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A Zero Stable Continuous Hybrid Methods for Direct Solution of Second Order Differential Equations

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Abstract: This study produces a zero stable hybrid three-step methods for a direct solution of general second order ordinary differential Eq of form *y" = f (x, y y')*. The differential system from the basis polynomial function to the problem is collocated at all the grid points and at an off-grid point. The basis function is interpolated at x_{n+i}, I = 0,1,2. The method is consistent and zero-stable. The efficiency and accuracy of the method are shown with some test examples.

Key words: Collocation, differential system, basis function, hybrid, symmetric, continuous method, zero stable

INTRODUCTION

The solution of higher order differential Eq of the form

$$y^{(m)} = f(t, y, y^1, y^{11}, ..., y^{(m-1)})y^{(s)}(t_0) = y_0^{(s)}, s = 0(1)m-1 \quad (1)$$

is considered in this study. It has been observed in literature that solutions of such Eq are usually reduced to system of first order Eq of the form

$$y' = f(y),\ y(t_0) = \mu,\ f \in C^1[a, b], y, t \in R^n \quad (2)$$

There are numerous numerical methods developed to handle the reduced Eq. 2 (Lambert, 1973; Goult *et al*. 1973; Jain, 1984; Lxaru, 1984; Jacques and Judd, 1987; Fatunla, 1988; Bun and Vasil' Yer, 1992; Awoyemi, 1992; Jaun, 2001; Chan *et al*., 2004). This approach has many disadvantages such as much of computational burden and computer time wastage. Hence, there is need for direct methods for solving Eq. 1 without reducing it to system of first order equations.

Awoyemi and Kayode (2003) highlighted some of the direct methods for solving (1), in which m = 2 and the derivative is absent in the right side.

In this study, a three-step hybrid numerical technique is proposed for a direct solution of initial value problems (1) in which m=2 to be of the form

$$y'' = f(x, y, y'),\ y(a) = \mu,\ y'(a) = \tau,\ f \in C^1[a, b], y, x \in R^n \quad (3)$$

MATERIALS AND METHODS

In this study, the development of the collocation methods for the solution of second order ordinary differential Eq. 3 directly without reducing it to first order system of Eq. is discussed. The method obtained is an order five hybrid linear multistep with continuous coefficients of the form

$$y_k(t) = \sum_{j=0}^{k-1} \alpha_j(t) y_{n+j} + \sum_{j=2}^{k} \beta_j(t) f_{n+j} + \tau(t) f_{n+u} t \in (0, 1] \text{ and } v \in (1, 2) \quad (4)$$

The approximate solution to problem (1) is taken to be a partial sum of a P- series of a single variable x in the form

$$y(x) = \sum_{j=0}^{2k} a_j x^j \quad a_j \in \Re,\ j = 0(1)2k,\ y \in C^m(a, b) \subset P(x) \quad (5)$$

It is assumed that the initial value problem (1) satisfies the hypotheses of existence and uniqueness theorem. The first and second derivatives of (5) are respectively taken as

$$y'(x) = \sum_{j=1}^{2k} j a_j x^{j-1} \quad (6)$$

$$y''(x) = \sum_{j=2}^{2k} j(j-1) a_j x^{j-2} \quad (7)$$

From (3) and (7), we have

$$\sum_{j=2}^{2k} j(j-1)a_j x^{j-2} = f(x, y(x), y'(x)) \qquad (8)$$

Thus, collocating Eq. 8 at the grid points x_{n+i}, I = 0,1,2,3,v, 1<v<2, and interpolating (5) at x_{n+i}, I = 0(1)k-1, for k = 3, yields a system of Eq.

$$\sum_{j=2}^{2k} j(j-1)a_j x_{n+i}^{j-2} = f_{n+i},\ i = 0(1)k \qquad (9)$$

$$\sum_{j=2}^{2k} j(j-1)a_j x_{n+i}^{j-2} = f_{n+v},\quad v\in(1,2) \qquad (10)$$

$$\sum_{j=0}^{2k} a_j x_{n+i}^{j} = y_{n+i},\ i = 0(1)k-1 \qquad (11)$$

whe re $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i})y_{n+i}$ is the numerical approximation to $y(x_{n+i})$ at x_{n+i}

and $x_{n+i} = x_n + ih.$

Solving Eq. 9, 10 and 11 to obtain the parameters a_j 's, *j*, and then substituting for these values into Eq. 3 produces a continuous method expressed as

$$y_k(x) = \sum_{j=0}^{k-1}\alpha_j(x)y_{n+j} + \sum_{j=2}^{k}\beta_j(x)f_{n+j} + \tau(x)f_{n+u} \qquad (12)$$

Using the transformations

$$t = \frac{1}{h}(x - x_{n+k-1})\text{ and }\frac{dt}{dx} = \frac{1}{h}\ t\in(0,1] \qquad (13)$$

the coefficients in the continuous method (12) are obtained, as a function of t, to be

$$\alpha_2(t) = \{1+t\}$$
$$\alpha_1(t) = -t$$
$$\alpha_0(t) = 0$$

$$\beta_3 = \frac{h^2}{360(3-v)}\{(8v-13)t + 20(2-v)t^3 + 5(8-3v)t^4 + 3(5-v)t^5 + 2t^6\}$$

$$\beta_2 = \frac{h^2}{120(2-v)}\{(75-43v)t + 60(2-v)t^2 + 10(4-v)t^3 + 5(2v-3)t^4 + 3(v-4)t^5 - 2t^6\}$$

$$\beta_v = \frac{h^2}{60v(3-v)(2-v)(v-1)}\{11t - 20t^3 - 5t^4 + 6t^5 + 2t^6\}$$

$$\beta_1 = \frac{h^2}{120(v-1)}\{11(2v-3)t + 20(2-v)t^3 + 5vt^4 + 3(v-3)t^5 - 2t^6\}$$

$$\beta_. = \frac{h^2}{360v}\{(11-7v)t + 10(v-2)t^3 - 5t^4 + 3(2-v)t^5 + 2t^6\} \qquad (14)$$

Taking the first derivatives of a_j, $$_j$, in (14) yields

$$\alpha'_2 = \frac{1}{h}$$

$$\alpha'_1 = -\frac{1}{h}$$

$$\beta'_3 = \frac{h}{360(3-v)}\{(8v-13) + 60(2-v)t^2 + 20(4-3v)t^3 + 15(5-v)t^4 + 12t^5$$

$$\beta'_2 = \frac{h}{120(2-v)}\{(75-43v) + 120(2-v)t + 30(4-v)t^2 + 20(2v-3)t^3 + 15(v-4)t^4 - 12t^5\}$$

$$\beta'_v = \frac{h}{60v(3-v)(2-v)(v-1)}\{11 - 60t^2 - 20t^3 + 30t^4 + 12t^5\}$$

$$\beta'_1 = \frac{h}{120(v-1)}\{11(2v-3) + 60(2-v)t^2 + 20vt^3 + 15(v-3)t^4 - 12t^5\}$$

$$\beta'_0 = \frac{h}{360v}\{(11-7v) + 30(v-2)t^2 - 40t^3 + 15(2-v)t^4 + 12t^5\} \qquad (15)$$

To obtain a sample discrete scheme from the continuous method (12), the values of t in (14) could be taken in the interval I = (0, 1]. Hence for the purpose of this research t is taken to be 1, which implies that $x = x_{n+3}$ from (13), to have a one-point hybrid discrete scheme as

$$y_{n+3} - 2_{n+2} + y_{n+1} = \frac{h^2}{60v(3-v)(2-v)(v-1)}(Af_{n+3} + Bf_{n+2} + Cf_{n+v} + Df_{n+1} + Ef_n) \qquad (16)$$

where

$$A = v(14-5v)(2-v)(v-1)$$
$$B = v(103-50v)(3-v)(v-1)$$
$$C = -6$$
$$D = v(5v-2)(3-v)(2-v)$$
$$E = -(3-v)(2-v)(v-1)$$

and from (15)

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{360v(3-v)(2-v)(v-1)} \{Ff_{n+3} + Gf_{n+2} + Hf_{n+v} + If_{n+1} + Jf\} \quad (17)$$

where

$$F = v(354-127v)(2-v)(v-1)$$
$$G = 3v(303-138v)(3-v)(v-1)$$
$$H = -162$$
$$I = 9v(10-v)(3-v)(2-v)$$
$$J = (8v-27)(3-v)(2-v)(v-1)$$

Taking the values of v in (16) and (17) are taken at three points 5/4, 3/2, 7/4, in the interval (Awoyemi, 1992; Awoyemi and Kayode, 2002) to obtain the following discrete schemes:

For v = 5/4:

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{2100}(155f_{n+3} + 1890f_{n+2} - 512f_{n+\frac{5}{4}} + 595f_{n+1} - 284f_n)$$

of order P = 5, error constant $C_{p+2} \approx -0.002292$ (18)

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{12600}(3905f_{n+3} + 18270f_{n+2} - 13824f_{n+\frac{5}{4}} + 11025f_{n+1} - 476f_n)$$

Order P = 5, $C_{p+2} = 0.006344$ (19)

For: v = 3/2

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{180}(13f_{n+3} + 168f_{n+2} - 32f_{n+\frac{3}{2}} + 33f_{n+1} - 2f_n)$$

Order P = 5, Error constant $C_{p+2} \approx -0.002083$ (20)

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h}{360}(109f_{n+3} + 576f_{n+2} - 288f_{n+\frac{3}{2}} + 153f_{n+1} - 10f_n)$$

Order P = 5, Error constant $C_{p+2} \approx 0.005407$ (21)

for v = 7/4:

$$y_{n+3} = 2_{n+2} - y_{n+1} + \frac{h^2}{2100}(147f_{n+3} + 2170f_{n+2} - 512f_{n+\frac{7}{4}} + 315f_{n+1} - 20f_n)$$

Order P = 5, Error constant $C_{p+2} \approx -0.001875$ (22)

$$y'_{n+3} = \frac{1}{h}(y_{n+2} - y_{n+1}) + \frac{h^2}{12600}(3689f_{n+3} + 25830f_{n+2} - 13824f_{n+\frac{7}{4}} + 3465f_{n+1} - 260f_n)$$

Order P = 5, Error constant $C_{p+2} \approx 0.004469$ (23)

Starting values for the methods: The set of implicit discrete schemes (18), (20) and (22) and their respective first derivatives (19) (21) and (23) are not self-starting. Thus to be able to implement them, some starting values, of the same order p = 5 and their derivatives are developed using the same technique for the main method (13). Thus at t = 1 and r = 5/4, 3/2, 7/4, the main starting values are:

For v = 5/4:

$$y_{n+3} = -\frac{23}{4}y_{n+2} + \frac{29}{2}y_{n+1} - \frac{31}{4}y_n + \frac{h^2}{240}(495f_{n+2} - 512f_{n+\frac{5}{4}} + 1990f_{n+1} + 127f_n)$$

having order p = 5 and $C_{p+2} \approx 0.0122396$ and (24)

$$y'_{n+3} = \frac{1}{24h}\{-757y_{n+2} + 1538y_{n+1} - 781y_n\} + \frac{h}{7200}\{45585f_{n+2} - 65024f_{n+\frac{5}{4}} + 248410f_{n+1} + 16129f_n\}$$

p = 5, $C_{p+2} \approx -0.054671$ (25)

For v = 3/2:

$$y_{n+3} = -\frac{9}{2}y_{n+2} + 12y_{n+1} - \frac{13}{2}y_n + \frac{h^2}{24}(51f_{n+2} - 32f_{n+\frac{3}{2}} + 150f_{n+1} + 11f_n) \quad (26)$$

of order p = 5 and $C_{p+2} \approx 0.011458$

$$y'_{n+3} = \frac{1}{4h}\{-105y_{n+2} + 214y_{n+1} - 109y_n\} + \frac{h}{720}\{4749f_{n+2} - 4064f_{n+\frac{3}{2}} + 18618f_{n+1} + 1397f_n\} \quad (27)$$

and of order p = 5, $C_{p+2} \approx -0.051364$

For7/4:

$$y_{n+3} = -\frac{13}{4}y_{n+2} + \frac{19}{2}y_{n+1} - \frac{21}{4}y_n + \frac{h^2}{336}(847f_{n+2} - 512f_{n+\frac{7}{4}} + 1638f_{n+1} + 127f_n) \qquad (28)$$

of order p = 5 and $C_{p+2} \approx 0.01015625$ and

$$y'_{n+3} = \frac{1}{24h}\{-503y_{n+2} + 1030y_{n+1} - 527y_n\} + \frac{h}{100080}\{83377f_{n+2} - 65024f_{n+\frac{7}{4}} + 201978f_{n+1} + 16129f_n\} \qquad (29)$$

also of order p = 5 and $C_{p+2} \approx -0.0458519$

Other starting values for y_{n+2}, y_{n+2} , y_{n+v}, y_{n+v}, y_{n+1}, y_{n+1} are obtained to be

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1}, \ p = 2, \ cp + 2 = 0.0833 \qquad (30)$$

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{6}(11f_{n+1} - 2f_n) \qquad (31)$$

p = 2, cp + 2 =- 0.375

The initial values y_n, y_n' are obtained in the given problem.

$$y_{n+j} = y_n + (jh)y'_n + \frac{(jh)^2}{2!}f_n + \frac{(jh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^4) \qquad (32)$$

$$y'_{n+j} = y'_n + (jh)f_n + \frac{(jh)^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^3) \qquad (33)$$

where j=1, $\frac{5}{4}$, $\frac{3}{2}$ and $\frac{7}{4}$

The initial values y_n, y'_n are obtained in the given preblem.

NUMERICAL EXPERIMENT

The accuracy of the continuous method developed for the direct solution of second order ordinary differential Eq tested with the following problems

$$y0 = 2y^3, \ y(1) = 1, \ y'(1) = -1;$$

Theoretical solution: y(x) = 1/x.

$$y0 = y + xe^{3x}$$

Theoretical solution: $y(x) = \frac{(4x-3)}{32\exp(-3x)}$.

$$y''(x) = \frac{(4x-3)}{32\exp(-3x)}y(\frac{\Pi}{6})$$

$$= \frac{1}{4}y(\frac{\Pi}{6}) = \frac{\sqrt{3}}{2}, \ h = \frac{1}{40}$$

Theoretical solution is given as $y(x) = \mathrm{Sin}^2 x$.

$$Y0 - x(y')^2 = 0, \ y(0) = 1, \ y'(0) = 1/2; \ h1/40$$

Theoretical solution is $y(x) = 1 + \frac{1}{2}\ln\{\frac{(2+x)}{(2-x)}\}$.

RESULTS

The absolute errors obtained from the method (15) for k = 3 are compared with those obtained from the method for k = 2 in Kayode (2004) for the problems (i)-(iv). The results are shown in the Table 1 and 2 below. The accuracy of the results is further illustrated graphically in the Fig. 1-4.

Table 1:Comparison of errors for problems (i) and (ii) for k = 2, 3

	Kayode (2004) for problem (i)	New Method (15) for problem (i)		Kayode (2004) for problem (ii)	New Method (15) for problem (ii)
x	Errors for k = 2	Errors for k = 3	x	Errors for k = 2	Errors for k = 3
1.1	0.5373197D-06	0.5263931D-08	0.1	0.9056176D-08	0.2086753D-09
1.2	0.4142659D-06	0.3720895D-08	0.2	0.8640876D-07	0.1923770D-09
1.3	0.3260932D-06	0.2704051D-08	0.3	0.2167778D-06	0.1391324D-09
1.4	0.2612683D-06	0.2012022D-08	0.4	0.4277919D-06	0.2508468D-10
1.5	0.2125477D-06	0.1527879D-08	0.5	0.7599223D-06	0.1857944D-09
1.6	0.1752254D-06	0.1180986D-08	0.6	0.1287868D-05	0.5588885D-09
1.7	0.1461539D-06	0.9271881D-09	0.7	0.2073414D-05	0.1157671D-08
1.8	0.1231736D-06	0.7380502D-09	0.8	0.3250983D-05	0.2107025D-08
1.9	0.1047692D-06	0.5947730D-09	0.9	0.4998729D-05	0.3578957D-08
2.0	0.8985620D-07	0.4846371D-09	1.0	0.7571481D-05	0.5822924D-08

Table 2:Comparison of errors for problem (iii) and (iv) for k = 2, 3

	Kayode (2004) for problem (iii)	New method (15) for problem (iv)		Kayode (2004) for problem (iii)	New method (15) for problem (iv)
x	Errors for k = 2	Errors for k = 3	x	Errors for k = 2	Errors for k = 3
1.1	0.5025381D-05	0.2282106D-05	1.1	0.1053972D-06	0.8047086D-07
1.2	0.6249908D-05	0.2893084D-05	1.2	0.2131542D-06	0.1625604D-06
1.3	0.7288316D-05	0.3453509D-05	1.3	0.3258333D-06	0.2480160D-06
1.4	0.8112927D-05	0.3954212D-05	1.4	0.4463340D-06	0.3387987D-06
1.5	0.8700464D-05	0.4384330D-05	1.5	0.5781210D-06	0.4372248D-06
1.6	0.9032439D-05	0.4731177D-05	1.6	0.7294657D-06	0.5490446D-06
1.7	0.9095639D-05	0.4980477D-05	1.7	0.8987956D-06	0.6725762D-06
1.8	0.8882690D-05	0.5116961D-05	1.8	0.1097337D-05	0.8153498D-06
1.9	0.8392690D-05	0.5125297D-05	1.9	0.1336004D-05	0.9842053D-06
2.0	0.7631870D-05	0.4991312D-05	2.0	0.1630750D-05	0.1188939D-05

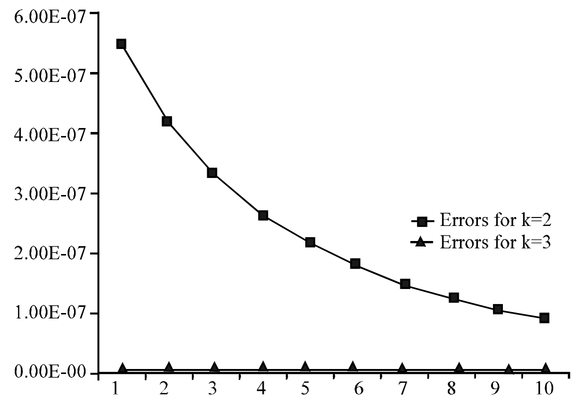

Fig 1: Comparison of errors for problem (i) for k = 2, 3

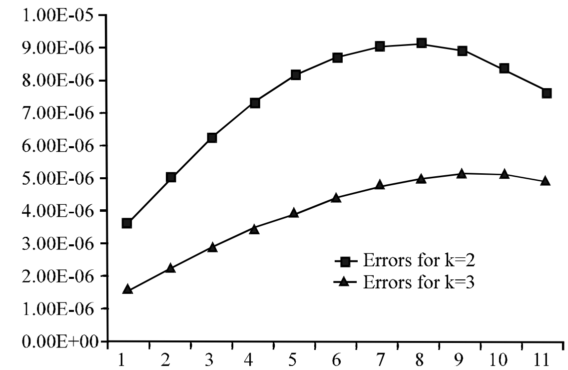

Fig 3: Comparison of errors for problem (iii) for k = 2, 3

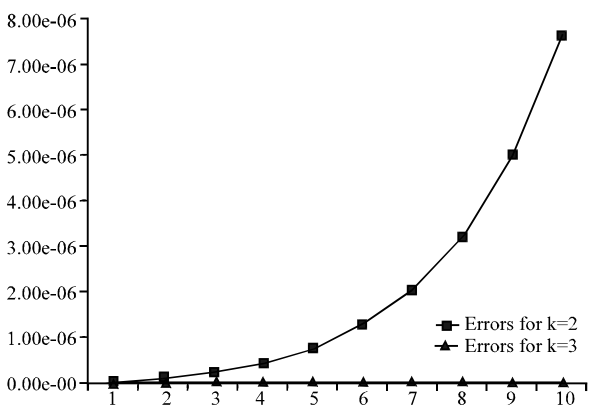

Fig 2: Comparison of errors for problem (ii) for k = 2, 3

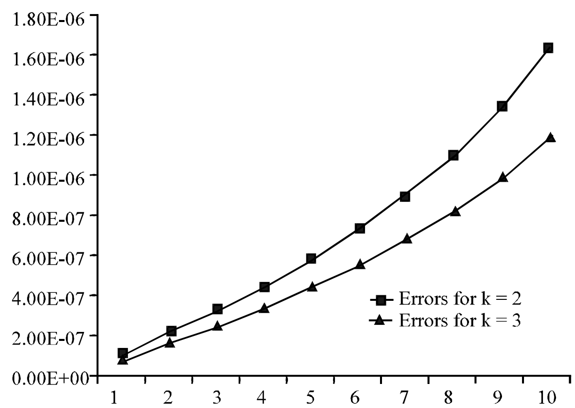

Fig 4: Comparison of errors for problem (iv) for k = 2, 3

CONCLUSION

This study has considered the development of a continuous hybrid numerical method with step number k = 3. A set of discrete schemes of the same order p = 5 are obtained from the continuous method. The major predictors for the methods are constructed to be of the same order p = 5 with the methods. The efficiency of the method is compared with existing order four method (Kayode 2004; 2006).

The comparison of the absolute errors obtained from the results for the test problems above are shown in Table 1 and 2 and also in Fig. 1 and 4. These errors show a considerable improvement in accuracy of the new method over Kayode (2004).

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