Research Journal of Applied Sciences 2 (7): 794-797, 2007

Continuous Hybrid Methods for Direct Solution of General Second Order Differential Equations

S.J. Kayode
Department of Mathematical Sciences, School of Sciences,
Federal University of Technology, Akure, Ondo State, Nigeria

Abstract: In this study, a two-step zero stable hybrid Numerov type method is developed for direct solution of general ordinary differential equations of the form

$$a_0(t)\frac{d^2y}{dt^2}+a_1(t)\frac{dy}{dt}+a_2(t)y=R(t)$$

The method based on predictor-corrector approach, is consistent and zero-stable. Efforts are made to develop a predictor having the same order as that of the corrector, thereby reducing the consequential effects of the predictor on the accuracy of the method. The accuracy of the method is tested with linear and non-linear problems.

Key words: Predictor-corrector, collocation, hybrid, symmetric, continuous method, stepsize, stepnumber, zero stable

INTRODUCTION

Methods of solving ordinary differential equations (odes) of higher order of the type

$$y^{(n)}=f(t,y,y^1,y^{11},...,y^{(n-1)}),\quad y^{(s)}(t_0)=y_0^{(s)}, s=0,1,2,...,n-1 \qquad (1)$$

have been considered by various mathematicians, both theoretically and numerically. Most of the methods for Eq. 1 are essentially to reduce it to system of first order equations of the form

$$y'=f(t,y),\ y(t_0)=y_0, f\in C^1[a,b], y,t\in R^n \qquad (2)$$

an approach, according to Awoyemi and Kayode (2002) considered to be uneconomical as a result of computational burden, human and computer time wastage. However, there are numerical methods developed to handle Eq. 1 directly. Awoyemi (1995, 1996) considered the solution of (1) using canonical polynomial and perturbation terms which would be determined according as the type of problem to be solved. The use of canonical polynomial is restricted to the type of order of the differential equation under consideration and the introduction of perturbation terms also makes the research more tedious.

In this research, a power series of order 2 (k+1), where k is an integer, is used as a basis function without any perturbation terms, in order to eliminate the problem-dependent nature of the canonical polynomial and the associated problems of perturbation terms. The method yields a two-step continuous hybrid algorithm for a direct solution of the second order initial value problems.

MATERIALS AND METHODS

In this research, the approximate solution of problem (1), with m = 2, is taking to be a partial sum of a Power series with single variable x in the form

$$y(x)=\sum_{j=0}^{2(k-1)}a_jx^j \qquad (3)$$

a_j is real, y is m-times differentiable in the given interval I = [a, b].

Differentiating Eq. 3 twice to the first and second derivatives, respectively as

$$y'(x)=\sum_{j=1}^{2(k-1)}ja_jx^{j-1} \qquad (4)$$

$$y''(x)=\sum_{j=2}^{2(k-1)}j(j-1)a_jx^{j-2} \qquad (5)$$

From (1) and (5), we have

$$\sum_{j=2}^{2(k-1)}j(j-1)a_jx^{j-2}=f(x,y(x),y'(x),...,y^{m-1}) \qquad (6)$$

Equation 3 and 6 are interpolated and collocated at the points x_{n+i}, i = 0, 1, 2,..., k-1 and

x_{n+i}, $i = 0, 1, 2, ..., r, s,$, respectively to have a system of equations

$$\left.\begin{aligned} &\sum_{j=0}^{2(k-1)} a_j x_{n+i}^j = y_{n+i},\ i = 0,1,2,...,k-1 \\ &\sum_{j=2}^{2(k-1)} j(j-1)a_j x_{n+i}^{j-2} = f_{n+i},\ i = 0,1,2,...,k \\ &\sum_{j=2}^{2(k-1)} j(j-1)a_j x_{n+i}^{j-2} = f_{n+v},\ \ r \in (1,2) \\ &\sum_{j=2}^{2(k-1)} j(j-1)a_j x_{n+i}^{j-2} = f_{n+v},\ \ s \in (k-1,k) \end{aligned}\right\} \quad (7)$$

Where $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, ..., y^{m-1})$ is the numerical approximation to $y(x_{n+i})$ at x_{n+i}, and $x_{n+i} = x_n + ih$, h is the stepsize.

Solving the Eq. 7 for the values of a_j's, j = 0,1,... and substituting these values in Eq. 3 produces the continuous hybrid method

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=2}^{k} \beta_j(x) f_{n+j} + \tau_1(x) f_{n+r} + \tau_2(x) f_{n+s} \quad (8)$$

By using the transformations in Kayode (2004)

$$t = \frac{1}{h}(x - x_{n+k-1}), \frac{dt}{dx} = \frac{1}{h}, \quad t \in (0,1] \quad (9)$$

and taking the stepnumber k to be 2 in the continuous method (8), the coefficients $\alpha_j, \beta_j, \tau_{,1}, \tau_2$ are obtained as functions of t to be

$$\alpha_1(t) = 1 + t$$

$$\alpha_0(t) = -t$$

$$\beta_2 = \frac{h^2}{120(2-r)(2-s)}\{(3s + 3r - 5rs - 2)t + 10(1 - r - s + rs)t^3 + 5(rs - 2s - 2r - 1)t^4 + 3(3 - r - s)t^5 + 2t^6\}$$

$$\beta_1 = \frac{h^2}{60(s-1)(1-r)}\{(18r + 18s - 25rs - 14)t - 30(1 - r - s + rs)t^2 + 10(2 - r - s)t^3 - 5(4 + r + s - rs)t^4 - 3(r + s - 2)t^5 + 2t^6\}$$

$$\beta_0 = \frac{h^2}{120rs}\{(4 - 7r - 7s + 15rs)t + 10(s - rs + r - 1)t^3 + 5(rs - 5)t^4 + 3(1 - r - s)t^5 + 2t^6\}$$

$$\tau_1 = \frac{h^2}{60r(2-r)(1-r)(s-r)}\{(7s - 4)t + 10(1 - s)t^3 + 25t^4 + 3(s - 1)t^5 - 2t^6\}$$

$$\tau_2 = \frac{h^2}{60s(2-s)(s-1)(s-r)}\{(7r - 4)t + 10(1 - r)t^3 + 25t^4 + 3(r - 1)t^5 - 2t^6\} \quad (10)$$

The first derivatives of $\alpha_j, \beta_j, \tau_{,1}, \tau_2$ in (10) yield

$$\alpha'_2 = \frac{1}{h}$$

$$\alpha'_1 = -\frac{1}{h}$$

$$\beta'_2 = \frac{h}{120(2-r)(2-s)}\{(3s + 3r - 5rs - 2) + 30(1 - r - s + rs)t^2 + 20(rs - 2s - 2r - 1)t^3 + 15(3 - r - s)t^4 + 12t^5\}$$

$$\beta'_1 = \frac{h}{60(s-1)(1-r)}\{(18r + 18s - 25rs - 14) - 60(1 - r - s + rs)t + 30(2 - r - s)t^2 - 20(4 + r + s - rs)t^3 - 15(r + s - 2)t^4 + 12t^5\}$$

$$\beta'_0 = \frac{h}{120rs}\{(4 - 7r - 7s + 15rs) + 30(s - rs + r - 1)t^2 + 20(rs - 5)t^3 + 15(1 - r - s)t^4 + 12t^5\}$$

$$\tau'_1 = \frac{h}{60r(2-r)(1-r)(s-r)}\{(7s - 4) + 30(1 - s)t^2 + 100t^3 + 15(s - 1)t^4 - 12t^5\}$$

$$\tau'_2 = \frac{h^2}{60s(2-s)(s-1)(s-r)}\{(7r - 4) + 30(1 - r)t^2 + 100t^3 + 15(r - 1)t^4 - 12t^5\} \quad (11)$$

An infinite number of discrete schemes could be obtained from (8) by taking different values of t in the interval I = (0, 1]. In this study , a sample discrete scheme is considered from the continuous method (8) taken t = 1, which from (9) implies that $x = x_{n+2}$, and this gives

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{D_1}\begin{pmatrix} P_1 f_{n+2} + Q_1 f_{n+s} + R_1 f_{n+1} \\ + S_1 f_{n+r} + T_1 f_n \end{pmatrix} \quad (12)$$

Where

$$D_1 = 60rs(1-r)(1-s)(2-r)(2-s)(r-s)$$

$$P_1 = rs(1-r)(1-s)(r-s)(7 + 5rs - 10r - 10s)$$

$$Q_1 = 2\ 6r(1-r)(2-r)$$

$$R_1 = -rs(2-r)(2-s)(r-s)(50r + 50s - 50rs - 76)$$

$$S_1 = -\ 26s(1-s)(2-s)$$

$$T_1 = (1-r)(1-s)(2-r)(2-s)(r-s)(5rs - 13)$$

and from (11)

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{\bar{D}_1}\left\{\begin{matrix} \bar{P}_1 f_{n+2} + \bar{Q}_1 f_{n+s} + \bar{R}_1 f_{n+1} \\ + \bar{S}_1 f_{n+r} + \bar{T}_1 f_n \end{matrix}\right\} \quad (13)$$

Where

$$\bar{D}_1 = 120rs(1-r)(1-s)(2-r)(2-s)(r-s)$$

$$\bar{P}_1 = rs(1-r)(1-s)(r-s)(65-82r-82s+45rs)$$

$$\bar{Q}_1 = 2r(1-r)(2-r)(99-8r)$$

$$\bar{R}_1 = -2rs(2-r)(2-s)(r-s)(73r+73s-65rs-172)$$

$$\bar{S}_1 = -2s(1-s)(2-s)(99-8r)$$

$$\bar{T}_1 = (1-r)(1-s)(2-r)(2-s)(r-s)(8r+8s+5rs-99)$$

THE PREDICTORS

A serious disadvantage of the predictor-corrector method is the problem of having the order of the predictor lower than that of the corrector, which eventually lowers the accuracy of the main method (the corrector). However, in this study an attempt has been made to obtain a predictor that has the same order of accuracy with the main method with a lower error constant and a wider interval of absolute stability than the corrector. The explicit predictor for the method is derived using the same power series (3) as an approximate solution of problem (1). Collocation is done at points

$$x_{n+\frac{3}{2}}, x_{n+1}, x_{n+\frac{1}{2}}$$

and x_n while interpolation is done at points x_{n+1}, x_n and taking t = 1, after necessary simplification, yields

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{D_2}(Q_2 f_{n+s} + R_2 f_{n+1} + S_2 f_{n+r} + T_2 f_n) \quad (14)$$

Where

$$D_2 = 60rs(1-r)(1-s)(r-s)$$
$$Q_2 = r(1-r)(20-10r)$$
$$R_2 = rs(r-s)(90-70r-70s-60rs)$$
$$S_2 = -10s(1-s)(2-s)$$
$$T_2 = 10(1-r)(1-s)(r-s)(r+s-2)$$

and from (11)

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{\bar{D}_2}\left(\bar{Q}_2 f_{n+s} + \bar{R}_2 f_{n+1} + \bar{S}_2 f_{n+r} + \bar{T}_2 f_n\right) \quad (15)$$

Where

$$\bar{D}_2 = 120rs(1-r)(1-s)(2-r)(2-s)(r-s)$$

$$\bar{Q}_2 = r(1-r)(82-45r)$$

$$\bar{R}_2 = rs(r-s)(237-155r-155s+110rs)$$

$$\bar{S}_2 = -s(1-s)(82-45s)$$

$$\bar{T} = (1-r)(1-s)(r-s)(45r+45s-20rs-82)$$

TEST SAMPLE

To be able to test the accuracy of the derived schemes, the values of r and s in (12), (13), are taking as 1/2 and 3/2, respectively. The following symmetric discrete schemes are obtained

$$y_{n+2} = 2_{n+1} - y_n - \frac{h^2}{180}(37f_{n+2} - 208f_{n+\frac{3}{2}} + 162f_{n+1} - 208f_{n+\frac{1}{2}} + 37f_n) \quad (16)$$

The order of (14) is P = 4 and the error constant C_{p+2} = 1/72. The interval of absolute stability h = (0, 8.182)The first derivative of (16) from (13) is

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) - \frac{h}{360}(261f_{n+2} - 1520f_{n+\frac{3}{2}} + 1794f_{n+1} - 1392f_{n+\frac{1}{2}} + 317f_n) \quad (17)$$

The order is also found to be P = 4 and C_{p+2} = 1/18.

The predictors for y_{n+2} and y'_{n+2} for the evaluation of f_{n+2} in (16) and (17) are, respectively obtained from (14) and (15) to be

$$y_{n+2} = 2_{n+1} - y_n + \frac{h^2}{3}(f_{n+\frac{3}{2}} + f_{n+1} + f_{n+\frac{1}{2}}) \quad (18)$$

of order P =4, C_{p+2} = 0.00104167 and the interval of absolute stability h = (-12, 0).
The first derivative of (16) is obtained to be

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{90}(119f_{n+\frac{3}{2}} - 57f_{n+1} + 87f_{n+\frac{1}{2}} - 14f_n) \quad (19)$$

of order is P = 4 and error constant C_{p+2} = 0.0102431.

It is interesting to note that while the coefficient of f_n in (14) is zero, its coefficient does not vanish (15). This is due to its coefficient in the continuous method when the value of t = 1.

Table 1: Results and errors for problem 1

X	YEX	YC	ERRold	ERRnew
1.1	0.31934375D+01	0.31934240D+01	0.28068825D-04	0.13450984D-04
1.3	0.63519902D+01	0.63519735D+01	0.35815041D-04	0.16684658D-04
1.5	0.10875132D+02	0.10875111D+02	0.44218834D-04	0.20399948D-04
1.7	0.17011061D+02	0.17011037D+02	0.53450135D-04	0.24673034D-04
1.9	0.25014396D+02	0.25014366D+02	0.63654802D-04	0.29579629D-04
2.0	0.29797471D+02	0.29797439D+02	0.69164197D-04	0.32294081D-04

Table 2: Results and errors for problem 2

X	YEX	YC	ERRold	ERRnew
2.1	0.52128820D+01	0.52128818D+01	0.80873408D-06	0.24305521D-06
2.5	0.61343820D+01	0.61343818D+01	0.80873236D-06	0.24305527D-06
2.9	0.72158820D+01	0.72158818D+01	0.80873109D-06	0.24305529D-06
3.3	0.84573820D+01	0.84573818D+01	0.80873015D-06	0.24305534D-06
3.7	0.98588820D+01	0.98588818D+01	0.80872945D-06	0.24305546D-06
4.0	0.11015007D+02	0.11015007D+02	0.80872903D-06	0.24305554D-06

Note: YEX = Y Exact, YC = Y Continuous, ERRold = Error in the old method, ERRnew = Error in the new method

The starting values for the evaluation of

$$y_{n+\frac{3}{2}}, y_{n+1} \text{ and } y_{n+\frac{1}{2}}$$

are adopted from Awoyemi (1996).

NUMERICAL EXPERIMENT

The following test problems are solved with the method (16)

(1) $$y'' - \frac{3y'}{x} + \frac{3y}{x^2 - 1} = 2x,\ y(1) = 2,\ y'(1) = 10$$

Theoretical solution:

$$y(x) = 3x^2 - 2x + x^2(1 + x\ln x)$$

(2) $$y'' - \frac{2y'}{x} - \frac{(y')^2}{x^2} = 0,\ y(2) = 5,\ y'(2) = 2$$

Theoretical solution:

$$y(x) = \frac{x^2}{2} + 3$$

The computed results of problems 1 and 2 are shown in Table 1 and 2. The results of the problems solved with the new method (16) are compared with that of Awoyemi (1996, 1997).

CONCLUSION

Attempts have been made in this study to present a simpler approach at generating a two-step numerical method for direct solution of general second order differential equations by using power series as a basis function instead of canonical polynomials in Awoyemi (1996, 1997) which is more complicated and restrictive in application. The accuracy of the derived method is tested with two problems (linear and non-linear) and the results were compared with Awoyemi (1996) of the same order four as shown in Table 1 and 2. The new method shows a better performance.

REFERENCES

Awoyemi, D.O., 1996. An efficient two-step numerical integrator for general second order ordinary differential equations. Abacus, 24: 31-44.

Awoyemi, D.O., 1997. A 6th-order continuous hybrid multistep method for initial value problems of general second order ordinary differential equations. NJTE., 14: 1-8.

Awoyemi, D.O. and P. Onumanyi, 1995. A two-step hybrid multistep method with continuous coefficients for initial value problems of general second order ordinary differential equations. Spectrum, 2: 56-63.

Awoyemi, D.O. and S.J. Kayode, 2002. An optimal order continuous multistep algorithm for initial value problems of special second order differential equations, J. Nig. Assoc. Math. Phys., 6: 285-292.

Kayode, S.J., 2004. A class of maximal order linear multistep collocation methods for direct solution of second order initial value problems of ordinary differential equations. Ph.D. Thesis, FUTA.