{\partial H_y}{\partial z} + C\right)\ dz = 0$$

Also, $H_y = [N]\{H_y\}$

Therefore,

$$\int N_i \frac{\partial [N]}{\partial z}\{H_y\}\ dz + \int N_i\ C\ dz = 0$$

$$\left[K_{ij}\right]\{H_y\}^e + F_i = 0$$

$$\left[K_{ij}\right]\{H_y\}^e = -F_i$$

Where K_{ij} is the stiffness matrix and F_i is the load matrix and they are

$$\int N_i \frac{\partial N_j}{\partial z} \quad \text{and} \quad -\int CN_i, \text{ respectively.}$$

We now assume a linear interpolation function $Hy = \alpha_1 + \alpha_2 z$.

Since we are dealing with one dimensional simplest element with the length, L. The nodes are denoted by i and j and the nodal values by H_{yi} and H_{yj}. The coefficients α_1 and α_2 can be determined using the nodal conditions below:

$$H_y = H_{y_i} \quad \text{at } z = z_i$$

and

$$H_y = H_{y_j} \quad \text{at } z = z_j$$

These nodal conditions result in the pair of equations

$$H_{y_i} = \alpha_1 + \alpha_2 z_i \qquad (*)$$

$$H_{y_j} = \alpha_1 + \alpha_2 z_j \qquad (**)$$

Which may be solved for as

$$\alpha_1 = \frac{H_{y_i} z_j - H_{y_j} z_i}{L}$$

$$\alpha_2 = \frac{H_{y_j} - H_{y_i}}{L}$$

When solved, substitution of the values for α_1 and α_2 into (* and **) produces

$$H_{y_i} z_j - H_{y_j} z_i$$

$$H_{y_i} = \frac{z_j - z}{L} + \left(\frac{H_{y_j} - H_{y_i}}{L}\right) z$$

which can be rearranged into

$$H_y = \left(\frac{z_j - z}{L}\right) H_{y_i} + \left(\frac{z - Z_i}{L}\right) H_{y_i}$$
$$= N_i H_{y_i} + N_j H_{y_j}$$

$$N_i = \frac{z_j - z}{L} \quad \text{and} \quad N_j = \frac{z - Z_i}{L}$$

$$\frac{\partial N_i}{\partial z} = -\frac{1}{L} \quad \text{and} \quad \frac{\partial N_j}{\partial z} = \frac{1}{L}$$

This can be substituted into our previous equation i.e.

$$\int N_i \frac{\partial N_j}{\partial z} dz + \int N_i C dz$$

$$\int N_i \frac{\partial N_j}{\partial z} dz = -\int N_i C dz$$

to generate stiffness and load matrices.
For example, consider an element

0 ———————— 1
z_1 L z_2

With nodal parameters H_{y1} and H_{y2} coordinates z_1 and z_2 with length L. Therefore, for the stiffness and load matrices, we have

$$K_{ij} = \int N_i \frac{\partial N_i}{\partial z} dz; \qquad F_i = -\int N_i \, C \, dz$$

therefore,

$$k_{11} = -\int_0^1 \left(\frac{z_1 - z}{L} . \frac{1}{L}\right) dz = \frac{1}{L^2}\left[\frac{z^2}{2} - z^2 z\right]_0^1 = -\frac{1}{2L^2}$$

$$k_{12} = \int_0^1 \left(\frac{z_2 - z}{L} . \frac{1}{L}\right) dz = \frac{1}{L^2}\left[z_2 z - \frac{z^2}{2}\right] = \frac{1}{2L^2}$$

$$k_{21} = -\int_0^1 \left(\frac{z - z_1}{L} . \frac{1}{L}\right) dz = \frac{1}{L^2}\left[z_1 z - \frac{z^2}{2}\right]_0^1 = -\frac{1}{2L^2}$$

$$k_{22} = \int_0^1 \left(\frac{z - z_2}{L} . \frac{1}{L}\right) dz = \frac{1}{L^2}\left[\frac{z^2}{2} - z_2 z\right]_0^1 = \frac{1}{2L^2}$$

also

$$F_1 = \frac{C}{L}\int_0^1 (z - z) dz = \frac{C}{L}\left(\frac{z^2}{2} - z_1 z\right) = \frac{C}{2L}$$

$$F_2 = \frac{C}{L}\int_0^1 (z - z_1) dz = \frac{C}{L}\left(\frac{z^2}{2} - z_1 z\right)_0^1 = \frac{C}{2L}$$

In matrix form

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2L^2} + \frac{C}{2L} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \left\{H_y\right\}^e = -C\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the same process is carried out on subsequent elements and the matrices are later assembled and linear equations that can be solved simultaneously by direct substitution are generated.

The same procedure is applied to the imaginary part of the magnetic field component.

CONCLUSION

Having derived expressions for both the real and imaginary parts of the electric and magnetic field components of the electromagnetic wave E_x and H_y ,their magnitudes and phases can be computed.

The magnitude of the electric field component is given as

$$|E| = \left[\left(E_{real}\right)^2 + \left(E_{imag}\right)^2\right]^{\frac{1}{2}}$$

and the phase is given by:

$$\tan\phi = \frac{E_{real}}{E_{imag}} \qquad \therefore \quad \phi = \text{arc } \tan\left[\frac{E_{real}}{E_{imag}}\right]$$

Similarly, for the magnetic field component,

$$|H| = \left[(H_{real})^2 + (H_{imag})^2\right]^{\frac{1}{2}}$$

and the phase is given by:

$$\tan\phi = \frac{H_{real}}{H_{imag}}; \qquad \therefore \quad \phi = \text{arc } \tan\left[\frac{H_{real}}{H_{imag}}\right]$$

Having computed the magnitudes of E_x and H_y, we can now find the values of the wave impedance which is given by:

$$Z = \frac{|E_x|}{|H_y|}$$

And the apparent conductivity σ_α can thus be computed from the equation stated below;

$$\sigma_a = \frac{jw\mu}{z^2}$$

The imaginary form in which σ_α is appearing merely indicates that there is a 45^0 phase difference between the oscillations in the magnetic and electric intensities. The next step is to write a computer program that will compute for us, all the steps treated above.

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