

## Assessment of Homotopy Perturbation Method to Nonlinear Fifth-Order Boundary Value Problem

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**Abstract:** Analytical solutions play a very important role in nonlinear boundary value problem. In this study, He's Homotopy Perturbation Method (HPM) has been applied to solve a nonlinear fifth-order boundary value problem. The concept of He's homotopy perturbation method are introduced briefly for applying this method for solving problem. The results of HPM as analytical solution are then compared with those derived from the exact solution in order to verify the accuracy of the proposed method. just one iteration results in highly accurate solution. The results reveal that the HPM is very effective and convenient in predicting the solution of such problems and it is predicted that HPM can have a found wide application in new engineering problems.

**Key words:** Homotopy Perturbation Method (HPM), nonlinear equation, fifth-order, boundary value problem, Iran

### INTRODUCTION

Nonlinear phenomena play a crucial role in applied Mathematics and Physics. We know that most of engineering problems are nonlinear and it is difficult to solve them analytically. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions.

The Homotopy Perturbation Method (HPM) was established by He (1999). The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series which converges rapidly to accurate solutions. Considerable research work has recently been conducted in applying this method to a class of linear and non-linear equations.

The following fifth-order boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences (Caglar *et al.*, 1999; Davies *et al.*, 1988; Karageoghis *et al.*, 1998):

$$y^{(v)}(x) = f(x, y, y', y'', y''', y^{(iv)}) \quad (1)$$

with suitable boundary conditions:

$$\begin{aligned} y(a) = A_1 \quad y'(a) = A_2 \\ y''(a) = A_3 \quad y(b) = B_1 \quad y'(b) = B_2 \end{aligned} \quad (2)$$

Where  $f$  is continuous function on  $(a, b)$  and the parameters  $A_1, A_2, A_3, B_1, B_2$  are real constants.

In this study, we develop He's homotopy perturbation method (Jalaal *et al.*, 2010; Khan and Wu, 2010; Babolian *et al.*, 2009; Javidi, 2009) by means of the computer for solving the system (1) with boundary condition (2). It is shown that this method is very easy to implement and it is more efficient than using sixth-degree B-spline functions Caglar *et al.* (1999) and variational iteration method Zhang (2009). An example is given to illustrate the performance of this method.

### MATERIALS AND METHODS

**Basic idea of homotopy perturbation method:** To explain this method, let us consider the following function:

$$S(u) - f(r) = 0, r \in \Omega \quad (3)$$

With the boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \quad (4)$$

Where:

- S = A general differential operator
- $f(r)$  = A known analytic function
- B = A boundary operator
- $\Gamma$  = The boundary of the domain  $\Omega$

The operator S can be generally divided into two operators, L and N where L is a linear and N a nonlinear operator. Equation 3 can be therefore, written as follows:

$$L(u) + N(u) - f(r) = 0 \tag{5}$$

Using the homotopy technique, the researchers constructed a homotopy  $v(r, p): \Omega \times [0, 1] \rightarrow \mathbb{R}$  which satisfies:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[S(v) - f(r)] = 0 \tag{6}$$

Or:

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{7}$$

Where:

- $p \in [0, 1]$  = Homotopy parameter
- $u_0$  = An initial approximation for the solution of Eq. 3

Which satisfies the boundary conditions. Obviously from Eq. 6 and 7 we will have:

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{8}$$

$$H(v, 1) = S(v) - f(r) = 0 \tag{9}$$

We can assume that the solution of Eq. 6 or 7 can be expressed as a series in p as follows:

$$v = v_0 + pv_1 + p^2v_2 + \dots \tag{10}$$

Setting  $p = 1$  results in the approximate solution of Eq. 3:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \tag{11}$$

**Implementation of HPM:** To illustrate the implementation of this method, we consider the same example by Wazwaz (2001).

We consider the nonlinear BVP:

$$y^{(v)}(x) = e^{-x}y^2(x) \quad 0 < x < 1 \tag{12}$$

with suitable boundary conditions:

$$y(0) = 1 \quad y'(0) = 1 \quad y''(0) = 1 \quad y(1) = e \quad y'(1) = e \tag{13}$$

To solve Eq. 12 with the initial condition Eq. 13, according to the homotopy perturbation, we construct the following homotopy:

$$L(v) = v^{(v)}(x) \tag{14}$$

$$N(v) = -e^{-x}v^2(x) \tag{15}$$

$$H(v, p) = v^{(v)}(x) - u_0^{(v)}(x) + pu_0^{(v)}(x) + p[-e^{-x}v^2(x)] = 0 \tag{16}$$

Substituting  $v = v_0 + pv_1 + p^2v_2 + \dots$  into Eq. 16 and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{d^5v_0(x)}{dx^5} - \frac{d^5u_0(x)}{dx^5} = 0 \tag{17}$$

$$p^1 : \frac{d^5v_1(x)}{dx^5} + \frac{d^5u_0(x)}{dx^5} - e^{-x}(v_0(x))^2 = 0 \tag{18}$$

with the following conditions:

$$\begin{aligned} v_0(0) &= 1 & v_0'(0) &= 1 & v_0''(0) &= 1 \\ v_0(1) &= e & v_0'(1) &= e \end{aligned} \tag{19}$$

$$\begin{aligned} v_1(0) &= 0 & v_1'(0) &= 0 & v_1''(0) &= 0 \\ v_1(1) &= 0 & v_1'(1) &= 0 \end{aligned} \tag{20}$$

With the effective initial approximation for  $v_0$  from the conditions (19), we obtain the solutions of Eq. 12:

$$v_0(x) = 0.0633x^4 + 0.1548x^3 + 0.5x^2 + x + 1 \tag{21}$$

$$\begin{aligned} v_1(x) &= (2730 + 2x^4 + 37x^3 + 315x^2 + 1410x)e^{-x+1} + \\ &(-7581 + -5.5x^4 - 102x^3 - 870.5x^2 - 3906x) \\ &e^{-x} + 21.05x^2 + 160 - 4.0466x^3 - 86.8x \end{aligned} \tag{22}$$

Having  $v_i, i = 0, 1$ , from Eq. 11, the solution  $y(x)$  is as follows:

$$y(x) = v_0(x) + v_1(x) \tag{23}$$

The exact solution of the system Eq. 12 and Eq. 13:

$$y(x) = e^x \tag{24}$$

**RESULTS AND DISCUSSION**

In this study we present the results with HPM to show the efficiency of the method. From Fig. 1, it is clear that solution of homotopy perturbation method nearly equals exact solution. Table 1 exhibits a comparison between the errors obtained by using the homotopy perturbation method and by using sixth-degree B-spline method Caglar *et al.* (1999) and variational iteration method Zhang (2009). From this Table 1, it is clear that an improvement is obtained by using the homotopy perturbation method. It is noticeable that the solution of HPM is from just one iteration. Higher accuracy can be obtained from higher iterations.

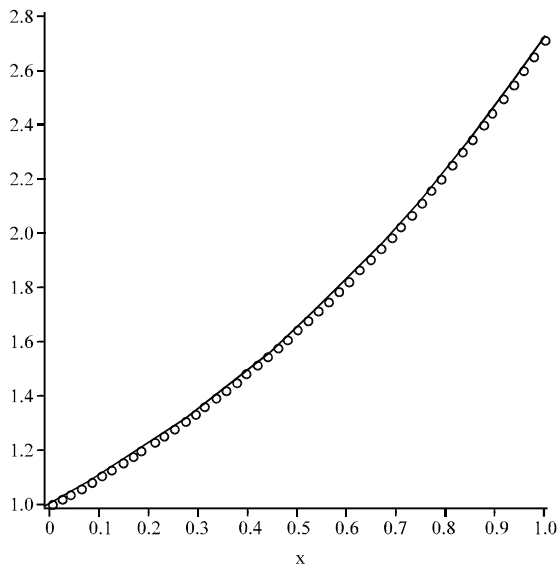


Fig. 1: Comparison of the exact solution (continuous line) with the HPM solution (circle line)

Table 1: Error estimates (Error = exact solution-analytical or numerical solutions)

| x   | Exact solution<br>y (x) | Error<br>(HPM) | Error (Variational<br>iteration method)<br>Zhang (2009) | Error (Sixth-degree<br>B-spline method)<br>Caglar <i>et al.</i> (1999) |
|-----|-------------------------|----------------|---|--|
| 0   | 1.0000                  | 0.0000         | 0.0000  | 0.0000   |
| 0.1 | 1.1052                  | -1.9E-6        | 0.0000  | -7.0E-4  |
| 0.2 | 1.2214                  | -2.0E-5        | 1.0E-5  | -7.2E-4  |
| 0.3 | 1.3499                  | -5.4E-5        | 1.0E-5  | 4.1E-4   |
| 0.4 | 1.4918                  | -9.9E-5        | 1.0E-4  | 4.6E-4   |
| 0.5 | 1.6487                  | -1.4E-4        | 3.2E-4  | 4.7E-4   |
| 0.6 | 1.8221                  | -1.6E-4        | 3.6E-4  | 4.8E-4   |
| 0.7 | 2.0138                  | -1.3E-4        | -1.4E-4   | 3.9E-4   |
| 0.8 | 2.2255                  | -1.1E-4        | -3.1E-4   | 3.1E-4   |
| 0.9 | 2.4596                  | -4.8E-5        | -5.8E-4   | 1.6E-4   |
| 1   | 2.7183                  | -1.4E-5        | -9.9E-5   | 0.0000   |

**CONCLUSION**

He's Homotopy Perturbation Method (HPM) has been successfully utilized to derive approximate explicit analytical solutions for nonlinear fifth order boundary value problem. The results show that this perturbation scheme provides excellent approximations to the solution of this nonlinear equation with high accuracy and avoids linearization and physically unrealistic assumptions. This new method accelerated the convergence to the solutions. The results show that when parameter of x increases, the error of HPM is lower than two methods and this method is stable for higher x in this problem.

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