

The Nonlinear Fractionally Oscillator with Strong Quadratic Damping Force

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Abstract: The equation of the fractionally nonlinear oscillator with strong quadratic damping force was considered with generalized damping term to Caputo fractional derivatives. The order of the derivatives considered for this problem was $0 \leq \nu \leq 1$. At the lower end $\nu = 0$ the linearly damped harmonic oscillator and at the upper end $\nu = 1$ a non-linearly damped harmonic oscillator or strong quadratic damping force were obtained. The method of Laplace transformations were used to obtain the analytical solution. Eighteen roots were obtained against the usual three for ordinary (i.e., damped, over damped and critically damped). Two solutions were obtained for both positive and negative δ . For six of these cases it was shown that the frequency of oscillation increases with increasing damping order before eventually falling to the limiting value given by the ordinary damped oscillator equation. For the other six cases the behavior is as expected, the frequency of oscillation decreases with increasing order of the derivative damping term.

Key words: Nonlinear oscillator, strong quadratic, damping force, fractionally, equation, Nigeria

INTRODUCTION

The simple harmonic oscillator is one of the central problems in physics, these systems of linear oscillator had been considered by many standard text books and many researchers this was due to their simplicity and existence of an exact analytical solution which had been discussed extensively but in reality the systems and damping are not linear. Most of the time, the two kinds of oscillator with damping are considered, either the system with nonlinear elasticity and linear damping or the system with linear elasticity and nonlinear damping.

The first group of the problem is widely discussed in their papers Mickens (2002), Waluya and van Horssen (2003), Andrianov and van Horssen (2006) and Pilipchuk (2007) while the second group of the problems on restriction to the smaller damping were considered by (Nayfeh and Mook, 1979) they applied the analytical method of multiple scale for solving lightly damped systems (Timoshenko *et al.*, 1974) gave an example of the application of the method of successive approximation for solving a differential with a small square damping term examined the same problem by modified the averaging method for solving the equation. Andronov *et al.* (1981) provided a qualitative analysis of the oscillator with strong damping quadratic. The trajectories for various initial energy values were plotted in the phase plane by Cveticanin (2004). The previously obtained result by Cveticanin (2009) was extended by Podlubny (1999) and Burov and Barkai (2007, 2008) examined an equation with

critical behaviour they a solution in terms of generalized Mittag-Leffler functions. Nonlinear fractional oscillator was studied numerical by (Zaslavsky *et al.*, 2006) he was interested in chaotic behaviours believing that the careful study of analytical solution to linearly fractional damped equation will explained better on properties of the nonlinear equation and be used for direct application of fractionally damped oscillation (Galucio *et al.*, 2006; Achar *et al.*, 2004), Naber (2009) applied the Caputo formula of fractional derivatives and Laplace transform of the two formation of the fractional derivatives to determine analytical solution and also applied the method of nonfractional case and found out that there nine distinct for fractionally damped equation as opposed to three cases for nonfractional equation. In this study the fractionally nonlinear oscillator with strong quadratic damping force is consider. The mathematical model to the systems is given as:

$$\begin{aligned} \ddot{X}(t) + \delta \dot{X}^\nu(t) | \dot{X}(t) | + \lambda^2 X(t) &= 0 \\ \text{or } \ddot{X}(t) + (\pm \delta) \dot{X}^{\nu+1}(t) + \lambda^2 X(t) &= 0 \end{aligned} \quad (1)$$

Whose order $\nu+1$ will be restricted to $0 \leq \nu \leq 1$. The Caputo formulation of the fractional derivative will be used. In particular the Caputo derivatives over the Riemann-Liouville derivative for physical reason shall be employed. Consider the Laplace transform of the two formulations of the fractional derivative for $0 < \nu < 1$.

$$\begin{aligned} L\left({}^{\text{RL}}_0 D_t^\nu f(t) \right) &= s^\nu F(s) - {}^{\text{RL}}_0 D_t^\nu f(t)|_0 \\ L\left({}^{\text{L}}_0 D_t^\nu f(t) \right) &= s^\nu F(s) - f(t)|_0 \end{aligned} \quad (2)$$

The constant term arise from the Laplace transform of the Caputo derivatives is just the initial value of the function which is not in Riemann-Liouville derivative also the constant arising from the Laplace has no physical interpretation therefore, the Caputo fractional will be more for the physical problem.

THE NONFRACTIONAL CASE

Laplace transform method is applied to obtain a solution to the fractionally nonlinear oscillator with strong quadratic damping force:

$$\ddot{X}(t) + (\pm\delta)\dot{X}(t) + \lambda^2 X(x) = 0 \quad (3)$$

Where:

- δ and λ = Constant of real and positive
- δ = The damping force per unit mass
- λ = The restoring force per unit mass for both case (the fractional and nonfractional

The following initial conditions will be applied:

$$X(0) = x_0, \quad \dot{X}(0) = x_1 \quad (4)$$

Applying the Laplace transform to Eq. 3 together with the initial conditions in Eq. 4 to obtain:

$$\begin{aligned} s^2 X(s) - sX(0) - \dot{X}(0) + (\pm\delta) \\ (sX(s) - X(0)) + \lambda^2 X(s) = 0 \end{aligned} \quad (5)$$

Or

$$X(s) = \frac{sX_0 + X_1 + (\pm\delta)X_0}{s^2 + (\pm\delta)s + \lambda^2} \quad (6)$$

Equation 6 can be inverted by using a relevant table but notwithstanding Eq. 6 can be inverted via the complex inversion integral. The components in the s variable in both terms are whole number. Thus, not a branch will be cut in the contour and the Bromwich contour shall be employed:

$$X(t) = \text{Residue} - \frac{1}{2\pi i} \int e^{st} X(s) ds \quad (7)$$

Bromwich contour is known to begin from $\gamma - i\infty$ and vertically to $\gamma + i\infty$ (where γ is chosen so that the poles lie

on the left of the vertical contour lines thus all poles will be concentrated within the contour) and then travels to a half circle (to the left, counter clockwise) back to $\gamma - i\infty$. For this particular case there is no contribution from the contour integral. The only contribution comes from the residue. The residue is generated from the roots of the quadratic equation:

$$\ddot{X}(t) + \delta\dot{X}(t) + \lambda^2 X(x) = 0 \quad (8)$$

Three cases were obtained, $\pm\delta > 2\lambda$, four unequal real roots that are two negatives $s_{1,2} = -\delta \pm \sqrt{\delta^2 - 4\lambda^2} / 2$ and two positives $s_{3,4} = \delta \pm \sqrt{\delta^2 - 4\lambda^2} / 2$, $\pm\delta = 2\lambda$, four repeated real roots that are both positive and negative:

$$s_5 = \frac{-\delta}{2} = -\lambda, \quad s_6 = \frac{\delta}{2} = \lambda$$

$\delta < 2\lambda$, four complex roots whose real part are both positive and negative:

$$s_{6,7} = \frac{-\delta \pm i\sqrt{4\lambda^2 - \delta^2}}{2}, \quad s_{8,9} = \frac{\delta \pm i\sqrt{4\lambda^2 - \delta^2}}{2}$$

Compute the residue for case one gives:

$$\begin{aligned} \text{Residue} &= \lim_{s \rightarrow s_{1,2}} (s - s_{1,2}) e^{st} \left(\frac{sX_0}{s^2 + \delta s + \lambda^2} + \frac{X_1 + \delta X_0}{s^2 + \delta s + \lambda^2} \right) \\ \text{Residue} &= \lim_{s \rightarrow s_{3,4}} (s - s_{3,4}) e^{st} \left(\frac{sX_0}{s^2 - \delta s + \lambda^2} + \frac{X_1 - \delta X_0}{s^2 - \delta s + \lambda^2} \right) \end{aligned} \quad (9)$$

$$\begin{aligned} X(t) &= \frac{e^{s_1 t}}{2s_1 + \delta} (s_1 X_0 + X_1 + \delta X_0) + \frac{e^{s_2 t}}{2s_2 + \delta} (s_2 X_0 + X_1 + \delta X_0) \\ X(t) &= \frac{e^{s_3 t}}{2s_3 - \delta} (s_3 X_0 + X_1 - \delta X_0) + \frac{e^{s_4 t}}{2s_4 - \delta} (s_4 X_0 + X_1 - \delta X_0) \end{aligned} \quad (10)$$

Since, s_1 and s_2 are both negative the solution will decay exponentially. This is called the over damped case. Case two the poles is of order 2 and the residue is given by:

$$\lim_{s \rightarrow \pm\lambda} \frac{d}{ds} \left((s \pm \lambda)^2 e^{st} \left(\frac{sX_0 + X_1 \pm \delta X_0}{s^2 \pm \delta s + \lambda^2} \right) \right) \quad (11)$$

Since, $\delta = 2\delta$ the denominator can be factored the limit then becomes:

$$\lim_{s \rightarrow \pm\lambda} \frac{d}{ds} \left(e^{st} (sX_0 + X_1 + 2\lambda X_0) \right) \quad (12)$$

$$X(t) = 2\lambda e^{\pm\lambda t} (t(\lambda x_0 + x_1) + x_0) \quad (13)$$

This is called the critically damped case. Case three is computed in the same way as case one since the poles are complex the exponential function can be expressed as sine and cosine including an overall exponential damping factor:

$$X(t) = e^{-\alpha t} \left(x_0 \cos(\beta t) + \frac{(2x_1 + (\pm\delta)x_0)}{2\beta} \sin(\beta t) \right) \quad (14)$$

where, $\alpha = \pm\delta/2$ and $\beta = \sqrt{4\lambda^2 \pm \delta^2} / 2$ the present of damping was notice to cause the effective angular velocity to be smaller than the undamped angular frequency in other words oscillation goes slower which might be expect if there were damping to impede the motion. This is referred to as the under damped case by comparison case one and case two can be viewed as having zero frequency or an infinite period.

THE FRACIONAL CASE

Consider Eq. 1 in this case the δ has the unit of time raise to the power $v-3$. Hence, the over all units of the second term remain the same as in Eq. 4. The case of shall be considered $0 \leq v \leq 1$ and $1 \leq v \leq 2$. The Laplace transform of Eq. 1 is:

$$s^2 X(s) - s x_0 + x_1 + (\pm\delta) \cdot (s^{v+1} X(s) - s^v x_0) = \lambda^2 X(s) = 0 \quad (15)$$

Or

$$X(s) = \frac{s x_0 + x_1 + (\pm\delta) s^v x_0}{s^2 + (\pm\delta) s^{v+1} + \lambda^2} \quad (16)$$

Equation 16 is inverted using the contour integral due to the fractional exponents on the complex variable s a branch cut is needed on the negative real axis. Hence, Hankel contour is applied.

This contour begins at $\gamma - i\infty$ and vertically to $\gamma + i\infty$ (where γ is chosen so that the poles lie on the left of the vertical contour lines, thus all poles will be concentrated within the contour) and then travels to a quarter circle (to the left, counter clockwise) back to $\gamma - i\infty$.

For this particular case there is no contribution from the contour integral. The only contribution comes from the residue. To find the poles quadratic equation of this for needs to be solved:

$$s^2 \pm \delta s^{v+1} + \lambda^2 = 0 \quad (17)$$

For which an arbitrary v is not a trivial problem. To determine the solution and the number of solutions let $s = r e^{i\phi}$ then Eq. 17 splits into 2 Equation a real and imaginary part:

$$\begin{aligned} r^2 \cos(2\phi) \pm \delta r^{v+1} \cos((v+1)\phi) + \lambda^2 &= 0 \\ r^2 \sin(2\phi) \pm \delta r^{v+1} \sin((v+1)\phi) &= 0 \end{aligned} \quad (18)$$

From the Eq. 17 the question that comes to the mind is that would there be solution on the positive axis the answer is no since $\phi = 0$ the first equation of (18) would be the sum of the three positive nonzero terms which would never is zero. Similarly would there be a solution on negative real axis, the answer is no in this case $\phi = \pi$ the second term of the second equation of (18) would never be zero. Using similar argument it can be shown that there no solutions in both positive and negative of the imaginary axes. It can also been shown that no solutions are in the right half plane since $0 < v < 1$ (both terms of the second would always be positive). Assume that there solution then they should be in pairs, complex conjugate with $\pi/2 < \phi < \pi$ and $-\pi < \phi < -\pi/2$. In an attempt to find a solution the second equation in Eq. 18 would be solve first to obtain a value for r and substitute in the first:

$$\begin{aligned} \left(\frac{-(\pm\delta) \sin((v+1)\phi)}{\sin(2\phi)} \right)^{\frac{2}{1-v}} \cos(2\phi) + \\ (\pm\delta) \left(\frac{-(\pm\delta) \sin((v+1)\phi)}{\sin(2\phi)} \right)^{1+\frac{2}{1-v}} \cos((v+1)\phi) + \lambda^2 &= 0 \end{aligned} \quad (19)$$

The negative sign, the negative and positive sign and the fractional exponent in Eq. 19 may be the worried of the reader nevertheless the restriction on the angular range $\pi/2 < \phi < \pi$ shall be considered since $\sin(2\phi)$ is always negative, so the root of the argument will always be positive. Assign values for v , δ and λ the Eq. 19 will be seen difficult to solve for ϕ . Equation 19 can be simplified to a likely preferred form.

$$\left(\frac{(\sin((v+1)\phi))^{v+1}}{(\sin(2\phi))^2} \right)^{\frac{1}{1-v}} \sin((1-v)\phi) = \left(\frac{\lambda}{(\pm\delta)^{\frac{1}{1-v}}} \right)^2 \quad (20)$$

Here there is a need to get a value of ϕ that satisfies Eq. 20. For this to be true that means that $\sin((1-v)\phi) = 0$ must be positive and this can only occur on the restricted domain $\pi/2 < \phi < \pi/(1-v)$. Based on the restriction on the domain the ϕ is chosen so that the left hand of Eq. 20 is large or small or vanish. Also no matter the values assign to δ , λ and v the value of ϕ can be find the Eq. 20:

$$\lim_{\phi \rightarrow \frac{\pi}{2}} \left(\frac{(\sin((v+1)\phi))^{v+1}}{(\sin(2\phi))^2} \right)^{\frac{1}{1-v}} \sin((1-v)\phi) = \infty \quad (21)$$

$$\lim_{\phi \rightarrow \frac{\pi}{(1-v)}} \left(\frac{(\sin((v+1)\phi))^{v+1}}{(\sin(2\phi))^2} \right)^{\frac{1}{1-v}} \sin((1-v)\phi) = 0$$

Since, the left hand side of Eq. 20 is continuous and two limits were obtained in Eq. 21 this gives an assurance that at least one solution to Eq. 20 could be obtained therefore there would be at least two poles for the calculation of the residue. To show that Eq. 20 has only one solution then it must be shown that the left hand side of Eq. 20 decreases monotonically in ϕ over the restricted domain, hence only two poles in residue calculation. For Eq. 20 to decrease monotonically then the derivatives with respect to ϕ must be negative:

$$\frac{\partial}{\partial \phi} \left(\left(\frac{(\sin((v+1)\phi))^{v+1}}{(\sin(2\phi))^2} \right)^{\frac{1}{1-v}} \sin((1-v)\phi) \right) < 0 \quad (22)$$

After some algebra and neglecting all the factors that are always positive then the derivative is:

$$(v+1)^2 \sin^2(2\phi) - 4(v+1)\sin(2\phi)\sin((v+1)\phi) \cos((1-v)\phi) + 4\sin^2((v+1)\phi) = 0 \quad (23)$$

Following the restriction on the domain $\sin((1+v)\phi) > 0$, $\sin(2\phi) < 0$ and $\cos((1-v)\phi) \leq 1$. Equation 23 reduces to:

$$(v+1)^2 \sin^2(2\phi) - 4(v+1)\sin(2\phi) \sin((v+1)\phi) + 4\sin^2((v+1)\phi) > 0 \quad (24)$$

Equation 24 is then written as a perfect square to prove the assertion on Eq. 22:

$$((v+1)\sin(2\phi) - 2\sin((v+1)\phi))^2 > 0 \quad (25)$$

On the restricted domain hence, the left hand of Eq. 20 decreases monotonically with ∞ as the upper bound and 0 the lower bound. Consequently only two poles shall be obtained for the residue calculation and this is the complex conjugate of each other. From the fractionally damped equation repeated roots are not possible the repeated roots can only occur when the order of the derivative is zero. The graphical representation of the location is shown in this study.

Now the pole solution for Eq. 1 can be generated by denote the two poles as $s_{6,7} = \theta + i\beta = re^{i\phi}$. Where θ and β are obtained from r and θ values that satisfy Eq. 20 $r = \sqrt{\theta^2 + \beta^2}$ and $\tan(\phi) = \theta/\beta$ here θ is negative this imply that both solutions are in the second and the third quadrants and s_7 is the complex conjugates of s_6 . Remember that θ is taken the place of $(\pm\delta)/2$ from the nonfractional case. The poles of order one and the residue is given by:

$$\begin{aligned} \text{Residue} &= \lim_{s \rightarrow s_6} (s - s_6) e^{st} \left(\frac{sx_0 + x_1 + (\pm\delta)s^v x_0}{s^2 + (\pm\delta)s^{v+1} + \lambda^2} \right) + \\ &= \lim_{s \rightarrow s_7} (s - s_7) e^{st} \left(\frac{sx_0 + x_1 + (\pm\delta)s^v x_0}{s^2 + (\pm\delta)s^{v+1} + \lambda^2} \right) \\ &= e^{s_6 t} \left(\frac{s_6 x_0 + x_1 + x_0 (\pm\delta) s_6^v}{s_6^2 + (\pm\delta) s_6^{v+1} + \lambda^2} \right) + e^{s_7 t} \left(\frac{\bar{s}_6 x_0 + x_1 + x_0 (\pm\delta) \bar{s}_6^v}{\bar{s}_6^2 + (\pm\delta) \bar{s}_6^{v+1} + \lambda^2} \right) \end{aligned} \quad (26)$$

s_6 was substituted for s_7 and Eq. 26 after some algebra reduces to:

$$\Gamma = x_1 (2r \cos(\phi) + (\pm\delta)(v+1)r^v \cos(v\phi)) \quad (27)$$

For the contour integral the only contribution comes from the path along the positive real axis:

$$\begin{aligned} &(Rx_1 - x_0) \sin((v+1)\pi) + \\ &\frac{(\pm\delta)}{\pi} \int_0^\infty \left(\frac{x_0}{R} \right) \frac{(R^2 + \lambda^2) \sin(\pi v)}{(R^2 + \lambda^2)^2 + 2(\pm\delta)R^{v+1}(R^2 + \lambda^2) + \cos((v+1)\pi) + ((\pm\delta)R^{v+1})^2} e^{Rt} R^{v+1} dR \end{aligned} \quad (28)$$

The solution to Eq. 1 is Eq. 28 subtracted from Eq. 27 which may be seems to be complicated but the solution in the general form shows that:

$$Ee^{\theta t} \cos(\beta t) + Fe^{\theta t} \sin(\beta t) - \text{decay function} \quad (29)$$

if v goes to zero or one then the decay function in Eq. 28 goes to zero in other words (1) goes to its nonfractional limits and the decay function vanish.

The oscillation frequency: Consider the frequency of the oscillation component of the solution, $\phi = \text{Im}(s_6)$ to know how the frequency equation changes with the order of the fractional damping or derivative we set v to zero a linear damped oscillator frequency is obtained with three case as discussed in under damped, critically damped and over damped so the frequency is nonzero that is:

$$\phi = 0 \quad (\pm\delta) \geq 2\lambda, \quad \phi = \frac{\sqrt{4\lambda^2 - (\pm\delta)^2}}{2}, \quad (\pm\delta) < 2\lambda \quad (30)$$

When v is set to be one:

$$\phi = \frac{\lambda}{\sqrt{1+(\pm\delta)}} \quad (32)$$

For $0 < v < 1$ this shows that there will always be a nonzero frequency:

$$2\lambda < \frac{\sqrt{4\lambda^2 - (\pm\delta)^2}}{2} < \frac{\lambda}{\sqrt{1+(\pm\delta)}}$$

In the nonfractional case increasing $(\pm\delta)$ causes the frequency of oscillation to become smaller, monotonically, until the critical cases are reached and the oscillation period becomes infinite (these are the critical and over-damped cases). In the fractional case the frequency of oscillation, $\phi = \text{Im}(s_0)$, now depends on the order of the derivative, v as well as $(\pm\delta)$ and λ . To determine the σ dependence α on these three parameters consider s to be a function of v , on $0 \leq v \leq 1$ defined by:

$$s^2 + (\pm\delta)s^{v+1} + \lambda^2 = 0 \quad (32)$$

for fixed values of $(\pm\delta)$ and λ (both being positive). Let us restrict our attention to the upper half plane for s . As such s will be one to one on $0 \leq v < 1$. Due to the fractional exponent causing a branch cut on the negative real axis s will not be one to one at $v = 1$. Now consider the derivative of Eq. 32 with respect to v and isolate ds/dv (remember, $(\pm\delta)$ and λ are being held fixed):

$$\frac{ds}{dv} = \frac{-(\pm\delta)s^{v+1} \ln(s)s^2}{s^2 + (\pm\delta)(v+1)s^{v+1}} \quad (33)$$

The imaginary part of this equation is:

$$\frac{ds}{d\phi} = \text{Im} \left(\frac{ds}{dv} \right) \quad (34)$$

Specifically, consider this equation at $v = 0$:

$$\left. \frac{d\phi}{dv} \right|_{v=0} = \text{Im} \left(\frac{(s^2 + \lambda^2) \ln(s) s^2}{2s^{2v} - (v+1)(s^2 + \lambda^2)} \right)_{v=0} = \frac{(\pm\delta) (\ln((\pm\delta) + \lambda^2))}{4\sqrt{(\pm\delta) + \lambda^2}} \quad (35)$$

This gives three initial slopes for the rate of change of σ with respect to v $\lambda/\sqrt{1+(\pm\delta)} < 1$. The frequency initially increases with increasing damping order $\lambda/\sqrt{1+(\pm\delta)} = 1$. The frequency initially is not changing with increasing damping order $\lambda/\sqrt{1+(\pm\delta)} > 1$. The frequency initially decreases with increasing damping order.

This is not entirely what might have been expected. In the first case the oscillation frequency increases and decreases in some cases. Hence, there will be some values of v for which the fractional damping will cause the oscillations to go faster than the linear damped oscillator (the damping will still cause the amplitude to decrease). Each of the above six cases can become any of the three Nonfractionally damped cases by letting $v \rightarrow 0$ Eq. 10-12. Hence, there are nine cases for the linear fractionally damped oscillator.

There are some graphs of solutions to the imaginary part of Eq. 32 (the oscillation frequency) for various values of v , δ and λ . In all six graphs the oscillation frequency is on the vertical axis and the order of the derivative is on the horizontal axis. The six graphs for each case correspond to what would be under-damped, critically damped and over-damped for a strong damped oscillator with whole order derivatives.

Figure 1 is a representative graph of case one $\lambda/\sqrt{1+(\pm\delta)} > 1$. The blue graph is for $\delta = \lambda = 1$ the red graph is for $\delta = 2, \lambda = 1$ and the green graph is for $\delta = 3, \lambda = 1$. The oscillation frequency decreases for different values of δ .

Figure 2 is a representative graph from case two $\lambda/\sqrt{1+(\pm\delta)} = 1$ a flat start. The blue graph is for $\delta = 2(\sqrt{2}-1), \lambda = \delta/2$, the red graph is for $\delta = 1/2, \lambda = 1/2$ and the green graph is for $\delta = 15/16, \lambda = 1/4$.

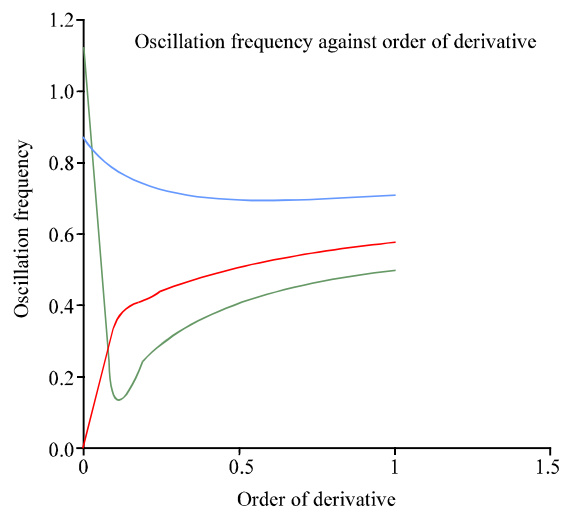


Fig. 1: The representative graph of case one $\lambda/\sqrt{1+(\pm\delta)} > 1$

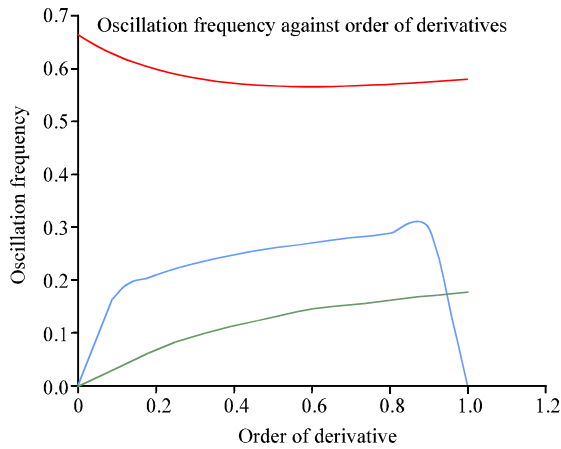


Fig. 2: The representative graph of case two $\lambda/\sqrt{1+(\delta)} > 1$

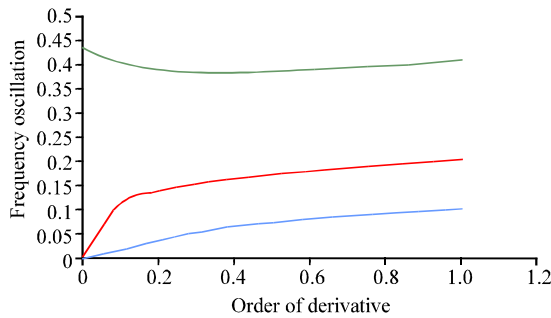


Fig. 3: Frequency oscillation against order of derivative

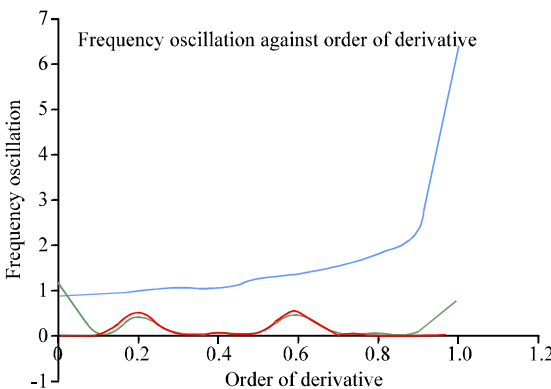


Fig. 4: The representative graph of case one $\lambda/\sqrt{1+(\delta)} > 1$

Figure 3 is a representative graph for case three. The blue graph is for $\delta = 1/2, \lambda = 1/8$, the red graph is for $\delta = 1/2, \lambda = 1/4$ and the green graph is for $\delta = 1/2$. For a constant value of δ and different value of λ and frequency oscillation increases for different values of λ .

Figure 4 is a representative graph of case one $\lambda/\sqrt{1+(\delta)} > 1$. The blue graph is for $-\delta = \lambda = 1$ the red graph is for $-\delta = 2, \lambda = 1$ and the green graph is for $-\delta = 3, \lambda = 1$. The oscillation frequency decreases for different values

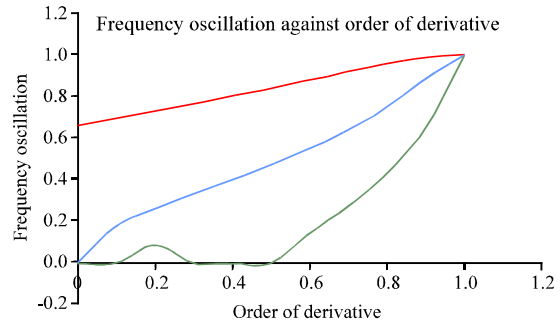


Fig. 5: The representative graph of case two $\lambda/\sqrt{1+(\delta)}$

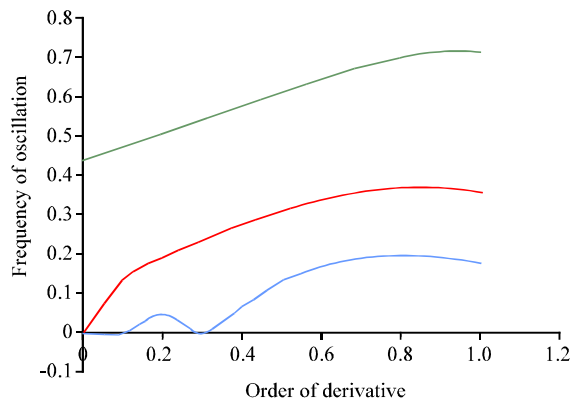


Fig. 6: The representative graph of case three

of δ . Figure 5 is a representative graph from case two, $\lambda/\sqrt{1+(\delta)} = 1$, a flat start. The blue graph is for $\delta = 2(\sqrt{2}-1), \lambda = \delta/2$, the red graph is for $\delta = 1/2, \lambda = 1/2$ and the green graph is for $\delta = 15/16, \lambda = 1/4$.

Figure 6 is a representative graph for case three. The blue graph is for $-\delta = 1/2, \lambda = 1/8$, the red graph is for $-\delta = 1/2, \lambda = 1/4$ and the green graph is for $\delta = 1/2, \lambda = 1/2$. For a constant value of λ and different value of δ and frequency oscillation increases for different values of λ .

CONCLUSION

In this study the strong quadratic nonlinear fractionally damped oscillator equation was solved analytically. It was found that the solution is very similar to the Nonfractional case (decayed oscillations but with the inclusion of an additional decay function). It was found that there are nine distinct cases both positive and negative damping as opposed to the usual three for the ordinary damped oscillator. An unexpected result was that for three of the cases the oscillation frequency actually increases with increasing order of derivative of the damping term and then the frequency decreases as expected.

REFERENCES

- Achar, B.N.N., J.W. Hanneken and T. Clarke, 2004. Damping characteristics of a fractional oscillator. *Phys. A Stat. Mech. Appl.*, 339: 311-319.
- Andrianov, I.V. and W.T. van Horssen, 2006. Analytical approximations of the period of a generalized nonlinear van der Pol oscillator. *J. Sound Vib.*, 295: 1099-1104.
- Andronov, A., A.A. Vitt and S.E. Hajkin, 1981. *Teorija Kolebanij*. Nauka publishers, Moskva, Russia.
- Burov, S. and E. Barkai, 2007. The critical exponent of the fractional langevin equation is $\alpha c - 0.402$. <http://arxiv.org/abs/0712.3407>.
- Burov, S. and E. Barkai, 2008. Fractional langevin equation: Over-damped, under-damped and critical behaviors. <http://arxiv.org/abs/0802.3777>.
- Cveticanin, L., 2004. Vibrations of the nonlinear oscillator with quadratic nonlinearity. *Physica A Stat. Mech. Appl.*, 341: 123-135.
- Cveticanin, L., 2009. Oscillator with strong quadratic damping force. *Publ. Del'Institut Math.*, 85: 119-130.
- Galucio, A.C., J.F. Deu and F. Dubois, 2006. On the use of fractional derivative operators to describe viscoelastic damping in structural dynamics-FE formulation of sandwich beams and approximation of fractional derivatives by using the Ga scheme. *Derivation Fractionnaire en Mecanique-Etat-de-L'art et Applications*, CNAM, Paris, 17th November, 2006.
- Mickens, R.E., 2002. Analysis of non-linear oscillators having non-polynomial elastic terms. *J. Sound Vib.*, 255: 789-792.
- Naber, M., 2009. Linear fractionally damped oscillator. *Int. J. Differ. Equations*, 2010: 12-12.
- Nayfeh, A.H. and D.T. Mook, 1979. *Nonlinear Oscillations*. John Wiley and Sons, New York.
- Pilipchuk, V.N., 2007. Strongly nonlinear vibrations of damped oscillators with two nonsmooth limits. *J. Sound Vib.*, 302: 398-402.
- Podlubny, I., 1999. *Fractional Differential Equations*. Vol. 198, Academic Press, San Diego, Calif, USA., ISBN: 9780125588409, Pages: 340.
- Timoshenko, S., D.H. Young and W. Weaver, 1974. *Vibration Problems in Engineering*. 4th Edn., John Wiley and Sons, New York, ISBN: 0471873152.
- Waluya, S.B. and W.T. van Horssen, 2003. On the periodic solutions of a generalized non-linear vander pol oscillator. *J. Sound Vib.*, 268: 209-215.
- Zaslavsky, G.M., A.A. Stanislavsky and M. Edelman, 2006. Chaotic and pseudochaotic attractors of perturbed fractional oscillator. *Chaos*, 16: 20-20.