Research Journal of Applied Sciences 10 (7): 287-293, 2015
ISSN: 1815-932X


# Discrimination Between Lifetime Distributions with Ratios of Maximized Likelihoods

Amusan Ajitoni Simeon, Zarina Mohd Khalid, Rashidah Ahmad and FadilahYoussuf
Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia, 81310 Skudai, Johor Bahru, Malaysia

**Abstract:** Major problem that often arises in the analysis of lifetime data is how to select the distribution that fits our data better among numerous models that apparently fit the data. The study investigates the procedure of ratio of the maximized likelihoods to discriminate between Weibull, Log-logistic and inverse Gaussian distributions and applies it to fit data on time-to-first-birth after marriage in Nigeria. We discriminate between two distributions at a time starting with Weibull and Log-logistic, then Weibull and inverse Gaussian and finally Log-logistic and inverse Gaussian. Ratios of maximized likelihoods computed for each of these combinations of distributions are all negative when the data set was analyzed. The study concludes by identifying inverse Gaussian distribution as the most suitable distribution to model data on time-to-first-birth in Nigeria having shown preference over Log-logistic which had initially been found to be more suitable than Weibull. The Kolmogorov-Smirnov (K-S) distance between the empirical cumulative distribution function (ecdf) and cumulative distribution function (cdf) of inverse Gaussian is 0.0873, the shortest of all the distributions investigated, points to inverse Gaussian as the most preferred distribution for the data.

**Key words:** Lifetime, distributions, discrimination, selection, ratio, log-likelihoods

## INTRODUCTION

Asymmetric and censoring natures of lifetime data render the use of normality assumptions for its analysis inappropriate. Among the distributions that are frequently used in modeling such positively skewed data are Weibull, Log-logistic and inverse Gaussian distributions. These three distributions are similar in various ways. For example, the hazard rate function of inverse Gaussian distribution has n-shape like those of Weibull and Log-logistic distributions for certain range of parameter values. That is the hazard rate functions of the three distributions can assume an inverted bath-tub shape (Lemeshko *et al*., 2010; Johnson *et al*., 1995) for more details.

Sometimes some distributions with similar properties like those in the foregoing may produce somewhat similar data fit for moderate sample sizes. This poses a problem of choice to researchers since, it is important to select the most appropriate model in order to achieve more precised inferences.

Testing whether some given observations follow a particular distribution is a classical problem and this was studied by Cox (1961, 1962) who developed a testing procedure for two families of distributions. Since then, several studies have been done in discriminating between distributions to fit data for more information, Atkinson (1969, 1970), Bromideh and Valizadeh (2014), Dumonceaux *et al*. (1973), Kundu *et al*. (2005), Kundu and Manglick (2004) and Lemeshko *et al*. (2010).

In this study, the problem of discriminating between two distributions as in each of the following combinations: Weibull and Log-logistic, Weibull and inverse Gaussian and Log-logistic and inverse Gaussian is considered with the sole aim of identifying which of the three distributions fits the data on survival time to first birth. We use likelihood ratio test to achieve this purpose.

## MATERIALS AND METHODS

Now, we describe the discrimination procedure based on a random sample $X = \{x_1, x_2, ..., x_n\}$. Here, we assume that the data have been generated from any of the three distributions, namely; Weibull, Log-logistic or inverse Gaussian. Characteristics of each of the distributions are as described:

**Weibull:** If X is any continuous non-negative random variable representing survival time, then X is Weibull with the parameters α and β denoted by $X \sim W(\alpha, \beta)$ if $X^{\alpha} \sim E(\beta)$. The distribution is characterized by two parameters scale, α and shape, β. Weibull is a generalization of exponential

**Corresponding Author:** Amusan Ajitoni Simeon, Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Skudai, Johor Bahru, Malaysia

but does not assume a constant hazard rate and therefore has a broader application. The model is flexible and has also been found to provide a good description of many forms of survival time data. It is popular in modeling time to event relating to cancer diseases, surgery and so on. The hazard function is monotone increasing when $\alpha>1$ decreasing when $\alpha<1$ and constant when $\alpha = 1$ (Kundu and Manglick, 2004; Lawless, 2003). The probability density function of the Weibull distribution is:

$$f_{WB}(X; \theta_1) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)^{\alpha}, \quad (1)$$
$$X>0, \theta_1 = (\alpha, \beta) \in \Re^1_+ X \Re^1_+ \subset \Re^2$$

where, $\theta_1 = (\alpha, \beta) \in \Re^1_+ X \Re^1_+ \subset \Re^2$ is the parameter space of Weibull distribution. The corresponding likelihood function of the observe sample $x_1, x_2, ..., x_n$ from Weibull is:

$$L_{WB}(\theta_1) = \alpha^n \beta^{-n\alpha} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} \exp\left(-\frac{x_i}{\beta}\right)^{\alpha} \quad (2)$$

The natural logarithm of Eq. 2 popular referred to as log-likelihood function is:

$$l_n(\theta_1) = -n\alpha - n\alpha \ln\beta + (\alpha-1)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} \quad (3)$$

By solving $\partial l_n(\theta_1)/\partial\beta = 0$ from Eq. 3, we have:

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{n} x_i^{\hat{\alpha}}}{n}\right)^{1/\alpha}$$

Similarly, solving $\partial l_n(\theta_1)/\partial\alpha = 0$ we obtain:

$$\hat{\alpha} = \left(\left(\sum_{i=1}^{n} x_i^{\hat{\alpha}} \ln x_i\right)\left(\sum_{i=1}^{n} x_i^{\hat{\alpha}}\right)^{-1} - \frac{\sum_{i=1}^{n} \ln x_i}{n}\right)^{-1}$$

are the maximum likelihood estimates, i.e., of $\alpha$ and $\beta$.

**Log-logistic:** Log-logistic is the distribution of a random variable whose logarithm has logistic distribution. It is a commonly used lifetime distribution in lifetime data analysis like Weibull Model. It has a fairly flexible functional form. Its hazard rate function may be decreasing, increasing or hump-shaped. The Log-logistic distribution is a 2-parameter distribution with parameters $\mu$ and $\sigma$, scale and shape parameters, respectively (Bennett, 1983; Kalbfleisch and Prentice, 2002). The probability distribution function (pdf) of this distribution is given as:

$$f_{LL}(X; \theta_2) = \frac{\exp\left[\frac{\ln x - \mu}{\sigma}\right]}{\sigma x\left(1+\exp\left[\frac{\ln x-\mu}{\sigma}\right]\right)^2}, X>0 \quad (4)$$
$$\theta_2 = (\mu, \sigma) \in \Re^1_+ X \Re^1_+ \subset \Re^2$$

where, $\theta_2 = (\mu, \sigma) \in \Re^1_+ X \Re^1_+ \subset \Re^2$ is the parameter space of the Log-logistic distribution and the corresponding likelihood function of the sample $x_1, x_2, ..., x_n$ is (Dey and Kundu, 2009a):

$$L_{LL}(\theta_2) = \prod_{i=1}^{n} \frac{\exp\left[\frac{\ln x_i - \mu}{\sigma}\right]}{\sigma x_i\left(1+\exp\left[\frac{\ln x_i-\mu}{\sigma}\right]\right)^2}$$
$$= \prod_{i=1}^{n}\left(\frac{1}{\sigma x_i}\right)\prod_{i=1}^{n}\exp\left[\frac{\ln x_i-\mu}{\sigma}\right]\prod_{i=1}^{n}\left(1+\exp\left[\frac{\ln x_i-\mu}{\sigma}\right]\right)^{-2} \quad (5)$$

The natural logarithm of Eq. 5 is the log-likelihood function given as:

$$l_n(\theta_2) = \sum_{i=1}^{n} \begin{array}{l}\log\left(\frac{1}{\sigma x_i}\right) + \sum_{i=1}^{n}\exp\left[\frac{\ln x_i-\mu}{\sigma}\right] \\ +\sum_{i=1}^{n}\log\left(1+\exp\left[\frac{\ln x_i-\mu}{\sigma}\right]\right)^{-2}\end{array}$$
$$l_n(\theta_2) = \frac{n\ln x}{\hat{\sigma}} - n\ln\hat{\sigma} - \frac{n\hat{\mu}}{\hat{\sigma}} - 2\sum_{i=1}^{n}\log\left(1+\exp\left[\frac{\ln x_i-\mu}{\sigma}\right]\right)' \quad (6)$$

$\partial l_n(\theta_2)/\partial\mu = 0; \partial l_n(\theta_2)/\partial\sigma = 0$ solved by maximizing Eq. 6.

**Inverse gaussian:** The inverse Gaussian distribution is a continuous probability distribution also known as Wald distribution. Its many similarities to standard Gaussian makes it popular among scientists in modeling diverse phenomena. And also, its hazard rate function is unimodal in shape which makes it a competitor of models like Log-normal, Log-logistic, generalized Weibull for a lifetime model (Lemeshko *et al*., 2010). The distribution is also characterized by two parameters, mean and shape. The tails of the distribution decrease more slowly than the normal distribution. It is therefore, suitable to model phenomena where numerically large values are more probable than is the case for the normal distribution (Folks and Chhikara, 1978).

If $x_1, x_2, ..., x_n$ are independent and identical random variables, $x_i$ follows the inverse Gaussian distribution if the probability density function (pdf) is defined as:

$$f_{IG}(X;\theta_3) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], X>0 \quad (7)$$
$$\theta_3 = (\mu, \lambda) \in \Re_+^1 X \Re_+^1 \subset \Re^2$$

where, $\theta_3 = (\mu, \lambda) \in \Re_+^1 X \Re_+^1 \subset \Re^2$ is the parameter space of the distribution. $\mu$ and $\lambda$ are the mean and the shape parameter, respectively and are of the same physical dimensions as X (Tweedie, 1957a, b). There are other forms of Eq. 7 obtainable through re-parameterization (Folks and Chhikara, 1978). The corresponding likelihood function of the sample $x_1, x_2, ..., x_n$ from the distribution is:

$$L_{IG}(\theta_3) = \prod_{i=1}^{n}\left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} \prod_{i=1}^{n}\exp\left[-\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right] \quad (8)$$

The natural logarithm of Eq. 8 known as log-likelihood function is:

$$l_n(\theta_3) = \frac{n}{2}\ln\lambda - \frac{n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i} \quad (9)$$

Maximum likelihood estimates (mle) of $\mu$ and $\lambda$ are obtained from Eq. 9 as following:

$$\hat{\mu} = \bar{x}; \hat{\lambda} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)\right]^{-1}$$

**Likelihood ratio:** Suppose two models 1 and 2 appear to fit samples of n-independent and identical data $x_1, x_2, ..., x_n$. In this process, we calculate the maximum likelihood estimates of the parameters based on the assumptions of the two distributions. If $(\gamma_1, \gamma_2)$ and $(\tau_1, \tau_2)$ are the parameters of model 1 and 2, respectively then their ratio of maximized likelihoods as defined by Dey and Kundu (2009b) and Goh *et al.* (2014) is:

$$L = \frac{L_1(\hat{\gamma}_1, \hat{\gamma}_2)}{L_2(\hat{\tau}_1, \hat{\tau}_2)} \quad (10)$$

where, $L_1(\hat{\gamma}_1, \hat{\gamma}_2)$ and $L_2(\hat{\tau}_1, \tau_2)$ are the maximum values of the likelihood functions for the two models 1 and 2, respectively. $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\tau}_1, \hat{\tau}_2$ are the MLEs of $\gamma_1$, $\gamma_2$, $\tau_1$, $\tau_2$, respectively. If $L_1(\hat{\gamma}_1, \hat{\gamma}_2) > L_2(\hat{\tau}_1, \hat{\tau}_2)$, then we select model 1 and 2 if otherwise.

We take the natural logarithm of L in Eq. 10 to form the basis of our decision rule in order to discriminate between the two models, ln(L) = T:

$$T = \ln(L) = \ln\left[\frac{L_1(\hat{\gamma}_1, \hat{\gamma}_2)}{L_2(\hat{\tau}_1, \hat{\tau}_2)}\right] \quad (11)$$

T is either positive or negative depending on the probability of correct selection of a particular model being better than the other. For example, the sign is positive if model 1 is preferably and correctly selected as being better than model 2. That is if lnL = T>0, data belong to model 1. If lnL = T<0, then the data belong to model 2 being the denominator of Eq. 11. The value of T, gives the direction of the Probability of Correct Selection (PCS) of a model. For more on the distribution of T and computation of probability of correct selection of model (Dumonceaux *et al.*, 1973; Dey and Kundu, 2009a). Equation 11 can be written as difference of the log-likelihoods of the models 1 and 2, viz:

$$T = l(\hat{\gamma}_1, \hat{\gamma}_2) - l(\hat{\tau}_1, \hat{\tau}_2) \quad (12)$$

Now to address the objective of this study, we discriminate between the three distributions paired-wisely; Weibull, Log-logistic and inverse Gaussian. We have the following combinations: between Weibull and Log-logistic, between Weibulland inverse Gaussian and between Log-logistic and inverse Gaussian.

We compute T which gives the direction of the probability of correctly selecting the best distribution for a given dataset for each combination of these distributions as following:

- Between Weibull and Log-logistic:

$$T_1 = \ln(L_1) = \ln\left[\frac{L_{WB}(\theta_1)}{L_{LL}(\theta_2)}\right] \quad (13)$$

$$T_1 = -n\alpha - n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - \frac{n\ln x}{\hat{\sigma}} + n\ln\hat{\sigma} + \frac{n\hat{\mu}}{\hat{\sigma}} + 2\sum_{i=1}^{n}\log\left(1+\exp\left[\frac{\ln x_i - \mu}{\sigma}\right]\right) \quad (14)$$

- Between Weibull and Log-logistic:

$$T_2 = \ln(L_1) = \ln\left[\frac{L_{WB}(\theta_1)}{L_{IG}(\theta_3)}\right] \quad (15)$$

$$T_2 = -n\alpha - n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - \frac{n}{2}\ln\lambda + \frac{n}{2}\ln(2\pi) + \frac{3}{2}\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i} \quad (16)$$

- For Log-logistic vs. inverse Gaussian:

$$T_3 = \ln(L_3) = \ln\left[\frac{L_{LL}(\theta_2)}{L_{IG}(\theta_3)}\right] \quad (17)$$

$$T_3 = \frac{n\ln x}{\hat{\sigma}} - n\ln\hat{\sigma} - \frac{n\hat{\mu}}{\hat{\sigma}} - 2\sum_{i=1}^{n}\log\left(1+\exp\left[\frac{\ln x_i - \mu}{\sigma}\right]\right) - \frac{n}{2}\ln\lambda + \frac{n}{2}\ln(2\pi) + \frac{3}{2}\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i} \quad (18)$$

## RESULTS

For illustrative purpose, in this study, we analyze the data on survival time to first birth to a woman after marriage in Nigeria. The data was extracted from the reports of National Demographic and Health Survey (NDHS). The data set comprises of 15,363 respondents. We apply the ratio of maximized likelihood functions to discriminate between Weibull, Log-logistic and inverse Gaussian distributions. We fit the three distribution functions and pair their results up for discrimination purposes. The MLEs of the parameters of the distribution functions and their corresponding log-likelihoods ratios are given in Table 1-3 as they are combined. In Table 1 as shown below where Weibull and Log-logistic are discriminated, the MLEs of their parameters are respectively, $\hat{\alpha} = 1.39$, $\hat{\beta} = 31.63$, $\hat{\sigma} = 0.42$ and $\hat{\mu} = 3.05$.

Also, the Kolmogorov-Smirnov (K-S) distances between the empirical density function and the fitted functions of Weibull and Log-logistic distributions with their respective p-values in brackets are 01720 (0.63) and 0.1094 (0.81). $T_1$ = 1399.1 is negative, an indication that Log-logistic distribution is preferably better than Weibull to model the data.

The empirical cumulative density function (ecdf) and cumulative density functions of Weibull and Log-logistic are plotted in Fig. 1. The density functions of the two distributions are as shown in Fig. 2.

Table 2 below presents the results of MLEs of inverse Gaussian and Weibull functions as $\hat{\alpha} = 1.39$, $\hat{\beta} = 31.63$, $\hat{\lambda} = 45.46$, $\hat{\mu} = 28.74$, respectively.

The values of the K-S distance between ecdf and cdf of Weibull and inverse Gaussian distributions with their p-value in brackets are respectively, 0.1720 (063) and 0.0873 (0.93). $T_2$ = -2059.3 which is negative an indication that inverse Gaussian distribution is preferably better than

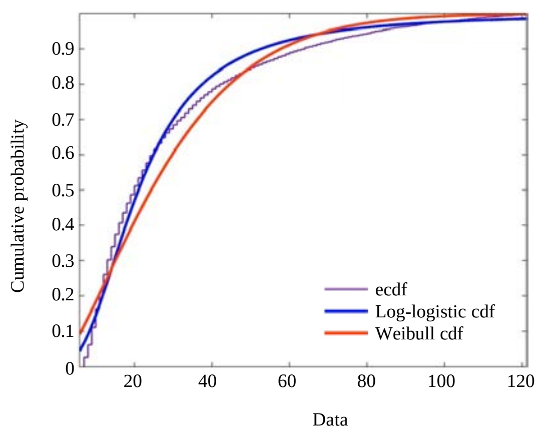


Fig. 1: The cdf of Weibull, Log-logistic and ecdf of the data

Table 1: MLE of parameters of Weibull and Log-logistic distributions with their respective Kolmogorov-Smirnov (K-S) goodness-of-fit statistics

| Distributions | Parameter estimates | Standard error | K-S distance | p-values | Log-likelihood | $T_1 = \ln(L_1)$ |
|---|---|---|---|---|---|---|
| Weibull | $\hat{\alpha} = 1.39$ | 0.2004 | 0.1720 | 0.63 | -60,367.9 | -1399.1 |
| | $\hat{\beta} = 31.63$ | 0.0084 | | | | |
| Log-logistic | $\hat{\sigma} = 0.42$ | 0.0029 | 0.1094 | 0.81 | -58,968.8 | |
| | $\hat{\mu} = 3.05$ | 0.0060 | | | | |

Table 2: MLEs of parameters of Weibull and inverse Gaussian distributions with their respective Kolmogorov-Smirnov (K-S) goodness-of-fit statistics

| Distributions | Parameter estimates | Standard error | K-S distance | p-values | Log-likelihood | $T_2 = \ln(L_2)$ |
|---|---|---|---|---|---|---|
| Weibull | $\hat{\alpha} = 1.39$ | 0.2004 | 0.1720 | 0.63 | -60367.9 | -2059.3 |
| | $\hat{\beta} = 31.63$ | 0.0084 | | | | |
| Inverse Gaussian | $\hat{\lambda} = 45.46$ | 0.5319 | 0.0873 | 0.93 | -58308.6 | |
| | $\hat{\mu} = 28.74$ | 0.1942 | | | | |

Table 3: MLE of parameters of Log-logistic and inverse Gaussian distributions with their respective Kolmogorov-Smirnov (K-S) goodness-of-fit statistics

| Distributions | Parameter estimates | Standard error | K-S distance | p-values | Log-likelihood | $T_3 = \ln(L_3)$ |
|---|---|---|---|---|---|---|
| Log-logistic | $\sigma = 0.42$ | 0.0029 | 0.1094 | 0.89 | -58968.8 | -660.2 |
| | $\mu = 3.05$ | 0.0060 | | | | |
| Inverse Gaussian | $\lambda = 45.46$ | 0.5319 | 0.0873 | 0.93 | -58308.6 | |
| | $\mu = 28.74$ | 0.1942 | | | | |

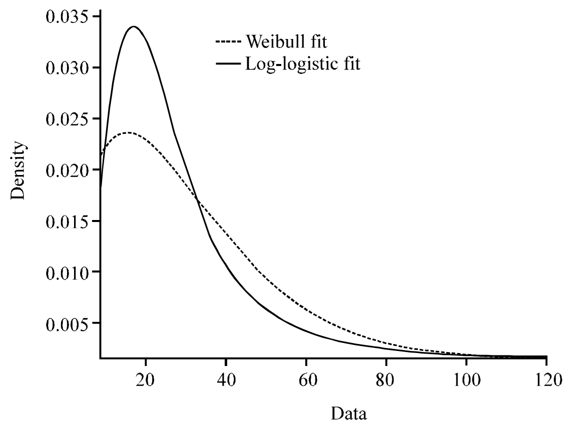


Fig. 2: The pdf of Weibull, Log-logistic and histogram of the data

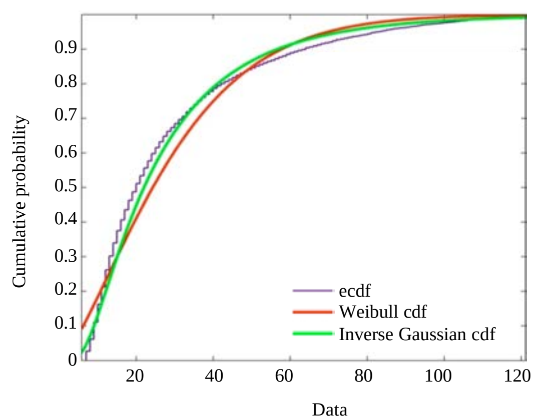


Fig. 3: The cdf of Weibull, inverse Gaussian and ecdf of the data

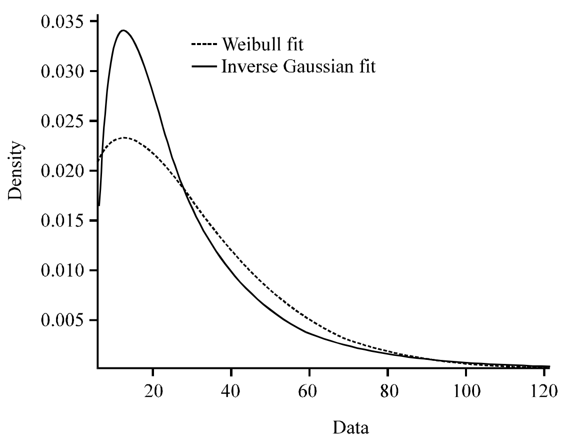


Fig. 4: The pdf of Weibull, inverse Gaussian and histogram of the data

Weibull to model the data. Figure 3 is the plot of ecdf and the cdf of Weibull and inverse Gaussian functions.

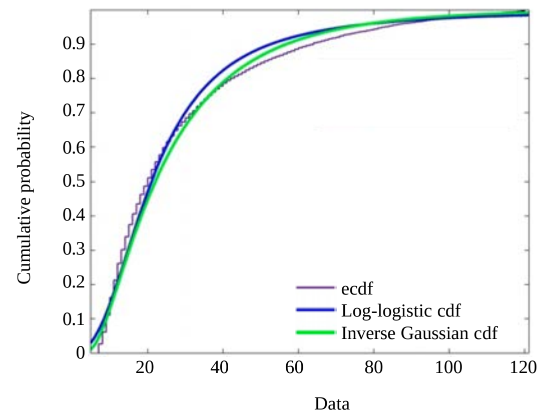


Fig. 5: The cdf of Log-logistic, inverse Gaussian and ecdf of the data

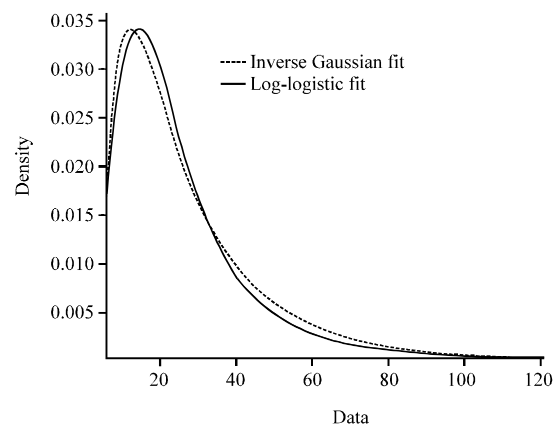


Fig. 6: The pdf of Log-logistic, inverse Gaussian and histogram of the data

The plot of density functions of Weibull and inverse Gaussian is as shown in Fig. 4. K-S distances between ecdf and cdf of Log-logistic and inverse Gaussian distributions with their p-values in brackets are respectively, 0.1094 (0.89) and 0.0873 (0.93). $T_3 = -660.2$ which is also negative, an indication that inverse Gaussian distribution is preferably better than Log-logistic to model the data (Fig. 5). Figure 6 is the plot of density functions of Log-logistic and inverse Gaussian distributions.

## DISCUSSION

The study investigates the procedure of ratio of the maximized likelihoods to discriminate between Weibull, Log-logistic and inverse Gaussian distribution functions. We discriminate between two distributions at a time starting with Weibull and Log-logistic, then Weibull and inverse Gaussian and finally with Log-logistic and inverse

Gaussian. The shapes of density functions plotted in Fig. 2, 4, 6 give credence to the fact that the two distributions in each combination describe the data in similar manners accordingly and hence discrimination between them can be carried out.

We compute the decision statistic in terms of what is designated as 'Probability of Correct Selection' (PCS) of a distribution as T by taking the natural logarithm of the ratio of likelihood functions of the two distributions under investigationand is interpreted by its sign. It is however important to note here that the computation of true value of PCS is beyond the scope of this study.

From the results in Table 1 for fitted Weibull and Log-logistic distributions in terms of $T_1$ is negative and the K-S distances between ecdf and cdf with Log-logistic having the shortest distance, all point to the fact that Log-logistic is more suitable to model the data than Weibull. This is also corroborated by the plot of ecdf and cdf of Log-logistic function which is more consistently lying on the ecdf than that of Weibull as shown in Fig. 1. The choice of Log-logistic as against Weibull is also evident in the larger value of log-likelihood function in Table 1.

In the similar manner in Table 2 where Weibull and inverse Gaussian are considered, inverse Gaussian is identified to be better fit for the data since, the sign of $T_2$ negative. The K-S distance between ecdf and the cdf of inverse Gaussian is shorter than that of Weibull. In Table 3, inverse Gaussian is a preferred choice as against Log-logistic distribution as also evident in the sign of $T_3$ which is negative, the shorter K-S distance between the ecdf and the cdf of fitted inverse Gaussian distribution than other two distributions and their values of the log-likelihoods. The ecdf and cdf of the inverse Gaussian plotted as shown in Fig. 3 and 5 also give credence to it as the most preferred distribution for the data as its cdf consistently lies on the ecdf than do the cdfs of other two distribution.

## CONCLUSION

The study concludes by identifying inverse Gaussian distribution as the most suitable distribution to model data on waiting time to first birth in Nigeria having shown preference over Log-logistic which has initially been found to be better than Weibull. The K-S distance between ecdf and cdf of inverse Gaussian recorded the lowest.

We recommend that the distribution of T, ratio of the maximized likelihoods be investigated further since, the findings in this study rely on the asymptotic property of MLEs and the sign of T. Knowing the distribution of the ratio of the maximized likelihoods is important in the computation of the actual value of probability of correction selectionof the most suitable distribution for a data set and also for the calculation of sample sizes relevant in discriminating between two models.

## ACKNOWLEDGEMENTS

Researchers wish to acknowledge the financial intervention of the Federal Ministry Education, Nigeria, through tertiary education funding program. We appreciate the management of the Universiti Teknologi Malaysia for the opportunity and atmosphere provided.

## REFERENCES

Atkinson, A.C., 1969. A test for discriminating between models. Biometrika, 56: 337-347.

Atkinson, A.C., 1970. A method for discriminating between models. J. R. Stat. Soc., 32: 323-353.

Bennett, S., 1983. Log-logistic regression models for survival data. Appl. Stat., 32: 165-171.

Bromideh, A.A. and R. Valizadeh, 2014. Discrimination between gamma and log-normal distributions by ratio of minimized kullback-leibler divergence. Pak. J. Stat. Oper. Res., 9: 443-453.

Cox, D.R., 1962. Further results on tests of separate families of hypotheses. J. R. Stat. Soc., 24: 406-424.

Dey, A.K. and D. Kundu, 2009a. Discriminating between the log-normal and log-logistic distributions. Commun. Stat. Theory Methods, 39: 280-292.

Dey, A.K. and D. Kundu, 2009b. Discriminating among the log-normal, weibull and generalized exponential distributions. Reliab. IEEE Trans., 58: 416-424.

Dumonceaux, R., C.E. Antle and G. Haas, 1973. Likelihood ratio test for discriminagon between two models with unknown location and scale parameters. Technometrics 15: 19-27.

Folks, J.L. and R.S. Chhikara, 1978. The inverse gaussian distribution and its statistical application: A review. J. R. Stat. Soc., 40: 263-289.

Goh, S., H.W. Kwon and M.Y. Choi, 2014. Discriminating between weibull distributions and log-normal distributions emerging in branching processes. J. Phys. Math. Theor., 47: 1-225101.

Johnson, N.L., S. Kotz and N. Balakrishnan, 1995. Continuous Univariate Distributions. 2nd Edn., John Wiley and Sons, New York.

Kalbfleisch, J.D. and R.L. Prentice, 2002. The Statistical Analysis of Failure Time Data. 2nd Edn., Wiley-Interscience, New York, ISBN: 047136357X.

Kundu, D. and A. Manglick, 2004. Discriminating between the weibull and log-normal distributions. Naval Res. Logistics, 51: 893-905.

Kundu, D., R.D. Gupta and A. Manglick, 2005. Discriminating between the log-normal and generalized exponential distributions. J. Stat. Plann. Infer., 127: 213-227.

Lawless, J.F., 2003. Statistical Models and Methods for Lifetime Data. 2nd Edn., John Wiley and Sons Inc., New York, USA.

Lemeshko, B.Y., S.B. Lemeshko, K.A. Akushkina, M.S. Nikulin, N. Saaidia, 2010. Inverse Gaussian Model and its Applications in Reliability and Survival Analysis. In: Mathematical and Statistical Models and Methods in Reliability. Rykov V.V., N. Balakrishnan and M.S. Nikulin (Eds.). Springer, ISBN: 978-0-8176-4970-8, New York USA., pp: 433-453.

Tweedie, M.C. 1957a. Statistical properties of inverse gaussian distributions: I. Ann. Math. Stat., 28: 362-377.

Tweedie, M.C., 1957b. Statistical properties of inverse gaussian distributions: II. Ann. Math. Stat., 28: 696-705.