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Own Oscillations of Transversally Isotropic Layer Between a Hard Surface and an Elastic Half-Space

Christina N. Stekhina, Dmitry N. Tumakov and Anastasia V. Anufrieva
Kazan Federal University, Kremlevskaya Street 18, Kazan, Russia

Abstract: We consider a semi-open elastic waveguide structure formed by a transversely isotropic layer which on the one hand is firmly fixed and on the other hand is linked with an isotropic half-space. A general solution of differential equation system is obtained describing the propagation of elastic waves in a transversely isotropic medium. Using the boundary conditions and conjugation conditions at the junction of a strip and a half-space as well as the explicit representations of the fields in each of the media, a characteristic equation for the eigenvalues (longitudinal permanents) is obtained concerning our waveguide structure. We considered separately the intervals of eigenvalues. The range in which the values of the longitudinal permanents form a discrete spectrum is specified. The dependence of the longitudinal permanents real values from oscillation frequencies is studied. It is noted that the waveguide modes may exist only if the substrate (half-space) is acoustically more rigid material than the layer. It is concluded that the eigenvalues are bounded above and below by the values corresponding to wave numbers of the attached media. Also, the range is specified in which the modes are originated. It is noted that the characteristic curves are not intersected anywhere. The claculation results are presented for a transversely isotropic layer, filled with sandstone and coupled with rather solid material close to the foundation.

Key words: Natural vibrations, transverse isotropic layer, the method of overdetermined boundary problem, range, sandstone

INTRODUCTION

The layered media have waveguide properties in such structures waves may propagate without sources. The elastic waves in layered media are described for example in the monograph (Brekhovskikh, 1973; Ewing *et al*., 1957) which focus on the problem of reflection and the refraction of elastic waves and the physical interpretation of the results. In Vdovina *et al*. (2008), the own waves of the isotropic half-open waveguide related to discrete and continuous spectrum are obtained. It is shown that the values of the longitudinal propagation constant (of the spectral parameter) form a complex plane set consisting of a vertical semiaxis, the horizontal segment and individual points. It is proved that the own waves of the half-open waveguide are orthogonal ones and form a complete system of modes on the basis of which any wave may be expanded propagating in an infinite half-open waveguide.

However for the most part, the media are anisotropic ones. The simplest case of anisotropy is the transversely isotropic media. These layers may be found both in nature and in machinery (Anufrieva and Tumakov, 2013). In this study, we studied the properties of natural waves for a waveguide structure which is formed by a full contact transverse isotropic elastic band and an isotropic elastic half-plane.

The spectrum analysis of waveguide structures are necessary most of all in conjugation problems when the field in each mating portion is represented as a superposition of the own waves for this structure. For example in diffraction problems at the junction of two waveguides it is convenient to search the fields in each of the waveguides as a series of modes (Stekhina and Tumakov, 2013). The issues of spectrum finding in geophysics are also of particular interest (Kipot *et al*., 2011). Also, the knowing of the discrete spectrum is necessary to explain the qualitative picture of the amplitude-frequency characteristics, resulting in the problems of diffraction waves on transversely isotropic layers (Anufrieva and Tumakov, 2014). In the present study, we demonstrated the presence of a waveguide continuous spectrum formed by a transversely isotropic layer and an isotropic elastic half-plane and the intervals of discrete elements possible existence are highlighted.

The presence of the discrete spectrum for a specific waveguide structure depends not only on the geometry

Corresponding Author: Christina N. Stekhina, Kazan Federal University, Kremlevskaya Street 18, Kazan, Russia

of a waveguide but also on the values of the media elastic parameters, forming it. For example, the works (Pleshchinskaya *et al.*, 2011, 2013) performed the analysis of some isotropic waveguide structures and developed the dispersion curves for a discrete spectrum. The dispersion equations for the calculation of eigenvalues concerning transverse isotropic waveguides is much more complicated than for the isotropic case and the studies of such structures are unknown for the researchers.

MATERIALS AND METHODS

Problem set: Let's consider the free elastic vibrations of waveguide structure (Fig. 1) which is a transverse isotropic layer (0<y<L) with constant density ρ_2 and an elastic modulus tensor:

$$K = \begin{pmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$$

We believe that the layer adheres firmly to the semi-infinite isotropic half-plane y>L with a constant density ρ_1 and Lame coefficients λ_1, μ_1. Let's look for the solution of a flat harmonic problem within elasticity theory at y>L:

$$\frac{\partial \sigma_{x1}}{\partial x} + \frac{\partial \tau_1}{\partial y} + \rho_1 \omega^2 u_{x1} = 0$$
$$\frac{\partial \tau_1}{\partial x} + \frac{\partial \sigma_{y1}}{\partial y} + \rho_1 \omega^2 u_{y1} = 0 \quad (1)$$

$$\sigma_{x1} = (\lambda_1 + 2\mu_1)\frac{\partial u_{x1}}{\partial x} + \lambda_1 \frac{\partial u_{y1}}{\partial y}$$
$$\sigma_{y1} = \lambda_1 \frac{\partial u_{x1}}{\partial x} + (\lambda_1 + 2\mu_1)\frac{\partial u_{y1}}{\partial y} \quad (2)$$
$$\tau_1 = \mu_1 \left(\frac{\partial u_{x1}}{\partial y} + \frac{\partial u_{y1}}{\partial x}\right)$$

Describing the propagation of elastic waves in an isotropic medium. The process of wave propagation in a homogeneous transverse isotropic medium (0<y<L) is described by the equations:

$$\frac{\partial \sigma_{x2}}{\partial x} + \frac{\partial \tau_2}{\partial y} + \rho_2 \omega^2 u_{x2} = 0$$
$$\frac{\partial \tau_2}{\partial x} + \frac{\partial \sigma_{y2}}{\partial y} + \rho_2 \omega^2 u_{y2} = 0 \quad (3)$$

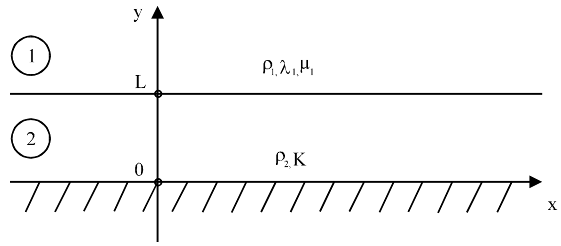

Fig. 1: The issue geometry

$$\sigma_{x2} = k_{11}\frac{\partial u_{x2}}{\partial x} + k_{12}\frac{\partial u_{y2}}{\partial y}$$
$$\sigma_{y2} = k_{12}\frac{\partial u_{x2}}{\partial x} + k_{22}\frac{\partial u_{y2}}{\partial y} \quad (4)$$
$$\tau_2 = k_{33}\left(\frac{\partial u_{x2}}{\partial y} + \frac{\partial u_{y2}}{\partial x}\right)$$

The conditions of conjugation should be performed at the medium boundary:

$$u_{x1}(x, L+0) = u_{x2}(x, L-0),\ u_{y1}(x, L+0) = u_{y2}(x, L-0)$$
$$\tau_1(x, L+0) = \tau_2(x, L-0),\quad \sigma_{y1}(x, L+0) = \sigma_{y2}(x, L-0) \quad (5)$$

at y = L. We will study, the case when a waveguide is mounted on a rigid base that corresponds to the conditions at y = 0:

$$u_{x2}(x, 0) = 0,\quad u_{y2}(x, 0) = 0 \quad (6)$$

Let's consider such solutions of the system Eq. 1 and 2 which are limited at $y \to +\infty$. All the required functions of the Eq. 1-4 are believed to be continuously differentiable in the areas 0<y<L, y>L and continuously extended to the boundaries of these areas.

Further let's assume that the dependence of the unknown functions of the coordinate x is the following one $\exp\{i\xi x\}$ where the complex number ξ represents the spectral parameter. Thus, the transition to the functions u_{xn}, u_{yn}, σ_{xn}, σ_{yn} and τ_n is performed which are the solutions of the following ordinary differential equations:

$$i\xi\sigma_{x1} + \tau_1' + \rho_1\omega^2 u_{x1} = 0,\ i\xi\tau_1 + \sigma_{y1}' + \rho_1\omega^2 u_{y1} = 0$$
$$\sigma_{x1} = i(\lambda_1 + 2\mu_1)\xi u_{x1} + \lambda_1 u_{y1}',\ \sigma_{y1} = i\lambda_1\xi u_{x1} + \quad (7)$$
$$(\lambda_1 + 2\mu_1)u_{y1}',\ \tau_1 = \mu_1(u_{x1}' + i\xi u_{y1})$$

for y>L and:

$$i\xi\sigma_{x2} + \tau_2' + \rho_2\omega^2 u_{x2} = 0,\ i\xi\tau_2 + \sigma_{y2}' +$$
$$\rho_2\omega^2 u_{y2} = 0,\ \sigma_{x2} = ik_{11}\xi u_{x2} + k_{12}u_{y2}', \quad (8)$$
$$\sigma_{y2} = ik_{12}\xi u_{x2} + k_{22}u_{y2}',\ \tau_2 = k_{33}(u_{x2}' + i\xi u_{y2})$$

for 0<y<L. In the system, Eq. 7 and 8 all the required functions are considered as the functions of the variable y and parameter ξ. For brevity, we will omit the dependence by y for example, instead of $u_x(\xi, y)$, we will put down $u_x(\xi)$.

Let's study the problem on the own complex values ξ, t which the system Eq. 7 and 8 with the terms of conjugation Eq. 5 and the boundary conditions Eq. 6 has non-trivial solutions (for the functions under the conditions Eq. 5 and 6, we also accept the above relationship by x in the form of exp $\{i\xi x\}$). These values of ξ define their own waves of a semi-open waveguide with transverse isotropic layer.

Elastic vibrations of transverse isotropic medium: Let us find the solution of the system Eq. 8. To do this, Eq. 8 let's exclude σ_{x2} and bring the present system to the following form $v' = Mv$ where $v = (u_{x2}, u_{y2}, \sigma_{y2}, \tau_2)$:

$$M = \begin{pmatrix} 0 & -i\xi & 0 & 1/k_{33} \\ -i\xi k_{12}/k_{22} & 0 & 1/k_{22} & 0 \\ 0 & -\rho\omega^2 & 0 & -i\xi \\ -\rho\omega^2 + \xi^2 k_{11} - \xi^2 k_{12}^2/k_{22} & 0 & -i\xi k_{12}/k_{22} & 0 \end{pmatrix}$$

Let:

$$\gamma_{2n} = \sqrt{\kappa_{2n}^2 - \xi^2}, \quad \kappa_{2n} = \sqrt{\frac{\rho_2}{k_{nn}}}\omega, \quad \Gamma_\pm = \frac{\Omega_\pm}{\sqrt{2k_{22}k_{33}}}$$

Where:

$$\Omega_\pm = \sqrt{P \pm \sqrt{P^2 - 4k_{11}k_{22}k_{33}^2\gamma_{21}^2\gamma_{23}^2}}$$
$$P = \xi^2(k_{12}^2 - k_{11}k_{22} + 2k_{12}k_{33}) + \rho_2\omega^2(k_{22} + k_{33})$$

The matrix M has the eigenvalues $\mp i\Gamma_+$ and $\mp i\Gamma_-$ the following eigenvectors correspond to them: Λ_+^{-1} ($\pm F_+$, G_+, $\pm H_+$, Λ_+) and Λ_-^{-1} ($\pm F_-$, G_-, $\pm H_-$, Λ_-):

$$F_\pm = i\Omega_\pm\sqrt{\frac{k_{22}}{2k_{33}}}(\Omega_\pm^2 - 2k_{33}(\rho_2\omega^2 + \xi^2 k_{12})),$$
$$G_\pm = i\xi(2k_{11}k_{22}k_{33}\gamma_{21}^2 + k_{12}\Omega_\pm^2),$$
$$H_\pm = \sqrt{2k_{22}k_{33}}\xi(k_{12}(\rho_2\omega^2 + \xi^2 k_{12}) + k_{11}k_{22}\gamma_{21}^2)\Omega_\pm,$$
$$\Lambda_\pm = \Omega_\pm^2(\xi^2 k_{12}^2 + k_{11}k_{22}\gamma_{21}^2) - 2\rho_2\omega^2 k_{11}k_{22}k_{33}\gamma_{21}^2$$

Thus, the general solution of the system describing the propagation of elastic waves in a transverse isotropic medium may be written as follows:

$$u_{x2} = A_2F_+e^{-i\Gamma_+y} - B_2F_+e^{i\Gamma_+y} + C_2F_-e^{-i\Gamma_-y} - D_2F_-e^{i\Gamma_-y},$$
$$u_{y2} = A_2G_+e^{-i\Gamma_+y} + B_2G_+e^{i\Gamma_+y} + C_2G_-e^{-i\Gamma_-y} + D_2G_-e^{i\Gamma_-y},$$
$$\sigma_{y2} = A_2H_+e^{-i\Gamma_+y} - B_2H_+e^{i\Gamma_+y} + C_2H_-e^{-i\Gamma_-y} - D_2H_-e^{i\Gamma_-y},$$
$$\tau_2 = A_2\Lambda_+e^{-i\Gamma_+y} + B_2\Lambda_+e^{i\Gamma_+y} + C_2\Lambda_-e^{-i\Gamma_-y} + D_2\Lambda_-e^{i\Gamma_-y} \tag{9}$$

where, A_2, B_2, C_2 and D_2 are arbitrary constants.

Own oscillations of transverse isotropic layer attached to elastic half-plane: The displacements and stresses as a general solution of the system Eq. 7 may be represented as follows (Vdovina *et al*., 2008):

$$u_{x1} = -\xi A_1e^{-i\gamma_{11}(y-L)} + \xi B_1e^{i\gamma_{11}(y-L)} + \gamma_{12}C_1e^{-i\gamma_{12}(y-L)} + \gamma_{12}D_1e^{i\gamma_{12}(y-L)},$$
$$u_{y1} = \gamma_{11}A_1e^{-i\gamma_{11}(y-L)} + \gamma_{11}B_1e^{i\gamma_{11}(y-L)} + \xi C_1e^{-i\gamma_{12}(y-L)} - \xi D_1e^{i\gamma_{12}(y-L)},$$
$$\sigma_{x1} = -QA_1e^{-i\gamma_{11}(y-L)} + QB_1e^{i\gamma_{11}(y-L)} - M_2C_1e^{-i\gamma_{12}(y-L)} - M_2D_1e^{i\gamma_{12}(y-L)},$$
$$\sigma_{y1} = -PA_1e^{-i\gamma_{11}(y-L)} + PB_1e^{i\gamma_{11}(y-L)} + M_2C_1e^{-i\gamma_{12}(y-L)} + M_2D_1e^{i\gamma_{12}(y-L)},$$
$$\tau_1 = -A_1M_1e^{-i\gamma_{11}(y-L)} - M_1B_1e^{i\gamma_{11}(y-L)} - PC_1e^{-i\gamma_{12}(y-L)} + PD_1e^{i\gamma_{12}(y-L)} \tag{10}$$

Where:

$$M_n = -2i\mu_1\xi\gamma_{1n}, \qquad Q = i((\lambda_1 + 2\mu_1\xi^2) + \lambda_1\gamma_{11}^2),$$
$$P = i(\rho_1\omega^2 - 2\mu_1\xi^2), \quad \gamma_{1n} = \sqrt{k_{1n}^2 - \xi^2},$$
$$k_{11} = \sqrt{\frac{\rho_1}{\lambda_1 + 2\mu_1}}\omega, \qquad k_{12} = \sqrt{\frac{\rho_1}{\mu_1}}\omega$$

The branches of the functions γ_{1n} are selected, so that the real and imaginary parts are non-negative ones. It should be noted that for a transverse isotropic medium the wave numbers in the plane of the elastic symmetry k_{2n} as well as k_{1n} may be expressed via λ_2 and μ_2 (Annin, 2009). The wave carrying energy in any given direction will be called the wave moving in this direction.

The research (Anufrieva *et al*., 2014) proves that the eigenwaves of a semi-open elastic waveguide moving to the right exist at the values of the parameter ξ, belonging to the positive imaginary semi-axis, the interval ($-k_{12}$, 0) and develop a continuous spectrum on these sets.

Let's consider the possible types of spectra. The dispersion equation may be obtained if the Eq. 9 and 10 satisfy the conditions of conjugation (Eq. 5) and the boundary condition (Eq. 6). The essential point here is the number of unknowns (amplitudes) at the six equations. The solutions (Eq. 10) must satisfy the term of a field limitation at infinity, so the behavior of γ_{1n} is important for the presentation of the solution at determined ξ.

Let's consider only those waves which transfer energy to the right, i.e., along the axis and (or) are damped in the same direction. At that the parameter ξ should be within the second quadrant of the complex plane with its borders (Re $\xi \leq 0$, Im $\xi \geq 0$). Note also that according to the positivity of the Lame coefficients: $k_{11}<k_{12}$.

Let $\xi \in (-k_{11}, 0)$, then all $\gamma_{1j}(\xi)$ will be real ones. In the Eq. 9 and 10 all (eight) of unknown factors will remain. Then in the system of six equations (Eq. 5 and 6) the will be two more unknown elements and hence, it is possible to find two linearly independent eigen waves of elastic waveguide related to the continuous spectrum.

At $\xi \in (-k_{12}, -k_{11})$, the value $\gamma_{12}(\xi)$ remains a real one and the value $\gamma_{11}(\xi)$ becomes purely imaginary with positive imaginary part. In this case, we assume that $A_1 = 0$ within the Eq. 10. Then seven unknown elements of the form (Eq. 5 and 6) remain in the system, one of which is an arbitrary constant in the solution. The interval $(-k_{12}, -k_{11})$ also relates to the continuous part of the studied spectrum but here only one mode corresponds to each value ξ.

At $\xi<-k_{12}$ the amplitudes A_1 and C_1 are = 0. Therefore in the system of six equations only six coefficients remain unknown ones. The nonzero solution exists when the matrix determinant of its coefficients is = 0. In Anufrieva *et al.* (2014), it was indicated that the discrete spectrum is possible only when $\xi<-k_{12}$. Its presence is determined by the elastic parameters and the thickness of transverse isotropic layer. Let's note that if the layer is an isotropic one, then as shown in (Vdovina *et al.*, 2008) the discrete spectrum is in the range between the wave numbers of two media transverse waves.

RESULTS AND DISCUSSION

Let's obtain the characteristic equation for the eigenvalues studied in the research of the waveguide structure and let's put it down in the following way:

$$\begin{aligned}&\begin{pmatrix} e^{ih(\Gamma_+ + \Gamma_-)}\left(T_1^- T_4^- - T_2^- T_3^-\right) + \\ e^{-ih(\Gamma_+ + \Gamma_-)}\left(T_1^+ T_4^+ - T_2^+ T_3^+\right)\end{pmatrix}\left(G_- F_+ - G_+ F_-\right) - \\ &\begin{pmatrix} e^{ih(\Gamma_+ - \Gamma_-)}\left(T_1^- T_4^+ + T_2^+ T_3^-\right) + \\ e^{-ih(\Gamma_+ - \Gamma_-)}\left(T_1^+ T_4^- + T_2^- T_3^+\right)\end{pmatrix}\left(G_- F_+ + G_+ F_-\right) + 2 \\ &\left(T_1^+ T_3^- + T_1^- T_3^+\right) G_- F_- + 2\left(T_2^- T_4^+ + T_2^+ T_4^-\right) G_+ F_+ = 0\end{aligned} \tag{11}$$

Where:

$$T_1^{\pm} = \frac{1}{2P}\left(H_+ \pm \frac{P\left(PG_+ + \xi\Lambda_+\right)}{-\xi M_1 + P\gamma_{11}} - \frac{\left(PF_+ - \xi H_+\right)M_2}{-\xi M_2 + P\gamma_{12}}\right),$$

$$T_2^{\pm} = \frac{1}{2P}\left(H_- \pm \frac{P\left(PG_- + \xi\Lambda_-\right)}{-\xi M_1 + P\gamma_{11}} - \frac{\left(PF_- - \xi H_-\right)M_2}{-\xi M_2 + P\gamma_{12}}\right),$$

$$T_3^{\pm} = \frac{1}{2}\left(\frac{G_+ M_1 + \Lambda_+ \gamma_{11}}{-\xi M_1 + P\gamma_{11}} \pm \frac{PF_+ - \xi H_+}{-\xi M_2 + P\gamma_{12}}\right),$$

$$T_4^{\pm} = \frac{1}{2}\left(\frac{G_- M_1 + \Lambda_- \gamma_{11}}{-\xi M_1 + P\gamma_{11}} \pm \frac{PF_- - \xi H_-}{-\xi M_2 + P\gamma_{12}}\right)$$

Let's study, the dependence of the real longitudinal constants ξ from the frequency ω. For different substances, the ratio of the elastic parameters of a layer and a substrate changes. The waveguide modes may exist only if a substrate is acoustically more rigid material than a layer. If this condition is not satisfied then, the modes are transferred to a substrate as radiative ones forming a continuous range.

Figure 2 demonstrates the dependence curves ξ on the frequency ω for the case ρ_2 = 2400 kg m^{-3}, k_{11} = 1.0910^{10} Pa, k_{12} = 0.3910^{10} Pa, k_{22} = 1.7910^{10} Pa, k_{33} = 0.5210^{10} Pa which corresponds to sandstone (taken from (Annin, 2009)). The half-plane is considered to be a sufficiently hard material close to the foundation with the parameters ρ_1 = 2600 kg m^{-3}, v_{p1} = 6000 m sec^{-1}, v_{s1} = 4300 m sec^{-1}. Let's note that

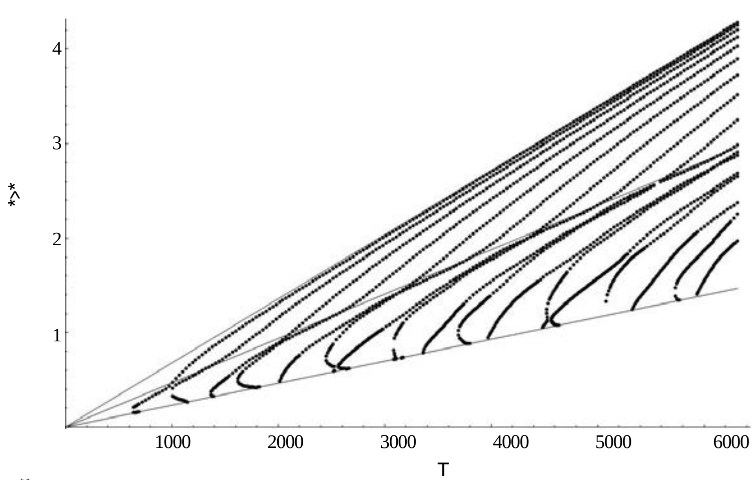

Fig. 2: Dependence o ξ on ω

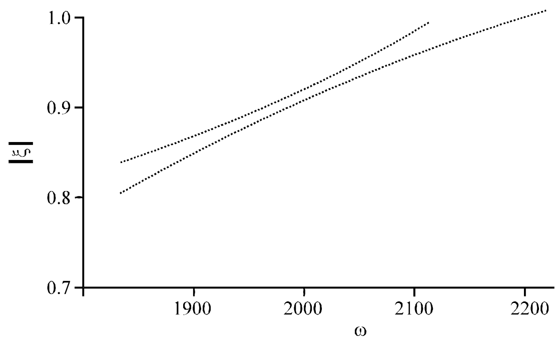

Fig. 3: Dependence of ξ on ω. The isolated fragment $\omega\in$ (1800, 2200) on $\xi\in$ ($1.7k_{12}$, k_{21}). Parameters $\rho_2 = 2400$ kg m^{-3}, $k_{11} = 1.0910^{10}$ Pa, $k_{12} = 0.3910^{10}$Pa, $k_{22} = 1.7910^{10}$ Pa, $k_{33} = 0.5210^{10}$ Pa, $\rho_1 = 2600$ kg m^{-3}, $v_{p1} = 6000$ m sec^{-1}, $v_{s1} = 4300$ m sec^{-1}

Lame coefficients λ_1 and μ_1 in the Eq. 10 are expressed in terms of elastic wave speed $\mu_1 = \rho_1 v_{s1}^2$, $\lambda_1 = \rho_1 v_{p1}^2 - 2\mu_1$.

Figure 2 demonstrates by the straight lines upward $|\xi| = k_{12}$, $|\xi| = k_{21}$ and $|\xi| = k_{23}$. The roots are cut by straight $|\xi| = k_{12}$ to the bottom and the top values fit close to the line $|\xi| = k_{23}$, condensing near it with the growth of ω. Let's note that the points of mode origin are between the values of k_{12} and k_{21}. The dispersion curves are not intersected. This is well illustrated by the example at $\omega\in$ (1800, 2200) on Fig. 3.

Summary: The representation for elastic waves propagating in transverse isotropic medium is obtained. The dispersion equation is derived for the modes of a waveguide structure formed by a transverse isotropic layer which on the one hand is rigidly fixed and on the other hand is related to the half-space.

CONCLUSION

It was found that the waveguide modes may exist only if a half-space is acoustically more rigid medium than a layer. The result of numerical experiments show that longitudinal permanents ξ which are the solutions of a characteristic equation are limited by the values k_{12} below and by k_{23} above. And the emergence of modes (the minimum value of ξ for each characteristic curve) belongs to the interval $k_{12}<\xi<k_{21}$. It is concluded that the characteristic curves are never intersected.

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