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Conjugate Polar Form of Cauchy-Riemann Equations

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Abstract: If we use the form z = (cosθ + I sinθ) and set f(z) = f ($re^{i\theta}$) = u(r, θ) + iv(r, θ), then the Cauuchy-Riemann equations are

$$u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta \qquad (r > 0)$$

In this study, we establish the conjugate forms of the above Cauchy-Riemann differential equations in polar coordinate. That is; if we use the conjugate polar form $\bar{z}$ =r (cosθ + I sinθ) and set f($\bar{z}$) = f ($re^{-i\theta}$) = u(r, θ) + iv(r, -θ), then the conjugate polar form Cauuchy-Riemann equations are

$$v_r = \frac{1}{r}u_\theta \text{ and } u_r = -\frac{1}{r}v_\theta \qquad (r > 0)$$

which is a "reflection" of Cauchy-Riemann Differential Equations in Polar coordinate.

Key words: Analyticity, Cauchy-Riemann, conjugate, polar, reflection

INTRODUCTION

Cantor, Dedekind and Weierstrass etc, extended the conception of rational numbers to a large field known as the real numbers which constitute rational as well as irrational numbers (Erwin, 2003). Evidently the system of real numbers is not sufficient for all mathematical needs e.g. there is no real number (rational or irrational) which satisfies $x^2 + 1 = 0$ (Erwin, 2003; Serge, 1993). It was therefore felt necessary by Euler Gauss, Hamilton, Cauchy, Riemann and Weierstrass etc to extend the field of real numbers to the still larger field of complex numbers. Euler for the first time introduced the symbol i with the property $i^2 = -1$ (Erwin, 2003; Louis and Lawrence, 1970) and then Gauss introduced a number of the form α + iβ which satisfies every algebraic equation with real coefficients (Erwin, 2003).

Such a number α + iβ with $i = \sqrt{-1}$ and α, β being real, is known as a complex number (Erwin, 2003; Louis and Lawrence, 1970).

At various times in mathematical analysis it becomes convenient to make use of many of the properties of complex conjugates. It is well known that the complex variable z and its conjugate are defined by Gupta (1993) and Louis and Lawrence (1970)

$$z = x + iy, \quad \bar{z} = x - iy \tag{1}$$

Now if we consider a function ϕ which is a function of z and $\bar{z}$, that is, ϕ = ϕ (z, $\bar{z}$), its derivatives with respect to x and y are (Gupta, 1993)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x} = \frac{\partial\phi}{\partial z}(1) + \frac{\partial\phi}{\partial\bar{z}}(1) = \frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial\bar{z}} \tag{2}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial y} = \frac{\partial\phi}{\partial z}(j) + \frac{\partial\phi}{\partial\bar{z}}(-j) = j(\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial\bar{z}}) \tag{3}$$

Now defining the operators

$$D_x = \frac{\partial}{\partial x} \quad D_y = \frac{\partial}{\partial y} \quad D_z = \frac{\partial}{\partial z} \quad D_{\bar{z}} = \frac{\partial}{\partial\bar{z}} \tag{4}$$

we may rewrite Eq. 2 and 3 as Gupta (1993)

$$D_x = D_z + D_{\bar{z}} \tag{5}$$

$$D_y = j(D_z - D_{\bar{z}}) \tag{6}$$

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or

$$2D_{\bar{z}} = D_x \;\; jD_y \qquad 2D_z = D_x - jD_y \tag{7}$$

From (7) we may readily obtain

$$(2D_{\bar{z}})(2D_z) = (D_x + jD_y)(D_x - jD_y) = D_x^2 + D_y^2 \tag{8}$$

Hence, we see that the Laplace operator may be expressed in the form Gupta (1993)

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (D_x^2 + D_y^2)\phi \tag{9}$$

THEORETICAL ANALYSIS

Complex numbers: An ordered pair of real numbers such as (x,y) is termed as a complex number. If we write z=(x,y) or x+iy ,where $i = \sqrt{-1}$ (Erwin, 2003; Louis and Lawrence, 2007; Serge, 1993), then x is called the real part and y imaginary part of the complex number z and denoted by Gupta (1993) and Erwin (2003)

$$x = R_z \text{ or } R(z) \text{ or } Re(z)$$
$$y = I_z \text{ or } I(z) \text{ or } Im(z)$$

Equality of complex numbers: Two complex numbers (x,y) and (x′,y′) are equal iff x=x′ and y=y′ (Erwin, 2003; Louis and Lawrence, 2007).

So we get this important result (Louis and Lawrence, 2007):

- The two real parts are equal.
- The two imaginary parts are equal.

Modulus of a complex number: If z = x+iy be a complex number then its modulus (or module) is denoted by $|z|$ and given by

$$|z| = |x + iy| = +\sqrt{x^2 + y^2} \text{ (Erwin, 2003)}$$
$$\text{Evidently } |z| = 0 \text{ if } x = 0, y = 0$$

Conjugate complex numbers: If z = x + iy, then x-iy is said to be the conjugate of complex number z and denoted by $\bar{z}$

Evidently

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$$
$$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$$
$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$$
$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\, R_z \text{ or } 2\, R(z)$$
$$z - \bar{z} = (x + iy) - (x - iy) = i\; 2y = 2i\; I_z \text{ or } 2i\; I(z)$$

Analytic (or regular or holomorphic or monogesic functions): A function f(z) which is single-valued and differentiable at every point of a domain D, is said to be regular in the domain D (Erwin, 2003). Let f be a function of an open set U, and write f in terms of its real and imaginary parts,

$$f(x + iy) = u(x,y) + i\, v(x,y)$$

It is reasonable to ask what the condition of differentiability means in terms of u and v (Serge, 1993).

The necessary and sufficient conditions for f(z) to be regular

Necessary conditions: If w = f(z), where w = u + iv and z = x + iy, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(Erwin, 2003; Gupta, 1993; Serge, 1993)

These two relations, which are necessary conditions for a function to be analytic, are called the Cauchy-Riemann Differential Equations (Erwin, 2003).

Sufficient conditions: The sufficient conditions for the function f(z) to be regular require the continuity of the four first partial derivatives of u and v (Erwin, 2003).

RESULTS AND DISCUSSION

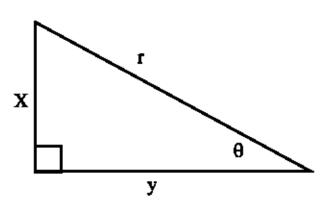
Derivation of conjugate form of Cauchy-riemann conditions in polar coordinates

Theorem 1:

If $\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$ and $w = f(\bar{z}) = u + iv$ is differentiable at $\bar{z}$,

then $\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$ and $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$

Proof:



From the above diagram,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \tag{10}$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \tag{11}$$

$$\tan\theta = \frac{y}{x} \tag{12}$$

Squaring Eq. 10 and 11

$$x^2 = r^2\cos^2\theta \tag{13}$$

$$y^2 = r^2\sin^2\theta \tag{14}$$

Add up

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2$$
$$\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2(1) = r^2$$

That is,

$$r^2 = x^2 + y^2 \tag{15}$$

Differentiating Eq. 15 with respect to x and y, respectively, we have

$$2r\partial r = 2x\partial x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$
$$2r\partial r = 2y\partial y \quad \Rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

That is,

$$\frac{\partial r}{\partial x} = \cos\theta \tag{16}$$

$$\frac{\partial r}{\partial y} = \sin\theta \tag{17}$$

Differentiating Eq. 12 with respect to x and y, respectively, we have

$$\sec^2\theta\,\partial\theta = -\frac{y}{x^2}\partial x \quad \Rightarrow \quad \frac{1}{\cos^2\theta}\partial\theta = -\frac{y}{x^2}\partial x \quad \Rightarrow$$
$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}\cos^2\theta = -\frac{r\sin\theta}{r^2\cos^2\theta}\cos^2\theta = -\frac{\sin\theta}{r}$$
$$\sec^2\theta\,\partial\theta = \frac{1}{x}\partial y \quad \Rightarrow \quad \frac{1}{\cos^2\theta}\partial\theta = \frac{1}{x}\partial y$$
$$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{x}\cos^2\theta = \frac{1}{r\cos\theta}\cos^2\theta = \frac{\cos\theta}{r}$$

That is,

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r} \tag{18}$$

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \tag{19}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} = u_r\cos\theta - \frac{u_\theta}{r}\sin\theta \tag{20}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} = u_r\sin\theta + \frac{u_\theta}{r}\cos\theta \tag{21}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x} = v_r\cos\theta - \frac{v_\theta}{r}\sin\theta \tag{22}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y} = v_r\sin\theta + \frac{v_\theta}{r}\cos\theta \tag{23}$$

By Conjugate Cartesian form of Cauchy-Riemann Conditions,

if $\overline{z} = x - iy$ and $w = f(\overline{z}) = u + iv$, then

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{24}$$

and

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \tag{25}$$

Substituting Eq. 22 and 21 into 24, we have

$$v_r\cos\theta - \frac{v_\theta}{r}\sin\theta = u_r\sin\theta + \frac{u_\theta}{r}\cos\theta \tag{26}$$

Substituting Eq. 23 and 20 into 25, we have

$$v_r\sin\theta + \frac{v_\theta}{r}\cos\theta = -u_r\cos\theta + \frac{u_\theta}{r}\sin\theta \tag{27}$$

Multiply Eq. 26 by cosθ and 27 by sinθ, we have

$$v_r\cos^2\theta - \frac{v_\theta}{r}\sin\theta\cos\theta = u_r\sin\theta\cos\theta + \frac{u_\theta}{r}\cos^2\theta \tag{28}$$

$$v_r\sin^2\theta + \frac{v_\theta}{r}\sin\theta\cos\theta = -u_r\sin\theta\cos\theta + \frac{u_\theta}{r}\sin^2\theta \tag{29}$$

Add up

$$v_r(\cos^2\theta + \sin^2\theta) + 0 = 0 + \frac{u_\theta}{r}$$
$$(\cos^2\theta + \sin^2\theta) \Rightarrow v_r(1) = \frac{u_\theta}{r}(1)$$

That is,

$$v_r = \frac{u_\theta}{r} \quad \text{or} \quad \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial\theta} \tag{30}$$

Similarly, multiply Eq. 26 by $\sin\theta$ and 27 by $\cos\theta$, we have

$$v_r \sin\theta\cos\theta - \frac{v_\theta}{r}\sin^2\theta = u_r \sin^2\theta + \frac{u_\theta}{r}\sin\theta\cos\theta \qquad (31)$$

$$v_r \sin\theta\cos\theta + \frac{v_\theta}{r}\cos^2\theta = -u_r \cos^2\theta + \frac{u_\theta}{r}\sin\theta\cos\theta \qquad (32)$$

Subtracting

$$0 - \frac{v_\theta}{r}(\sin^2\theta + \cos^2\theta) = u_r(\sin^2\theta + \cos^2\theta)$$

$$+0 \Rightarrow -\frac{v_\theta}{r}(1) = u_r(1)$$

That is,

$$u_r = -\frac{1}{r}v_\theta \quad \text{or} \quad \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (33)$$

Hence, by Eq. 30 and 33 we have the theorem
By Cauchy-Riemann Differential Equations,

$$\text{if } z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$\text{and } w = f(z) = f(re^{i\theta}) = u + iv,$$
$$\text{then } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(Erwin, 2003; Gupta,1993; Serge, 1993)

In this study, we have been able to derive the conditions that the conjugate of the above satisfy, that is;

$$\text{if } \bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$
$$\text{and } w = f(\bar{z}) = f(re^{-i\theta}) = u + iv,$$
$$\text{then } \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$$

CONCLUSION

We know that all logarithm functions satisfy Cauchy-Riemann conditions in Polar coordinate. Therefore, logarithm of conjugate functions satisfy the above theorem.
That is,

$\log_e$ (nz) for all $n \in N$ satisfy Cauchy-Riemann Differential equations.

Similarly,

$\log_e$ (nz) for all $n \in N$ satisfy the above theorem.

Comparing the above result with Cauchy-Riemann Differential Equations, we conclude that for every z such that

$$\text{if } w = f(z) = f(re^{i\theta}), \text{ where } w = u + iv$$
$$\text{and } z = x + iy, \text{ then}$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

There exists the corresponding conjugate of z such that

$$\text{if } w = f(\bar{z}) = f(re^{-i\theta}), \text{ where } w = u + iv$$
$$\text{and } \bar{z} = x - iy, \text{ then}$$
$$\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$$

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