Research Journal of Applied Sciences 2 (4): 372-376, 2007

The Representation of Real Characters of Finite Groups

H.S. Ndakwo
Department of Mathematical Sciences, Nasarawa State University, Keffi, Nasarawa State, Nigeria

Abstract: Let A(x) be the representation of an element x in a group G. The representation A(x) may be real or complex. The aim of this study is to distinguish when the character of A(x) is real and when it is not. This distinction is linked with the notion of bilinear invariants and to find out the situation in which if A(x) is complex for some x whether it is equivalent as a representation to Q(x) such that Q(x) has a real coefficients for all xG. This notion is equivalent to finding an invertible matrix T such that Q(x) = TA(x) T and Q(x) is real. It was also proved in this study that for any complex irreducible orthogonal representation of a finite group G, the representation Q(x) for every xG is equivalent to a real orthogonal representation.

Key words: Representation, irreducible, bilinear invariants, invertible matrix, orthogonal representation

INTRODUCTION

Let A be a real or complex matrix with its transpose denoted by A and its complex conjugate as. For a row vector x = (denote its quadratic form as q = xAx for the case when A is symmetric. We observed that for a quadratic for (Ledermann, 1977) there exists an invertible matrix K such that

$$A = KK^1 \quad (1)$$

For A(x) which is a representation of xG, its contragradient representation is given as

$$A(x) = (A(x^{-1}))^1$$

Its character which is the character of A(x) is equivalent to the conjugate of the character of A(x).

Definition 1: Let A(x) be a representation of a group G. The invertible matrix T is said to be a bilinear invariant of A(x) if

$$A(x)\, TA^1 (x) = T \quad (2)$$

Lemma 2: A real representation of a finite group possesses a bilinear invariant if and only if its character is real.

Proof: If $\varkappa(x)$ is the character of A(x), xG. Then the contragradient representation

$$A^{+}(x) = (A(x^{-1}))^1, x \in G. \quad (3)$$

Has character $\varkappa (x^{-1})$, which is equal to $\bar{\chi}$ (x). Hence $\varkappa$ is real if and only if the representation (3) is equivalent. This implies that there exists an invertible matrix T such that

$$T^1A(x)\, T = (A(x^{-1}))^1 = (A(x))^1 \quad (4)$$

Equation (4) can be written as

$$A(x)\, TA^1 (x) = T$$

which means that T is a bilinear invariant of A by Issacs (1976).

Lemma 3: Suppose that A is a real or complex irreducible representation of a finite group G, with character $\varkappa$. Then if α_1 and α_2 are bilinear invariants of A, then

$$\alpha_2 = k\alpha_1,$$

Where k is a non-zero number which may be complex or real.

Proof: Suppose that α_1 and α_2 are bilinear invariants of A. Then by (4)

$$\alpha_1^1 A(x)\alpha_1 = \alpha_2^{-1} A(x)\alpha_2 = A^{+}(x), x \in G$$

Hence,

$$A(x)(\alpha_1\alpha_2^{\,1}) = (\alpha_1\alpha_2^{-1})A(x)$$

By the corollary to Schur's Lemma.
$\alpha_1\alpha_2^{\,1}$ = kl where k is a scalar, which is clearly non-zero.

Lemma 4: Let A be a real or complex irreducible representation of a finite group G, with character $\varkappa$. Then if α is a bilinear invariant of A, we have that either α is symmetric or it is skew- symmetric.

Proof: By hypothesis

$$A(x)\alpha A^1 (x) = \alpha$$

Transposing this equation we have that

$$A(x) \alpha A^1 (x) = \alpha^1$$

Thus if α is a bilinear invariant of A. So is α^1. It follows from Lemma 3 that

$$\alpha^1 = r\alpha,$$

where r is a number. And by transposing

$$\alpha = r \alpha^1.$$

On eliminating α^1 between these equations we find that

$$\alpha = r^2 \alpha$$

Therefore $r^2 = 1$ so that either r = 1 or r = -1. This implies that either $\alpha = \alpha^1$ (symmetric) or $\alpha = -\alpha^1$ (skew symmetric).

Theorem 5: Let $q = xAx^1$ be a non-zero quadratic form, where A is a symmetric matrix which may be real or complex. Then there exists an integer r satisfying $1 \le r \le n$ and an invertible matrix K such that

$$q = Z^2{}_1 + Z^2{}_2 + \ldots Z_{r^2}, \text{ where } x = zK$$

when r =n, we have that

$$A = KK^1 \tag{5}$$

The real quadratic form q = xAx and the real symmetric matrix A are said to be positive definite if for every non-zero vector u, we have that uAu>0. We then have,

Lemma 6 (Ledermann, 1977): Let A be a positive definite real matrix. Then, there exists a real invertible matrix M such that

$$A = MM^1$$

We recall that a square matrix N is said to be orthogonal if

$$NN^1 = N^1N = I$$

where I is the identity matrix.

A complex representation J(x) of G is called orthogonal if for each x∈G, we have that

$$J(x) J^1 (x) = J^1 (x)J(x) = I \tag{6}$$

An nxn matrix H is called Hermitian if

$$\overline{H^1} = H \tag{7}$$

Now if H is a Hermitian matrix and if u is a complex row vector, then

$$h(u) = uHu^1$$

is a real number and H is said to be positive definite if h(u)>0 where $u \neq 0$. We also recall that every positive definite Hermitian matrix is invertible and if H is positive definite so are $\overline{H}, \overline{H}$ and H^1. The Hermitian matrix K is said to be a hermitian invariant of the representation A(x) for x∈G if K is positive definite and

$$A(x)K A^{-1}\mathbf{A} (x) = K, \text{ for } x \in G$$

Theorem 7: Every representation A(x) of a finite group G possesses Hermitian invariants.

Proof: For A(x), let

$$K = \sum_{b \in G} A(b)\bar{A}(b) \tag{8}$$

And it is easily verified that K is a Hermitian invariant of A(x). We note that if K is a Hermitian invariant of A(x), so is

$$H = \beta K$$

whereβ is an arbitrary constant.

We state without proof the Schur's Lemma and its corollary which states that if A(x) and B(x) are two irreducible representations over a field f of a group G and that there exists a constant matrix T over f such that

$$TA(x) = B(x)T$$

for x∈G, then either T = 0 or T is non- singular so that $A(x) = T^1B(x)T$ then T= 0I, where I is the identity matrix.

RESULTS

Theorem 8: Let A be an irreducible real or complex representation of a finite group G with ϰ as its character. Then ϰ is real and that A is equivalent to a real representation if and only if it has a real or complex symmetric bilinear invariant.

Proof: Suppose that ϰ is real and that a is equivalent to a real representation. Then there exists an invertible matrix T such that

$$T^1 A(x)\, T = B(x),\ x \in G \qquad (9)$$
$$Q = \sum_{y \in G} B(y) B^1(y)$$

Where B(x) is real. Let

Clearly Q is a real symmetric matrix which is positive definite and therefore invertible. As in Ledermann(1977) it can be shown that

$$B(x) Q B^1(x) = Q,\ x \epsilon G$$

Substituting for B(x) from (9) we obtain that,

$$A(x) \subset A^1(x) = C, x \in G \qquad (10)$$

Where C = TQ T^1

Evidently C is an invertible real or complex symmetric. Thus (10) establishes the fact that A has a symmetric bilinear invariant.

Conversely, suppose that (10) holds, where C is an invertible symmetric matrix. By (1), there exists an invertible matrix D such that

$$C = DD^1$$

We can therefore rewrite (10) as

$$(D^{-1} A(x) D)(D^{-1} A(x) D^{-1} = I$$

Thus the representation

$$E(x) = D^{-1} A(x) D$$

Which is equivalent to A(x), is a real or complex orthogonal representation. Thus A(x) is equivalent to a real representation and its character ϰ is also real.

Theorem 9: Let A be an irreducible real or complex representation of a finite group G with character ϰ. Then ϰ is real and A is not equivalent to a real representation if and only if it has a real or complex skew-symmetric bilinear invariant.

Proof: Suppose that A(x) is not equivalent to a real representation but its character is real. Then A(x) is equivalent to $\bar{A}$ (x), so the character of A(x) is real. By Lemma 2, A(x) has a bilinear invariant which must be either symmetric or skew-symmetric. But it cannot be symmetric, because this would imply that A(x) is of Theorem 8. Hence A(x) has a skew-symmetric invariant.

Conversely, suppose that A(x) has a skew-symmetric invariant. Then it is not like Theorem 8, because it cannot also have a symmetric invariant as in Lemma 4. Since A has a bilinear invariant, its character is real. Therefore A is equivalent to $\bar{A}$. Thus the Theorem.

Theorem 10: Let A be an irreducible real or complex representation of a finite group G with character ϰ. Then ϰ is complex, A and $\bar{A}$ are inequivalent and neither is equivalent to a real representation if and only if it has no bilinear invariant.

Proof: Let ϰ be the character of A. Then both $\bar{A}$ and A^+ by Eq. (3) we have the character $\bar{\chi}$ and are therefore equivalent. Now the hypothesis of the theorem holds if and only if A is equivalent to A^+, that is A is not equivalent to $\bar{A}$. Hence, A and $\bar{A}$ are inequivalent irreducible representations. By Schur's Lemma the only solution of

$$A(x) T = T A^{-1}(x),\ x \epsilon G$$

Is T = 0. Hence the Theorem.

Theorem 11: Let A(x) be a complex irreducible orthogonal representation of a finite group G, then by (Morris, 1968) A(x) is equivalent to a real orthogonal representation.
The proof will be given in steps as follows:

Step 1: Let

$$C = \sum_{b \in G} B(b) \overline{B^1}(b)$$

And

$$D = B^{\frac{-1}{2}} C$$

Then β is real positive number and D is both Hermitian and orthogonal. That is from Eq. (7), C is Hermitian invariant and we have that

$$B(x) C \overline{B^1}(x) = C \qquad (11)$$

On taking the complex conjugate of (6), we obtain that

$$\bar{B}(x)\overline{B^1}(x)=\overline{B^1}(x)\bar{B}(x)=1 \quad (12)$$

Hence Eq. 11 can be written as

$$B(x)\ C = C\ \bar{B}(x) \quad (13)$$

Transposing this equation we have,

$$C^1B^1(x)=\overline{B^1}(x)C^1$$

Or equivalently, by (6) and (12)

$$\bar{B}(x)C=CB(x) \quad (14)$$

Using (13) and (14) we find that

$$B(x)(CC^1 = (B(x)(C)C^1 = C(\overline{B^1}(x)C^1)=(CC^1)B(x)$$

By corollary to Schur's Lemma it follows that

$$CC^1 = \beta_1 \quad (15)$$

Where β is a number. We need to show that β is real and positive. From (15) we have that

$$CC^1C = \beta C$$

Hence if u is an arbitrary non-zero vector, we obtain that

$$(uC)C^1(\overline{Cu^1})=\beta(u\overline{Cu^1}) \quad (16)$$

Now let v = uC
Then ((16) becomes

$$(vC^1\bar{v}=\beta(u\overline{Cu^1})$$

And it is now obvious that β >0, because both C and C^1 are positive definite and v≠0. From H =β K, we may replace C by the Hermitian invariant D given by

$$D = B^{\frac{-1}{2}}C \quad (17)$$

We have that

$$\overline{D^1} \text{ and } u\overline{Du^1} > 0\ u \neq 0.$$

$$B(x)D\overline{B^1}(x)=D \text{ for } x \in G$$

And by substituting (17) into (15), we have that

$$D^1D = I = DD^1$$

And we have that D is both Hermitian and orthogonal.

Step 2: Let E(x) be define by

$$E(x)=(I+D^1)B(x)=(I+D)^{-1}$$

Where D, I and B are as above, then E(x) is real. Since D is positive definite, so is $I+D^1$. Hence $I+D^1$ is invertible and thus E(x) as defined above is equivalent to B(x). Taking conjugates of E(x) and noting that $D^1 = D$ we find that

$$\bar{E}(x)=(I+D)\bar{B}(x)=(I+D)^{-1}$$

and

$$\bar{B}(x)=D^1B(x)D=D^1B(x)(D^1)^{-1}$$

Substituting, we have

$$\bar{E}(x)=(I+D)D^1B(x)\{(I+D)D^1\}^{-1}=(D^1+I)B(x)(D^1+I)^{-1}=E(x)$$

Since $D^1D = I = DD^1$ and E(x) is real.

So far, we establishes that E(x) is real and equivalent to B(x). But E(x) is not orthogonal. However this last step establishes that it is in fact equivalent to a real representation.

Step 3: sLet E(x) be as above. Then there exists a representation P(x) which is equivalent to B(x).We want to show that there exists an invertible matrix T such that

$$P(x) = T^1\ E(x)\ T$$

Let

$$Q = (I+D^1)(I+D) \quad (18)$$

Then

$$E(x)QE^1(x) = Q \quad (19)$$

Now using $\bar{D} = D$ and DD = I = DD, we have that

$$Q = 2I+D+D = 2I+D+\bar{D}$$

This shows that Q is real, symmetric and positive definite because I, D and $\bar{D}$ are positive definite.

Hence by Lemma 6 there exists and invertible matrix T such that

$$Q = TT^1$$

and

$$E(x)QE^1(x) = Q$$

becomes

$$\{T^1 E(x)T\}\{T^1E(x)T\}^1 = 1$$

Thus from P(x) = T E(x) T,

The representation E(x) is equivalent to B(x). So E(x) and B(x) are both equivalent to a real and orthogonal representation. This completes the proof of Theorem 11.

REFERENCES

Dornhoff, L., 1971. Group Representation Theory. Marcel Dekker.

Issacs, L.M., 1976. Character Theory of finite Groups. Academic Press, New York.

Ledermann, W., 1977. Introduction to Group Characters. Cambridge University Press.

Morris, N., 1968. Matrix Representation of Groups. National Bureau of Standards. Applied Mathematics Series, pp: 60.

Puttaswamaiah, B.M., 1977. Modular Representation of Groups. Academic Press. New York.