Research Journal of Applied Sciences 9 (12): 1134-1142, 2014
ISSN: 1815-932X

Isochromes in Conoscopic Patterns of Uniaxial Crystals under Normal's Random Orientation in Relation to Optical Axis

[1]A.I. Kolesnikov, [1]I.A. Kaplunov, [1]S.A. Tretyakov, [1]R.M. Grechishkin,
[1]K.A. Morozova, [2]V.Y. Molchanov and [3]A.A. Kolesnikov
[1]Tver State University, 170100 Tver, Russia
[2]National University of Science and Technology "MISIS", 119049 Moscow, Russia
[3]Business Control Systems Ltd., 398001 Lipetsk, Russia

Abstract: Researchers establish the equation of the curve described by the wave vector of the extraordinary wave on the second surface of the plane parallel piece cut from a uniaxial crystal with optical axis's random orientation in relation to the normal to the piece, under the ray's rotation around the normal (this ray descents at any constant angle onto the first surface). This helps to obtain and analyze, without any simplications, used in known works, the equation describing the form of isochromes of any order in conoscopic patterns of uniaxial crystals. The calculated isochromes patterns for some angles between axis and normal which have the most obvious difference from the ones which were prognosticated before are verified experimentally on paratellurite crystal with special orientations of two pairs of faces.

Key words: Conoscopy method, linearly polarized light isochromes, uniaxial crystals, paratellurite crystals, path difference, wave vector, ordinal wave, extraordinary wave, Frenel equation

INTRODUCTION

Over the last years, we observe growth of interest in an old and seemingly elaborate method of research of anisotropic crystals in converging and divergent linearly polarized light a conoscopy method. Some works (Wen *et al*., 1995; Mamedov *et al*., 2003; Rudoi *et al*., 2003) cover even such delicate aspects of the method as optical activity's influence on the type of isochromes of uniaxial and biaxial crystals. The method is practically interesting because of new possibilities of its using in studying of optical effects in nanosized structures (Saito *et al*., 2000) in mineralogy (Punin and Shtukenberg, 2005) in holography (Sirat and Psaltis, 1985). Conoscopy is used for defining of category of crystal (low or medium) and its optical sign (Stoiber and Morse, 1972). In cases when the perfect crystal structure is known, the method can be used for checking of accuracy of orientation of faces which are orthogonal in relation to optical axis of an uniaxial crystal or bisector of an angle between the biaxial ones' axes (Born and Wolf, 1985; Moskalev *et al*., 2002). The method has the technical significance for polarization compensators and polarization interferometers (Moskalev *et al*., 2002). Data on angles of anomalous biaxiality, obtained with the help of conoscopy, allow to estimate mechanical stresses in an uniaxial crystal which lead to distortion of its optical indicatrix (Sirotin and Shaskolskaya, 1975). The most important practical meaning of the Conoscopy Method lies in the possibility to find optical inhomogeneities in crystals via analysis of forms of isochromes lines of equal path difference between ordinary and extraordinary rays in conoscopic patterns (Punin and Shtukenberg, 2005; Kolesnikov *et al*., 2013a). Isogyres in the form of dark areas bear less information, obstruct observation of isochromes and there are special methods for elimination of them (Stoiber and Morse, 1972). Anomalous biaxiality and also all isochromes lines' departure from theoretical form speak for large optical anomalies to a scale of the whole crystal. Fractures on an isochrome or series of such fractures which correspond to minor wavefront distortions in the order of wave length deciles are well observed in conoscopic patterns and allow to find optical anomalies and estimate the number of variations of refraction indices located in the crystal's minute volumes. In its turn, analysis of optical anomalies found with the help of the conoscopy method allows to correct the crystal growing technology to obtain the material's maximum homogeneity. Thus, conoscopy is a quite up-to-date, developing, multifunctional and delicate technique of diagnosing, defectoscopy and metrology of monocrystals used in optics.

Corresponding Author: A.I. Kolesnikov, Tver State University, 170100 Tver, Russia

For correct interpretation of conoscopic patterns form, we need a precise physical theory and corresponding precise mathematical tools allowing to estimate form of isochromes of any order for any crystal with known structure, sizes, principal values of refraction indices and orientation of faces which are parallel with each other in relation to axis (axes). Meanwhile, analysis of known works connected with the method theory shows that even in case of uniaxial crystals scientists still use fundamental equations established earlier with certain simplications (Born and Wolf, 1985). When trying to describe uniaxial crystals isochromes forms in more detail, they used further simplications (Rudoi *et al*., 2003; Moskalev *et al*., 2002; Sirotin and Shaskolskaya, 1975; Shuvalov *et al*., 1981; Bajor *et al*., 1998; Konstantinova *et al*., 1995, 2003) and that gave isochromes forms which did not correspond to conscopic patterns which were observed in reality in course of studying of crystals of paratellurite and lithium niobate, a high optical and structural quality of which had been ascertained in other ways.

The purposes of this research were to establish a precise (without any simplications) equation for isochromes of any order in plane of observation of uniaxial crystal conoscopic patterns under random orientation of a normal to piece's surfaces in relation to optical axis and also to verify obtained correlations experimentally.

ANALYSIS OF APPROXIMATE METHODS OF ISOCHROMES FORMS ESTIMATION

It seems that the first detailed and the most precise (among the approximate ones) mathematical description of forms of isochromes in uniaxial crystal conoscopic patterns was given by Born and Wolf (1985) in research. Calculation of phase difference between wave fronts of ordinary and extraordinary rays is illustrated by Born and Wolf (1985) by the figure, given below with only some designations changed (Fig. 1). Here SA, AB_0, AB_e are wave normals to an incident and two refracted waves at the point A, Cr is a crystal.

Equations below have the following designations: λ is a wave length in the first medium (air), $\lambda_0 = \lambda/n_0$, $\lambda_e = \lambda/n_e$ are lengths of both refracted waves, α, β_0 and β_e are correspondingly an angle of incidence and two angles of refraction, θ is the average value of β_0 and β_e, L is a lens (or a projection system), F is a point in the focal plane of a lens (or a projection system) where ordinary and extraordinary waves interfere. Rays come out of a crystal being parallel with each other and a wave of the normal to an incident wave with phase difference:

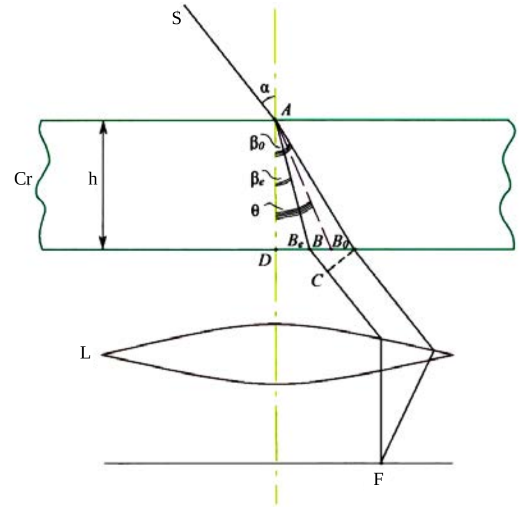

Fig. 1: To estimation of phase difference gained by two waves passed through a parallel-sided plate made from a uniaxial positive crystal (Born and Wolf, 1985)

$$\delta = 2\pi\left[\frac{AB_e}{\lambda_e} + \frac{B_eC}{\lambda} - \frac{AB_0}{\lambda_0}\right] \quad (1)$$

Where:

$$AB_0 = \frac{h}{\cos\beta_0},\ AB_e = \frac{f}{\cos\beta_e}\ \text{and} \quad (2)$$
$$B_eC = B_eB_0\sin\alpha = h\sin\alpha(tg\beta_0 - tg\beta_e)$$

After substitution of Eq. 2 into 1 and applying the refraction law ($\sin\alpha/\lambda = \sin\beta_o/\lambda_e = \sin\beta_o/\lambda_o = \sin\alpha/\lambda$) by Born and Wolf (1985), we obtain an absolutely precise expression for phase difference:

$$\delta = \frac{2\pi h}{\lambda}(n_e\cos\beta_e - n_0\cos\beta_0) \quad (3)$$

In this expression, we know (h, λ, n_0) or we can easily calculate ($\cos\beta_0$) all values except for refraction angle β_e and extraordinary refraction index corresponding to it. Moreover, no known work gives precise expressions for n_e depending on the angle of incidence of α for the set angle ψ between the normal to surface of the crystal or its optical axis. Frenel equation allows to easily calculate refraction index n_e of the extraordinary wave already spreading in the known direction in the crystal. But in the case this direction itself should be defined. By Born and Wolf (1985) at first, we apply approximation lying in the fact that we introduce some average angle θ between refraction angles β_0 and β_e. The point B in Fig. 1 shows the exit of such average normal of wave fronts on the crystal's lower surface. Altogether, Born and Wolf (1985) uses only two approximations, therefore conclusions presented in this work are the most real ones and have no expressly erratic provisions. All other works connected with the conoscopy of uniaxial (let alone biaxial) crystals (Wen *et al*., 1995; Mamedov *et al*., 2003; Rudoi *et al*., 2003; Moskalev *et al*., 2002; Sirotin and Shaskolskaya, 1975; Shuvalov *et al*., 1981; Konstantinova *et al*., 1995, 2003) except for Kolesnikov *et al*. (2013a) and Bajor *et al*. (1998) use additional approximations and as a result there are conclusions distorting true form of isochromes and even making possible gross errors in the course of estimating of optical homogeneity, optical indicatrix, values of mechanical stresses and their distribution in crystals.

By Born and Wolf (1985) Eq. 3 is replaced by an approximate expression on the basis of insignificance of difference n_e-n_0 in comparison with n_e and n_0 and after that the phase difference δ is equal to:

$$\delta = \frac{2\pi h}{\lambda \cos\theta}(n_e\text{-}n_0) \quad (4)$$

where, θ is an average value of angles β_0 and β_e and value $h/\cos\theta$ is an average path of two waves in a crystal and after multiplying by (n_e-n_0) it is a corresponding optical path difference.

All isochromes by Born and Wolf (1985) gain, creating around the point A of the surface of a constant phase difference $\delta(h, \theta)$ = const wherefore, a polar radius is used:

$$\rho = AB = \frac{h}{\cos\theta} \quad (5)$$

and also an approximate angle υ which AB generates with bearing of the optical axis. In an uniaxial crystal refraction indices corresponding to the direction of the wave normal generating the angle υ with the optical axis are connected, according to Born and Wolf (1985), via formula:

$$\frac{1}{{n_0}^2}-\frac{1}{{n_e}^2} = \left(\frac{1}{{N_0}^2}-\frac{1}{{N_e}^2}\right)\sin^2\upsilon \quad (6)$$

where, $N_0 = n_0$ and N_e are principal values of refraction indices of ordinary and extraordinary waves. Here by Max (1985), we make the second approximation. Due to insignificance of difference of refraction indices n_e and n_0 in comparison with their values, Eq. 6 is replaced by an approximate one:

$$n_e\text{-}N_0 = (N_e\text{-}N_0)\sin^2\upsilon \quad (7)$$

After substitution of Eq. 5 into 4, taking into account (Eq. 6), we obtain the formula for phase difference ρ:

$$\delta = \frac{2\pi\rho}{\lambda}(N_e\text{-}N_0)\sin^2\upsilon = \frac{2\pi h}{\lambda\cos\theta}(N_e\text{-}N_0)\sin^2\upsilon \quad (8)$$

which includes now two approximate values θ and υ refraction angle and angle between the optical axis and the wave fronts average normal correspondingly. Then, we write the equation for the constant phase difference surface:

$$\rho\sin^2\upsilon = c(c = \text{const}) \quad (9)$$

Then, we introduce a Cartesian coordinate system with axis z, directed along the optical axis and according to Eq. 9, constant phase difference surfaces are estimated with the equation:

$$(x^2+y^2)^2 = c^2(x^2+y^2+z^2) \quad (10)$$

Then by born and Wolf (1985), it is noted that all isochromes can be defined using sections of the surface (Eq. 10) as planes located at different distances h from the coordinates origin. Analysis of the corresponding intersectional curves form by Born and Wolf (1985) is of a qualitative character. It contains a correct statement, according to which in case of coincidence of the normal to faces with the optical axis isochromes take the form of circles. Less precise conclusions lie in that fact that if the normal to faces generates a slight angle with the optical axis isochromes compress and evolve into ellipses and if this normal generates a wide angle with the optical axis isochromes verge towards hyperbolas. The formula for the equal phases surface, determined with two approximations is a surface of the fourth order and in case of planes intersecting it, curves in general cannot be curves of the second order (Shuvalov *et al*., 1981) includes the same approximate expression for the phase difference δ between ordinary and extraordinary waves as (Born and Wolf, 1985) and then researchers made two more approximations. Due to littleness of refraction angles it was taken that $1/\cos\Theta \approx 1+/2\sin^2\theta$. The next approximation lay in decomposition of a square root into series in the expression for polar radius $\rho = \sqrt{x^2+y^2+z^2}$; z = h after introduction of coordinates system xyz in which plane xy is a plane of observing of a conoscopic pattern. As a result of specified approximations by Shuvalov *et al*. (1981), we see the expression for phase difference:

$$\delta=\frac{2\pi}{h\lambda}(N_e\text{-}N_0)\left[x^2\left(\cos^2\psi+\frac{1}{2}\sin^2\psi\right)+y^2\left(\mathrm{Cos}^2\psi+\frac{1}{2}\sin^2\psi\right)\text{-}hx\sin^2\psi+h^2\sin^2\psi\right] \quad (11)$$

where, ψ is an angle between the optical axis and the normal to the crystal. The surface of equal phases δ = const in the work Shuvalov *et al.* (1981) as distinct from (Max, 1985) is a surface of not the fourth order but of the second one. Analysis of Eq. 10 shows that under $\psi = 0$ isochromes are as in the (Born and Wolf, 1985), circles. Under $tg\psi<\sqrt{2}$ isochromes should be ellipses and under $tg\psi>\sqrt{2}$ they are hyperbolas. The matter of changing the form of isochromes depending on their order, i.e., when changing δ for conoscopic pattern of one and the same crystal neither by Shuvalov *et al.* (1981) nor by Born and Wolf (1985) was not considered.

Methods of estimation of isochromes forms used in researches (Wen *et al*., 1995; Mamedov *et al*., 2003; Rudoi *et al*., 2003; Moskalev *et al*., 2002; Sirotin and Shaskolskaya, 1975; Konstantinova *et al*., 1995) do not differ from the method described by Shuvalov *et al.* (1981), neither in formulas (with an accuracy to designations) nor in conclusions. Closing the review of publications which give estimations of forms of isochromes in uniaxial crystals conoscopic patterns, we can draw the following conclusions:

- There are no precise formulas for refraction angle of extraordinary wave β_e and its refraction index n_e in case of obtaining expression for path difference Δ between extraordinary and ordinary waves in any known work
- Instead of them scientists use approximate relationships for average refraction angle and average angle between the optical axis and refracted waves
- Depending on amount and character of approximations in the most mathematically exact work (Born and Wolf, 1985) equal phases surfaces are the fourth order surfaces, the rest of the works which mention them specify them as the second order surfaces. Nonetheless, all works mention only curves of the second order (circles, ellipses and hyperbolas) as possible isochromes as lines of intersection of equal phases' surfaces by planes
- No work contains analysis of change of forms of isochromes depending on their order in a crystal's conoscopic pattern
- No work represents such experimentally obtained conoscopic patterns which support isochromes forms which are theoretically prognosticated in this work. There are only two, the most trivial cases which are exceptions: an optical axis is perpendicular to crystal's faces; an optical axis lies in a plane which is parallel to crystal's faces

DERIVATION OF A PRECISE EQUATION FOR UNIAXIAL CRYSTAL'S ISOCHROMES

Let us write Frenel equation for uniaxial crystal in the classic form (Sirotin and Shaskolskaya, 1975):

$$(n^2\text{-}N^2)\left\{n^2\left(k_1^2+k_2^2\right)\text{-}N_0^2N_e^2\right\}=0 \quad (12)$$

Where:
n = A refraction index
k_i = Directional cosines of a wave vector of a refracted wave in a crystallographic coordinates system

Figure 2 shows crystallographic coordinates system xyz in a uniaxial crystal with optical axis z which generates angle $\psi\neq0$ with normal vector $\bar{m}$ which are axis of cone of rays descending on to the crystal at the angle α. Behind the crystal we see a lens (projection of lens L) with focal distance f at which we see observation plane (screen) with coordinates system X'O'Y' and point A'(X', Y') for which we calculate path difference between ordinary and extraordinary waves. Also, the figure shows wave vector of incident wave $\bar{l}$, wave vector of extraordinary wave, $\bar{k}$, X, Y coordinates of point A of departure of extraordinary wave on the crystal's second surface in the transformed coordinates system XYZ.

Due to the fact that our interest is in the extraordinary wave direction, let us take n as the refraction index of extraordinary wave $n = n_e = \sin\alpha/\sin\beta$ and in (Eq. 12) the expression in curly brackets = 0. Then, using forms $\sin^2\alpha = 1\text{-}\cos^2\alpha$; $\sin^2\beta_e = 1\text{-}\cos^2\beta_e$, we obtain:

$$N_0^2N_e^2(1\text{-}\cos^2\beta_e)=(1\text{-}\cos^2\alpha)\left[N_0^2\left(k_1^2+k_2^2\right)+N_e^2k_3^2\right] \quad (13)$$

and after that expressing $\cos^2\alpha$ and $\cos^2\beta_e$ via scalar products of vectors $\bar{m}\bar{l}$ and $\bar{m}\bar{k}$ correspondingly, we obtain Frenel equation in the following form:

$$\frac{1\text{-}(m_1k_1+m_2k_2+m_3k_3)^2}{1\text{-}(m_1l_1+m_2l_2+m_3l_3)^2}=\frac{k_1^2+k_2^2}{N_e^2}+\frac{k_2^2}{N_0^2} \quad (14)$$

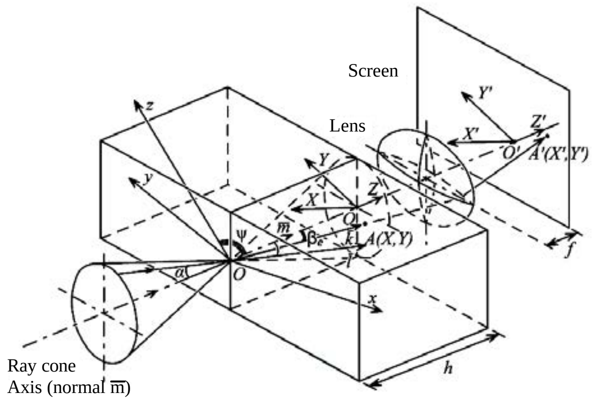

Fig. 2: Scheme explaining mutual arrangement of uniaxial crystal, its crystallographic coordinates system xyz, coordinates system XOY on the crystal's second surface, projection system (lens) and coordinates system X'O'Y' in screen plane screen

Coordinates of point of intersection A of extraordinary wave wavevector with output (second) plane of the crystal XOY are found in the second coordinates system XYZ where axis Z is a continuation of the normal $\bar{m}$ to the first surface and axis Y is a projection of optical axis z on to the second plane.

Reverse transition from the coordinates system XYZ to crystallographic system xyz is carried out with the help of two turns at first a turn to the angle ψ up to alignment of axis Z with axis z and then a turn around the axis z to some angle γ up to alignment of two other axes with axes x and y.

Directional cosines M_i, L_i and K_i (i = 1, 2, 3) of normal, incident wave and refracted wave in the coordinates system XYZ equal to:

$$M_1 = 0;\ M_2 = 0;\ M_3 = 1$$
$$L_1 = l;\ L_2 = \sqrt{1-l^2-\cos^2\alpha};$$
$$L_3 = \cos\alpha (l\text{-cosine of angle between } \bar{l} \text{ and } z)$$
$$K_1 = \frac{x}{\sqrt{X^2+Y^2+h^2}};\ K_2 = \frac{Y}{\sqrt{X^2+Y^2+h^2}};\ K_3 = \frac{h}{\sqrt{X^2+Y^2+h^2}} \quad (15)$$

where, h is the crystal's width. Matrix T_1 of the first turn to angle ψ is given by:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \quad (16)$$

After this turn directrices of the cosine in a new Intermediate Coordinates System M_{1i}, L_{1i} and K_{1i} (i = 1, 2, 3) equal to:

$$M_{11} = 0;\ M_{12} = -\sin\psi;\ M_{13} = -\cos\psi;$$
$$L_{11} = l;\ L_{12} = \sqrt{1-l^2-\cos^2\alpha}\cos\psi - \cos\alpha\cos\psi;$$
$$L_{13} = \sqrt{1-l^2-\cos^2\alpha}\sin\psi + \cos\alpha\cos\psi;$$
$$K_{11} = \frac{X}{\sqrt{X^2+Y^2+h^2}};\ K_{12} = \frac{Y\cos\psi}{\sqrt{X^2+Y^2+h^2}} - \frac{h\sin\psi}{\sqrt{X^2+Y^2+h^2}};$$
$$K_{13} = \frac{Y\sin\psi}{\sqrt{X^2+Y^2+h^2}} - \frac{h\cos\psi}{\sqrt{X^2+Y^2+h^2}}$$

After the second turn to the angle γ up to alignment of two other axes of the coordinates system with axes of crystallographic coordinates system, matrix of which T_2 is given by:

$$T_2 = \begin{bmatrix} \cos\gamma & -\sin\psi & 0 \\ \sin\psi & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

We finally obtain values of directional cosines of normal, incident and refracted extraordinary wave m_i, l_i and k_i (i = 1, 2, 3) besides k_i are expressed now via coordinates of the point of intersection of extraordinary wave vector with the crystal second surface:

$$m_1 = \sin\psi\sin\gamma;\ m_2 = -\sin\psi\cos\gamma;\ m_3 = \cos\psi;$$
$$l_1 = l\cos\gamma - \left(\sqrt{1-l^2-\cos^2\alpha}\cos\psi - \cos\alpha\sin\psi\right)\sin\gamma;$$
$$l_2 = l\sin\gamma + \left(\sqrt{1-l^2-\cos^2\alpha}\cos\psi - \cos\alpha\sin\psi\right)\cos\gamma;$$
$$l_3 = \sqrt{1-l^2-\cos^2\alpha}\sin\psi + \cos\alpha\cos\psi;$$
$$k_1 = \frac{X\cos\gamma}{\sqrt{X^2+Y^2+h^2}} - \left(\frac{Y\cos\psi}{\sqrt{X^2+Y^2+h^2}} - \frac{h\sin\psi}{\sqrt{X^2+Y^2+h^2}}\right)\sin\gamma; \quad (18)$$
$$k_2 = \frac{X\sin\gamma}{\sqrt{X^2+Y^2+h^2}} + \left(\frac{Y\cos\psi}{\sqrt{X^2+Y^2+h^2}} - \frac{h\sin\psi}{\sqrt{X^2+Y^2+h^2}}\right)\cos\gamma;$$
$$k_3 = \frac{Y\sin\psi}{\sqrt{X^2+Y^2+h^2}} + \frac{h\cos\psi}{\sqrt{X^2+Y^2+h^2}}$$

Equation 18 for directional cosines we substitute into Frenel equation written in the form of Eq. 14 and after necessary transformations we obtain an equation of formally 4th degree which is not presented because of extreme bulkiness. Now, it has no introduced unknown intermediate values l and γ. A corresponding polynomial in two letters X and Y can be decomposed into two multiplicand, one of which is multiplier ($X^2+Y^2+h^2$) by which the reducing may be performed because equation $+Y^2+h^2 = 0$ has no real solutions. After reducing we have the following equation of the second order:

$$A_1x^2+A_2y^2+A_3y+A_4=0 \quad (19)$$

Where:

$$A_1=N_0^2\left(N_e^2\text{-}\sin^2\alpha\right),$$
$$A_2=N_e^2N_0^2\text{-}\sin^2\alpha\left(N_e^2\sin^2\psi+N_0^2\cos^2\psi\right),$$
$$A_3=2h\sin\psi\cos\psi\sin^2\alpha\left(N_0^2\text{-}N_e^2\right),$$
$$A_4=\text{-}h^2\sin^2\alpha\left(N_0^2\sin^2\psi+N_e^2\cos^2\psi\right)$$

Equation 19 should be regarded as an equation of a curve described by wave vector $\bar{k}$ of an extraordinary wave (more exactly, its continuation) on the crystal's output surface under incident ray rotation at constant angle α around the normal to the crystal. In case when $\psi = 0$ (the optical axis coincides with the normal to the crystal) Eq. 19 is an equation of a circle:

$$X^2+Y^2=\frac{N_e^2h^2\sin^2\alpha}{N_0^2\left(n_e^2\text{-}\sin^2\alpha\right)} \quad (20)$$

In all other cases ($0<\psi\leq\pi/2$) we have an equation of an ellipsis. Analysis of Eq. 19 shows that ellipses centres do not pass through the coordinates origin, except for the case when the optical axis is orthogonal to the normal ($\psi = \pi/2$). In the last case ellipses have a maximum eccentricity.

Coordinates X and Y from Eq. 19 are used for writing of the formula of path difference Δ and then we express Δ via coordinates X' and Y' of the point of intersection of ordinary and extraordinary waves on the conoscopic pattern observation plane. Let us rewrite Eq. 3, replacing phase difference δ with path difference Δ for isochromes of m order:

$$\Delta = m\lambda = h(n_e\cos\beta_e\text{-}N_0\cos\beta_0) \quad (21)$$

Using refraction laws ($\sin\alpha/\sin\beta_0 = N_0$; $\sin\alpha/\sin\beta_e = n_e$), let us bring Eq. 24 to the following form:

$$\Delta=m\lambda=h\left(\frac{\sin\alpha\cos\beta_e}{\sqrt{1\text{-}\cos^2\beta_e}}\text{-}\sqrt{N_0^2\text{-}\sin^2\alpha}\right) \quad (22)$$

To write the value of $\cos\beta_e$ let us use coordinates X and Y of output of the extraordinary wave vector on to the crystal's second surface with the help of a trivial relationship $\cos\beta_e = h/\sqrt{X^2+Y^2}$ and after substitution into Eq. 22 and simple transformations we obtain a precise expression for the path difference:

$$\Delta = m\lambda = \frac{(X^2+Y^2+h^2)\sin\alpha}{\sqrt{X^2+Y^2}}+\sin\alpha\left(\frac{h\sin\alpha}{\sqrt{N_0^2\text{-}\sin^2\alpha}}\text{-}\sqrt{X^2+Y^2}\right)\text{-}\frac{hN_0^2}{\sqrt{N_0^2\text{-}\sin^2\alpha}} \quad (23)$$

Axes X' and Y' of the coordinates system on the screen showing isochromes are chosen in such a way that they are parallel to axes X and Y on the crystal's second surface. Therefore, the angles generated with the axes of of the section OA and O'A', sketched from the coordinates system origins to points A and A' of outputs of extraordinary wave vector on the crystal second surface and refracted angle on the screen located in focal plane are equal. Coordinates of points A and A' in both coordinates system are connected by Eq. 24:

$$Y=Y'\frac{\sqrt{X^2+Y^2}}{\sqrt{(X')^2+(Y')^2}};X=X'\frac{\sqrt{X^2+Y^2}}{\sqrt{(X')^2+(Y')^2}} \quad (24)$$

Besides, $\sqrt{(X')^2+(Y')^2}/f = tg\alpha$ wherefrom formulas are derived:

$$X^2+Y^2=\frac{X^2f^2}{(X')^2ctg^2\alpha};$$
$$X=\frac{X'ctg\alpha\sqrt{X^2+Y^2}}{f};\ Y=\frac{Y'ctg\alpha\sqrt{X^2+Y^2}}{f} \quad (25)$$

Having noted that ΔX and Y are included into the path difference equation only as sum of squares and having denoted this sum as B^2, we rewrite both Eq. 22 and 18, obtaining the two equations system:

$$\Delta=m\lambda=\frac{(B^2+h^2)\sin\alpha}{B}+\sin\alpha\left(\frac{h\sin\alpha}{\sqrt{N_0^2\text{-}\sin^2\alpha}}\text{-}B\right)-\frac{hN_0^2}{\sqrt{N_0^2\text{-}\sin^2\alpha}} \quad (26)$$

$$\left\{\frac{ctg^2\alpha}{f^2}[A_1(X')^2]\right\}B^2+\left(\frac{A_3(Y')ctg\alpha}{f}\right)B+A_4=0 \quad (27)$$

When solving quadratic relative B Eq. 27, let's take plus sign before the root (because $X^2+Y^2\geq0$) and substitute the found value of B into Eq. 26 the path difference equation after which we obtain at last the uniaxial crystal isochromes equation. As a result of necessary transformations Eq. 26 gains its final form (for obvious convenience of recording, prime marks of coordinates X' and Y' are now omitted):

$$\left(N_0^2\text{-}N_e^2\right)\left[\frac{Y\sin\psi}{\frac{m\lambda\sqrt{X^2+Y^2+f^2}}{h}+\sqrt{N_0^2(X^2+Y^2+f^2)\text{-}X^2\text{-}Y^2}}\cos\psi\right]^2 = N_0^2\left[\frac{X^2+Y^2\text{-}N_e^2(X^2+Y^2+f^2)}{\left(\frac{m\lambda\sqrt{X^2+Y^2+f^2}}{h}+\sqrt{N_0^2(X^2+Y^2+f^2)\text{-}X^2\text{-}Y^2}\right)^2}+1\right] \quad (28)$$

The uniaxial crystal isochromes equation, obtained without any approximations in the form Eq. 28, specifically is the most space-saving and convenient for practical computer calculations of form of isochromes of any order under random angles ψ between the optical axis and the normal to crystal's faces. For final solution of an important theoretical problem of order of curves isochromes of uniaxial crystals for general case of a random angle ψ between the optical axis and the normal, Eq. 28 should be brought to the canonical form in which on the left side there is a polynomial in two letters X and Y:

$$\begin{aligned}&B_1X^8+B_2X^6Y^2+B_3X^6+B_4X^4Y^4+B_5X^4Y^2+\\&B_6X^4+B_7X^2Y^{-6}+B_8X^2Y^4+B_9X^2Y^2+B_{10}X^2+\\&B_{11}Y^8+B_{12}Y^6+B_{13}Y^4+B_{14}Y^2+B_2Y^2+B_{15}=0\end{aligned} \quad (29)$$

where, B_i is coefficients including values depending on properties and sizes of a crystal, parameters of the optical system and isochromes order. These expressions are very cumbersome and due to this here they are omitted.

Thus, in general case of a random angle ψ between the optical axis and the normal, a precise equation for uniaxial crystal isochromes is an equation of not the second and not even the fourth order as it appears from the known works but of the eighth order (Kolesnikov *et al.*, 2013b). This is especially important for studying of crystals optical quality with the help of conoscopy method. Isochromes forms, appearing under certain angles ψ and other parameters of the test which do not correspond to any curves of the second order, result from the precise theory and should not be regarded as evidence of gross breaches of crystal optical indicatrix.

In case of coincidence of optical axis direction with the normal to crystal (ψ = 0) Eq. 28 gives circles to isochromes; these circles have radii R_m which depend on order m and other test parameters in the following way:

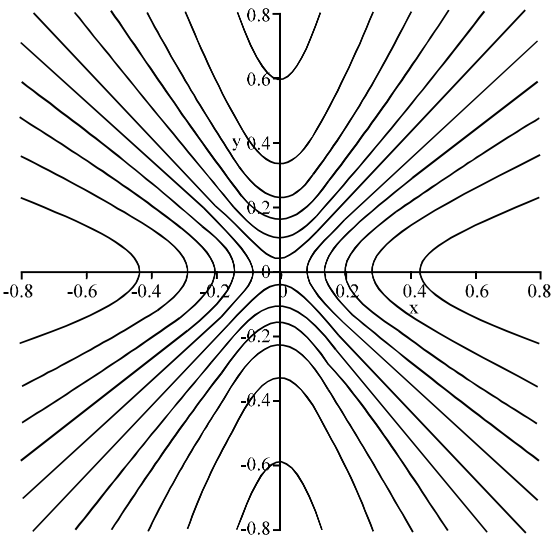

Fig. 3: Isochromes in a conoscopic pattern of a crystal which is cut parallely to the optical axis (ψ = 90°), estimated according to Eq. 28 and 31

$$R_m = fN_e\left\{\frac{\left[\left(4h^2m\lambda N_0^3N_e^2\text{-}\lambda^3m^3N_e^2\text{-}h^2\lambda mN_0^2\text{-}h^2\lambda mN_e^2+\sqrt[2]{h^4N_0^2\left(h^2N_0^4\text{-}2h^2N_0^2N_e^2+h^2N_e^4+\lambda^2m^2N_e^2\right)}\right)\right]m\lambda}{\lambda^4m^4N_e^4\text{-}4h^2\lambda^2m^2N_0^2N_e^4+2h^2\lambda^2m^2N_0^2N_e^2+2h^2\lambda^2m^2N_e^4+h^4N_0^4\text{-}2h^4N_0^2N_e^2+h^4N_e^4}\right\}^{1/2} \quad (30)$$

There is only one of the known works by Landsberg (2003) which correctly, though without mathematical manipulations, notes that contrary to established opinion in case when an optical axis is orthogonal to another optical axis (ψ = 90°) isochromes are not hyperbolas. Landsberg (2003) says the following words about these curves: "almost hyperbolas". Analysis of Eq. 28 gives a comprehensive insight into isochromes for case ψ = 90°: after completing necessary transformations we obtain an equation of the fourth order as follows:

$$B_1x^4+B_2y^4+B_3x^2y^2+B_4x^2+B_5y^2+B_6=0 \quad (31)$$

where, coefficients B_i in the polynomial on the left side are quite cumbersome expressions. Figure 3 shows isochromes in a conoscopic pattern of a crystal which is cut parallely to the optical axis (ψ = 90°), estimated according to Eq. 31 under the following parameters of the crystal and the optical arrangement: h = 0.02 m; N_0 = 2.2931; N_e = 2.452; $\lambda = 5.46\times10^{-7}$ m; f = 0.2 m.

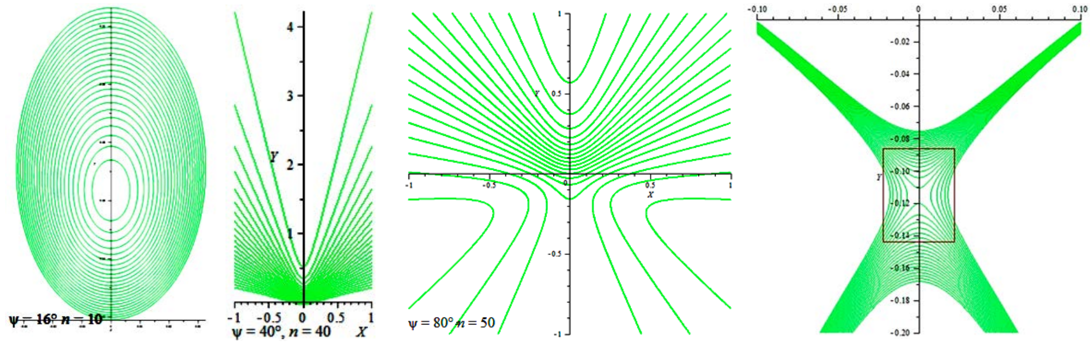

Fig. 4: Isochromes in uniaxial crystals conoscopic patterns, estimated according to Eq. 27 under different angles ψ between the normal and the optical axis (n space between orders)

Notwithstanding the formal resemblance to hyperbolas isochromes are not hyperbolas in fact they are curves of the fourth order.

Computer analysis of Eq. 28 for borderline cases when the angle between the axis and the lines' normal is within 0-90°, shows that under one and the same value of ψ in a conoscopic pattern there can be isochromes of different orders of completely different forms, including closed and unclosed ones. Corresponding regularities of isochromes forms evolution under increase of the angle between the axis and the normal from 0-90° are shown in Fig. 3 which includes not only values of ψ but also space n between calculated isochromes orders (Fig. 4).

EXPERIMENTAL VERIFICATION OF DERIVATIVE RELATIONS

For obtaining conoscopic patterns, we used the optical arrangement including laser, polarizer, lens which transformed laser beam into a cone one, analyzer, projecting lens and rear-projection screen, behind which there were a digital camera for recording of isochromes images. Experimental conoscopic patterns of paratellurite crystals for angles between the optical axis and the normal ψ = 16° and ψ = 84° are shown in Fig. 5.

Thus, calculated and experimentally verified forms of isochromes coincide within the limits of noise influence in real images connected with nonideality of optical arrangement elements and laser beam structure with speckles and with minor inhomogeneities in internal volumes of the crystal and its surfaces.

RESUME

Only in one case when the axis coincides with the normal, uniaxial crystals isochromes are curves of the second order. In all other cases, they are curves of at least the fourth order, described by the eighth order equation in some cases having reducible solutions. These curves look like ellipses, parabolas or hyperbolas merely occasionally but in fact they are curves of the fourth order. In addition, the form of curves-isochromes depends not only on the angle ψ but also on maxima order. A well-known assertion that under $\psi<\text{arctg}\sqrt{2}$ isochromes are ellipses and under $\psi>\text{arctg}\sqrt{2}$ isochromes are hyperbolas is wrong.

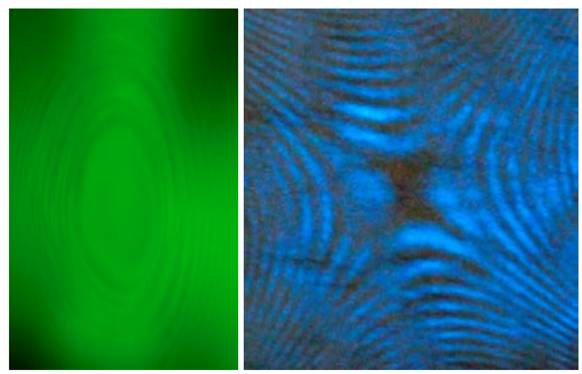

Fig. 5: Experimental conoscopic patterns of paratellurite crystals with faces, normals to which generate angle ψ = 16° and ψ = 84° with optical axis [001]

CONCLUSION

High sensitivity of form and location of isochromes in uniaxial crystals conoscopic patterns to minor changes of their physical properties, state and structure makes it promising to elaborate theoretical and technical aspects of the conoscopy method. To create mathematical tools allowing the most accurate estimation of location and form of isochromes under any orientation of the normal to optical faces in relation to the crystal's optical axis, we established the equation of the curve described by the vector of the extraordinary wave on the output surface. Without approximations which are usually made, it helped to establish the equation for isochromes of any order in the uniaxial crystal conoscopic pattern observation plane. Because it is of the eighth not of the second or the fourth order, presence of isochromes in form of curves of order which is higher than the second one is physically founded.

ACKNOWLEDGEMENTS

Research has been performed in Tver State University as a part of implementation of the Federal Target Program "Research and development works on priority orientations of evolution of Russia Scientific Technological Complex for 2014-2020".

REFERENCES

Bajor, A.L., L. Salbut, A. Szwedowski and T. Piatkowski, 1998. Imaging conoscope for investigation of optical inhomogeneity in large boules of uniaxial crystals. Rev. Scient. Instrum., 69: 1476-1487.

Born, M. and E. Wolf, 1985. Principles of Optics. Pergamon Press, New York, pp: 640-648.

Kolesnikov, A.I., R.M. Grechishkin, S.A. Tretiakov, V.Y. Molchanov and A.I. Ivanova, E.I. Kaplunova and E.Y. Vorontsova, 2013a. Laser conoscopy of large-sized optical crystals. IOP Conf. Seri.: Mater. Sci. Eng., Vol. 49. 10.1088/1757-899X/49/1/012037

Kolesnikov, A., R. Grechishkin, O. Malyshkina, Y. Malyshkin, J. Dec, T. Lukasiewicz and A. Ivanova, 2013b. Conoscopic study of strontium-barium niobate single crystals. IOP Conf. Ser.: Mater. Sci. Eng., Vol. 49. 10.1088/1757-899X/49/1/012010.

Konstantinova, A.F., B.N. Grechushnikov, B.V. Bokut and E.G. Valyashko, 1995. [Optical properties of crystals]. Science and Technology, Minsk, pp: 98-105, (In Russian).

Konstantinova, A.F., K.A. Rudoy, B.V. Nabatov, E.A. Evdishchenko, V.I. Stroganov and O.Y. Pikul, 2003. The influence of optical activity on the intensity and polarization parameters of transmitted light in crystals. Crystallogr. Rep., 48: 823-831.

Landsberg, G.S., 2003. Optics. Fizmatlit, Moscow, Pages: 848.

Mamedov, N., N. Yamamoto, Y. Shim, Y. Ninomiya and T. Takizawa, 2003. Extended application of light figures to optically active materials with transversally isotropic dielectric function. Jpn. J. Applied Phys., 42: 5145-5152.

Moskalev, V.A., N.A. Polushkina, V.L. Rudin and I.M. Nagibina, 2002. [Applied Physical Optics]. Vyschaya Shkola, Moscow, pp: 279-286.

Punin, Y.O. and A.G. Shtukenberg, 2005. Anomalous crystal optics of heterogeneous crystals. Crystallogr. Rep., 50: 297-307.

Rudoi, K.A., B.V. Nabatov, V.I. Stroganov, A.F. Konstantinova, L.V. Alekseeva, E.A. Evdishchenko and B.I. Kidyarov, 2003. Conoscopic figures of optically active uniaxial crystals. Crystallogr. Rep., 48: 300-304.

Saito, K., S. Sugawara, J.Q. Guo and A.P. Tsai, 2000. Determination of quasi-crystallographic orientations of Al-Pd-Mn icosahedral phase by means of light figure method. Jpn. J. Applied Phys., 39: 5173-5176.

Shuvalov, L.A., A.A. Urusovskaya and I.S. Zholudev, 1981. Physical Properties of Crystals. In: Modern Crystallography Volume 4, Vainshtein, B.K. (Ed.)., Nauka, Moscow, pp: 350-357.

Sirat, G. and D. Psaltis, 1985. Conoscopic holography. Optics Lett., 10: 4-6.

Sirotin, U.I. and M.P. Shaskolskaya, 1975. Crystallography Basics. Nauka, Moscow, pp: 257-259.

Stoiber, R.E. and S.A. Morse, 1972. Microscopic Identification of Crystals. Ronald Press Co., New York, Pages: 278.

Wen, T.D., Y.S. Raptis, E. Anastassakis, I.J. Lalov and A.I. Miteva, 1995. Interference patterns under normal incidence in birefringent, optically active plane parallel plates. J. Applied Phys. D: Applied Phys., 28: 2128-2134.