Research Journal of Applied Sciences 2 (3): 314-318, 2007

On the Use of Orthogonality Relations to Construct Character Tables of Finite Group

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Abstract: Group characters are very fundamental in the study of group representation. In this paper we employed the orthogonality relation in constructing the character table which shows the difference irreducible characters for each conjugacy class. The first kind is used along the rows, while the second kind is used along the columns. Relevant details of this method are described following.

Key words: Group characters, representation, orthogonality relation, irreducible, conjugacy class, inner product.

INTRODUCTION

The complete information about the character of G is conveniently displayed by Isaacs, (1956) in a character table which lists the values of the K simple character for all elements. We know that a character is constant on each of the conjugacy classes C_a. If x∈ C_a, we put $x(x) = x^{(a)}$. Thus it is sufficient to record the values$\chi^{(a)}$, $1\le a\le k$

Denoting the number of elements in C_a by $h_k^{(a)}$, we have the class equation

$$h^{(1)} + h^{(2)} +...+ h^{(k)} = g \quad (1)$$

Unless the contrary is stated. We adhere to the convention that G = {1} and that F is the trivial representation F_1 (x) = 1, x∈G

The Table 1 below represent a typical character table, the body of the table is a K square matrix whose rows correspond to the different characters, while each column contain the values of all simple character for a particular conjugacy class.

Definition 1: A generalized character of a group G is a function of the form

$$\sum_{i=1}^{s} a_i x_i$$

	C^1	C^2...	C^A...	C^K
	$h^{(1)}$	$h^{(2)}$...	$h^{(a)}$...	$h^{()}$
χ^1	f^1	1...	1...	1
χ^2	f_2	$\chi_2^{(2)}$...	$\chi_2^{(a)}$...	$\chi_2^{(k)}$
:	:	:	:	:
:	:	:	:	:
χ^1	f_i	$\chi_i^{(2)}$	$\chi_k^{(a)}$	$\chi_k^{(k}$
:	:	:	:	:
:	:	:	:	:
χ^k	f_k	$\chi_k^{(2)}$...	$\chi_k^{(a)}$...	$\chi_k^{(k}$

where x_1 are irreducible and a_i ∈Z. Generalized characters are class functions that are constant on any conjugacy class.

Definition 2: Characters of degree one are called linear characters by Feit (1971). In particular, the function with constant value 1 on G is a linear F- character. It is called the principal F- character.

Definition 3: Let χbe an F- representation of G. Then the F- character χ of g afforded by x is a function given by $\chi(g) = trx(g)$.

Lemma 4

(a) Similar F- representation of G afforded equal characters.
(b) Characters are constant on the conjugacy classes of a group.

Proof

(a)If P is non singular then
$tr(P^{-1}AP) = tr(PP^{-1}A) = tr(A)$. Both (a) and (b) follow from this observations.
(b)Observe that $\chi(h^{-1}gh) = \chi(h)^{-1}\chi(g)\chi(h)$ if χ is a representation of G and hence
$tr(\chi(h^{-1}gh)) = tr(\chi(g))$. Hence the result.

Corollary 5

The group G is abelian if and only if every irreducible character is linear.

Proof

Let k be the number of classes of G. Then
K = |G| if and only if G is abelian.
Now

	S^3		
	C_1	C_2	C_3
$h^{(a)}$	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$|G| = \sum_{i=1}^{k} \chi_i(1)^2 \text{ and } \chi_i(1) \geq 1$$

and for all i it follows that K = |G| if and only if χ_1 (1) (1) = 1 for all i. Hence the result.

NOTE
For a particular group G, the irreducible characters are usually presented in a character table. A square array of complex numbers whose rows correspond to the χ_1 and whose columns correspond to the classes C^1.

Example 6: An example of a character table is the accompanying one for the symmetric groups S^3 on three symbols. It has three conjugacy classes namely;

C^1 = {1}
C^2 = {(12), (13), (23)}
C^3 = {(123), (132)}

Therefore, S^3has threeirreducible representations. The character Table 2 of $S^3$is given below

This method for computing the coefficient is by using orthogonality relations which we are about to derive. These relations are useful in the construction of character tables.

The key to the orthogonality relations is to compute explicitly the coefficients of the group elements in the C^1's in terms of the characters.

Definition 7:
Let ϕ and ψbe class functions on a group G. Then (Dornhoff, 1971)

$$[\phi, \psi] = \frac{1}{|G|} \sum_{g \varepsilon G} \phi(g)\psi(g)$$

is the inner product of $\phi$and ψ.

Properties of inner product 8

$$[\phi, \psi] = \overline{[\phi, \psi]}$$
$$[\phi, \phi] > 0 \text{Unless} \phi \neq 0$$
$$[C_1\phi_1 + C_2\phi, \psi] = C_1[\phi_1, \psi] + C_2[\phi, \psi]$$
$$[\phi, C_1\psi_1 + C_2\psi_2] = \overline{C}_1[\phi, \psi_1] + \overline{C}_2[\phi, \psi_2]$$

Remark 9
The [ ]has all the properties usually used to define an inner product in linear algebra and analysis. We know that irr(g) is a basis for the space of class functions and it is the content of the orthogonality relation which is in fact an orthonormal basis that is
$[\chi_i, \chi_j] = \delta_{ij}$ This yields the promised method for expressing an arbitrary class functions in terms of the irreducible characters, for if,

$$[\phi, \chi_i] = C_i. \text{Then}$$
$$\phi = \sum C_i \chi_i$$

Definition 10 (Alternative form of the inner product)

If X lies in C^a, we let

$$\phi(x) = \phi(a)$$
$$\psi(x) = \psi(a)$$
$$\psi(x^{-1}) = \overline{\psi}(a)$$

Then the inner product of $\phi$and $\Psi$is

$$[\phi, \psi] = \frac{1}{g} \sum_{x \varepsilon g} \phi(x)\psi(x^{-1}) = \frac{1}{g} \sum_{a=1}^{k} h^{(a)}\phi(a)\psi(a) \quad (2)$$

Note
If $\chi_1$and $\chi_2$are simple characters, the character relation of the first kind by (Ledermann, 1976) states that

$$\frac{1}{g} \sum_{a=1}^{k} h^{(a)} \chi_1^{(a)} \chi_2^{(a)} = \begin{cases} 1, \chi_1 = \chi_2 \\ 0, \chi_1 \neq \chi_2 \end{cases}$$

Applying the result to all the rows of the character Table 2 we have that

$$\frac{1}{g} \sum_{a=1}^{k} h^{(a)} \chi_1^{(a)} \chi_2^{(a)} = \delta_{ij} \quad (3)$$

Where I, j = 1,2,…,K. We can express this information in a more compact form by introducing the kxk matrix,

$$(u_{ia}) = \chi_i^{(a)} \sqrt{\frac{h^{(a)}}{g}} \quad (4)$$

Then Eq. (4) states that the rows of U form a unitary matrix that is

$$U\overline{U}^{1} = I \quad (5)$$

Where t denotes transposition. U and $U^t$are inverse to each other and therefore commute. Hence

$$U\overline{U}^{-1} = I$$

And on taking the conjugate complex, we have
This means that the columns of U also form a unitary orthogonal system, that is

$$U\overline{U}^{-1} = I$$

Or in terms of characters,

$$\sum_{i=1}^{k} U_{ia}\,\overline{U}_{ib} = \frac{g}{h^{(a)}}\delta_{ab} \qquad (6)$$

$$\sum_{i=1}^{k} \chi_i^{(a)}\overline{\chi}_i^{(b)} = \frac{g}{h^{(a)}}\delta_{ab}$$

Equation (6) is called orthogonality relation of the second kind.

MAIN RESULT

HEOREM 11 (ORTHOGONALITY RELATIONS)
(i) If ij then

$$\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = 0$$

Alternatively,

$$\sum_{k=1}^{s} h_k\chi_i(g)\chi_j(\chi_k^{-1}) = |G|, k\in G$$

(ii) $\sum_{g\in G}\chi_i(g)\chi_i(g^{-1}) = |G|$.

Alternatively

$$\sum_{k=1}^{s} h_k\chi_i(\chi)\chi_j(\chi_k^{-1}) = |G|$$

(iii) $\sum_{k=1}^{s}\chi_k(\chi_i)\chi_k(\chi_j^{-1}) = 0$

if $i \neq j$

(iv) $\sum_{k=1}^{s}\chi_k(\chi_i)\chi_k(\chi_i^{-1}) = \frac{|G|}{h_i} = |c(x_i)|$

(v)The $\chi_i$are linearly independent over C as functions from the set of conjugacy class of C.

(vi) $\sum_{k=1}^{s}\chi_k^{2}(1) = |G|$

Proof
We have

$$\chi_i(g) = tr(\psi_i(g)) = tr(a_{ij}(g)) = \sum_m a_{mm}(g)$$

and similarly,

$$\chi_j(g) = \sum_n b_m(g).$$

$$\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = \sum_g\left(\sum_m a_{mm}(g)\right)\left(\sum_n b_{nn}(g^{-1})\right)$$
$$= \sum_{m,n}\sum_g a_{mm}(g)b_{nn}(g)^{-1} = 0$$

If $i\neq j$ (for $V_i \cong V_j$). Hence (i)

$$\sum_m a_{mm}(g)\sum_g b_{nn}(g^{-1}) = 0$$

(ii). Unless m = n.The case where i=j and a = b we get.

$$\sum_m\sum_g a_{mm}(g)a_{mm}(g^{-1}) = \sum_m \delta_{mm}\delta_{mm}\frac{|G|}{\dim V_i} =$$
$$\sum_m \frac{|G|}{\dim V_i} = \dim V_i\frac{|G|}{\dim V_i} = |G|$$

Hence (ii).
Let Y be the matrix with (i, j) $\chi_j(x_i^{-1})$entry. and χ are the matrix with (i, j) entry $h_i\chi_i$ (x_j). But
X and Y are square matrix. So XY = I |G|YX=I Hence (iii) and (iv)
(v) Is a consequence of (i) and (ii).
Suppose χ_i were linear combination, say,

$$\chi_i = a_2\chi_2 + a_3\chi_3 + \ldots + a_n\chi_n \text{ then,}$$
$$\sum h_m\chi_i(x_m)\chi_j(x_m^{-1}) = 0 \text{ for all}$$
$$\sum h_m\chi_i(x_m)\left(\sum_{j=2}^{s} a_j\chi_j(x_m^{-1})\right) = 0 \Rightarrow$$
$$\sum h_m\chi_i(x_m)\chi_j(x_m^{-1}) = 0$$

But the left hand side is equal to |G| by (ii) and $|G| \neq 0$. Thus a contradiction. (vi) Is a consequence of (iv) with

$$\chi_i = 1 = \chi_i^{-1} \text{ but } \chi_i(1) = \dim V_i = n_i \text{ andwehave}$$
$$h_1^{2} + h_2^{2} + \ldots + h_n^{2} = |G|$$

Example 12 (Construction of character tables)

(i)Using the above theorem to construct the character table for symmetric group of degree 3.

	S^3		
	C_1	C_2	C_3
$h^{(a)}$	1	3	2
χ_1	1	1	1
χ_2	a	c	d
χ_3	b	e	f

The symmetric group is the group of all permutations of the. It has six elements and three conjugacy classes namely,

$C_1 = \{1\}$
$C_2 = \{(12), (13), (23)\}$
$C_3 = \{(123), (132)\}$

Therefore the symmetric group of degree three has 3 irreducible representations.

Let the values of χ_2 for the classes C_1, C_2 and C_3 be a, c and d respectively and the values for χ_3 of the classes C_1, C_2 and C_3 be b, e and f respectively.
Thus the character Table 3 or the symmertwric group of degree three is
Now to find a, b, c, d, e and f. However,
$1^2 + a^2 + b^2 = 6 \Rightarrow a^2 + b^2 = 5$
That is $1^2 + 2^2 = 5 \Rightarrow a = 1, b = 2$.

By using the orthogonality relations of the second kind and taking the inner products to find the remaining variables.
Now if a = 1, b =2, we have

$$\chi_1^{(1)} \cdot \overline{\chi}_1^{(2)} + \chi_2^{(1)} \cdot \overline{\chi}_2^{2} + \chi_3^{(1)} \cdot \overline{\chi}_3^{(2)} = 0$$
$$1.1 + 1.c + 2.e = 0 \tag{7}$$
$$1 + c + 2e$$

If a = 2, b = 2, we have

$$\chi_1^{(2)} \cdot \overline{\chi}_1^{(2)} + \chi_2^{(2)} \cdot \overline{\chi}_2^{2} + \chi_3^{(2)} \cdot \overline{\chi}_3^{(2)} = \frac{6}{3}$$
$$1.1 + c.c + e.e = 2$$
$$c^2 + e^2 = 1$$

But c = -1-2e so (8) becomes

$$(-1-2e)^2 + e^2 = 1$$
$$1 + 4e + 5e^2 = 1$$
$$4e + 5e^2 = 0$$

e = 0 or -4/5 put e = 0 in Eq. 7 we have c = -1. And if a = 1, b = 3 we have,

$$\chi_1^{(1)} \cdot \overline{\chi}_1^{(3)} + \chi_2^{(1)} \overline{\chi}_2^{(3)} + \chi_3^{(1)} \cdot \overline{\chi}_3^{(3)} = 0$$
$$1.1 + 1.d + 2.f = 0, \tag{9}$$
$$d = -1-2f \text{ if}$$

a = 3, b = 3, we have

$$\chi_1^{(3)} \cdot \overline{\chi}_1^{(3)} . + \chi_2^{(3)} \cdot \overline{\chi}_2^{(3)} . + \chi_3^{(3)} \cdot \overline{\chi}_3^{(3)} = \frac{6}{2}$$
$$1.1 + d^2 + f^2 = 3 \Rightarrow d^2 + f^2 = 3$$
$$\text{But } d = -1 - 2f$$
$$(-1-2f)^2 + f^2 = 2$$
$$5f^2 + 4f - 1 = 0$$
$$f = -1 \text{ or } \frac{1}{5}$$

Put f = -1 in Eq. (9) we have d = 1.

Therefore the required character Table 4 for the sysmmetric group of degree threeis

	S^3		
	C_1	C_2	C_3
$h^{(a)}$	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

THEOREM 13

Let N be a normal subgroup of G and suppose that $A_o(N_x)$ is a representation of degree m of the group, G/N then by (Curtis and Reiner (1962) we have the following
$A(x) = A_o(N_x)$, $x \in G$, defines a representation of G "lifted from G/N".
If $\phi_o(N_x)$is the character of, $A_o(N_x)$ we have that
$\phi(x) = \phi_o(N_x)$, $x \in G$, is the lifted character of A(x).
The lifting operation preserves irreducibility.

Proof

Suppose that $A_o(N_x)$, is a representation of G/N having degree m and that
$A(x) = A_o(N_x)$, $x \in G$,
$A(x)A(y) = A_o(N_x)A_o(N_y), = A_o(N_x N_y), = A(N_{xy}) = A(xy)$, which implies that A(x) is a Representation.
From $A(x) = A_o(N_x)$ it follows trivially that
$\phi(x) = \phi_o(N_x)$, .

Since A(x) consisting of the same matrices of, $A_o(N_x)$, it is clear that A(x) is irreducible if and only if $A_o(N_x)$, is irreducible.

An example to illustrate the use of this lifting process in the construction of character table.
Example 14 (character table for Alternating group of degree four)

	Z_3		
χ_0	1	Z	Z^2
χ_1	1	1	1
χ_2	1	W	W^2
χ_3	1	W^2	W

	S^3			
	C_1	C_2	C_3	C_4
$h^{(a)}$	1	4	4	3
χ_1	1	1	1	1
χ_2	1	W	W^2	
χ_3	1	W^2	W	1
χ_4	3	A	B	D

	A_4			
	C_1	C_2	C_3	C_4
$h^{(a)}$	1	4	4	3
χ_1	1	1	1	1
χ_2	1	W	W^2	1
χ_3	1	W^2	W	1
χ_4	3	0	0	-1

The alternating group of degree four () is of order 12 and consists of the following elements
1, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23).
We first split A_4into conjugacy classes, namely,

C_1 = {1}
C_2 = {(123), (142), (134), (234)}
C_3 = {(132), (124), (143), (234)}
C_4 = {(12)(34), (13)(24), (14)(23)}

We know that by Hill (1976) posses A_4 four irreducible representations. It so happen that U = C_1u $C_4$is a normal subgroup of A_4. The quotient group A_4/V is of order three and therefore isomorphic to Z_3 whose character Table 5 is
Where $W = e^{2\Pi / \frac{i}{3}}$

According to Theorem 13 we can list the three characters of Z_3 all of which are linear to obtain three characters χ_1, χ_2 and χ_3 of χ_4. The kernel being V. It remains to find χ_4.
Since, $(\chi_1)^2 + (\chi_2)^2 + (\chi_3)^2 + (\chi_4)^2 = 12$ that is $\chi_1 = \chi_2 = \chi_3 = 1$ which implies that $\chi_4 = 3$.
Let the values of χ_4 for the classes, C_1, C_2 and C_3 be a, b, c respectively.Thus the character Table 6 forA_4 is,

The values of a, b and d are derived from orthogonality of the columns (character relations of the second kind). Thus it is found that a = b = 0 because $1 + W + W^2 = 0$ and 3d + 3 = 0.Hence, d = -1 and the complete character Table 7 for is given.

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