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Heterogeneous Stationary Nonlinear Filtration Problem with Degeneration If a Point Source is Available

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Abstract: This study discusses a nonuniform stationary nonlinear filtering problem with degeneration at the presence of a point source. The filtration law has a linear growth at infinity and is monotonous one. The pressure is considered to be known at the boundary. The solution to this problem exists and has an additive representation with an explicit peculiarity in one summand, generated by the presence of a concentrated source. A variation problem is formulated for a second term and the method of simple iteration is applied. The filtration area proves the convergence of the iteration process at a geometric rate in the uniform norm and the value of an optimal parameter is obtained. Then, we prove the Holder continuity of the second summand within the field of filtration.

Key words: Filtering problem with degeneration, nonlinear problem, point source, nonuniform, stationary

INTRODUCTION

The research studied the iterative method for inhomogeneous nonlinear stationary problem of filtration of an incompressible fluid in a limited domain at the presence of a point source (Zadvornov and Zadvornova, 2014). The pressure was considered to be known one at the boundary. The defining relation between the pressure gradient and filtration rate was assumed to to be depended on spatial coordinates which is strongly monotone and which has a linear growth at infinity. The existence of a solution for this problem was proved in (Zadvornov, 2010) and the research (Zadvornov and Zadvornova, 2014) proved the convergence of the iteration method in a continuous norm for any subdomain.

In this study, the problem of filtration with limiting gradient was considered. The solution to this problem exists (Zadvornova, 2012) and has an additive representation with a particular determination of a peculiarity in one summand generated by the presence of a concentrated source. As to the second (more regular) term which belongs to the area $\mu \in W_2$ is formulated and the method of simple iteration is used for its solution. The investigation of iteration process convergence is carried out by the methods described in the monographs (Koshelev, 1986; Koshelev and Chelkak, 2000). The Holder continuity of the second term is proved and the convergence of the iteration process at the optimum value of the parameter at a geometric rate in the norm of the area C^{γ} where the filtering occurs. Note that at the other conditions on determining law, the Holder continuity of the second term is set by Koshelev (1986), Koshelev and Chelkak (2000), Zadvornov and Zadvornova (2012) and Zadvornova (2012).

PROBLEM SETTING

A boundary problem is studied describing the steady process of incompressible fluid filtration in a porous heterogeneous environment. The filtration occurs in the area $\Omega \subset R^n$, $n \geq 2$ with Lipschitz continuous boundary $\partial\Omega$ where the pressure is considered to be known if the point intensity source q is considered available at the origin of coordinates (let the coordinate origin is the internal point Ω):

$$-\mathrm{div}\left(\frac{g\left(x,|\nabla w(x)|\right)}{|\nabla w(x)|}\nabla w(x)\right) = q\delta(x),\ x \in \Omega \quad (1)$$

$$w(x) = w_{\partial\Omega}(x), \quad x \in \partial\Omega \quad (2)$$

We assume that for each $s \geq 0$, the function $x \rightarrow g(x, s)$, describing the Filtration Law is measurable at Ω and is represented as follows:

$$g(x,s) = \begin{cases} 0, & s < s_* \\ \tilde{g}(x, s - s_*), & s \geq s_* \end{cases} \quad (3)$$

There are the constants $L \geq \mu > 0$, $p \geq 2$, k_0, C, $\varepsilon_0 > 0$ and the function $d \in L_p(\Omega)$ such that the following inequalities are performed:

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$$L(z-s) \geq \tilde{g}(x,z) - \tilde{g}(x,s) \geq \mu(z-s),\ z \geq s,\ \forall x \in \Omega \quad (4)$$

At:

$$\beta > \frac{(p-1)}{p} n - 1$$

and:

$$x \in B_{\varepsilon_0} = \{x \in R^n : |x| < \varepsilon_0\} \subset \Omega$$

$$|g(x,s) - k_0 s| \leq C|x|^\beta s + d(x)\ s \geq 0 \quad (5)$$

We believe that there is the function $\tilde{w} \in W_p^{(1)}(\Omega)$ with a track, satisfying the equality:

$$\tilde{w}(x) = w_{\partial\Omega}(x),\ x \in \partial\Omega \quad (6)$$

Under these assumptions, the boundary Eq. 1 and 2 has the solution (Zadvornov, 2010) in the sense that there is the function $w \in W_1^{(1)}(\Omega)$ with a trace which satisfies the equality (Eq. 2) and the following variation equality is performed for w:

$$\int_\Omega \left(\frac{g(x,|\nabla w(x)|)}{|\nabla w(x)|} \nabla w(x),\ \nabla\eta(x) \right) = q\eta(0),\ \forall \eta \in C_0^\infty(\Omega) \quad (7)$$

It follows from (Zadvornov, 2010) that this decision is represented as w = u+ξ where $u \in \overset{0}{W}{}_2^{(1)}(\Omega)$ and the function ξ is the linear problem solution (k_0 is the constant from (Eq. 5)):

$$k_0 \Delta\xi(x) = q\delta(x),\ x \in \Omega,\quad \xi(x) = w_\gamma(x),\ x \in \partial\Omega \quad (8)$$

ITERATIVE METHOD OF PROBLEM SOLUTION

Let's start the solution of Eq. 1 and 2 with the definition of the Eq. 8 solution. To this end, we solve a linear boundary problem:

$$k_0 \Delta\tilde{\xi}(x) = 0,\ x \in \Omega,\ \tilde{\xi}(x) = w_\gamma(x) - k_0^{-1} q\phi(x),\ x \in \partial\Omega \quad (9)$$

Then, $\xi(x) = \tilde{\xi}(x) + k_0^{-1} q\varphi(x)$ where the function $\phi(x)$ is the fundamental decision of Laplace operator:

$$\phi(x) = \frac{1}{2\pi}\ln(|x|),\ n = 2;$$

$$\phi(x) = \frac{-1}{(n-2)\sigma_n |x|^{n-2}},\ n \geq 3$$

σ_n single sphere measure in R^n. Regarding the function $u_m \in \overset{0}{W}{}_2^{(1)}(\Omega)$ let's put down the Eq. 7 in an equivalent form:

$$u \in \overset{0}{W}{}_2^{(1)}(\Omega): \int_\Omega \left(G\nabla(\xi + u)(x)) - k_0 \nabla\xi(x), \nabla\eta \right) dx = 0\ \forall \eta \in C_0^\infty(\Omega) \quad (10)$$

And develop the sequence $u_m \in \overset{0}{W}{}_2^{(1)}(\Omega)$, m = 0, 1, 2..., ($u_0$ is set arbitrary) for its solution, using simple iteration method (τ>0 iteration parameter):

$$\int_\Omega (\nabla u_{m+1}, \nabla\eta(x))\, dx = \int_\Omega (\nabla u_m, \nabla\eta)\, dx - \tau \int_\Omega (G(x, \nabla u_m), \nabla\eta)\, dx \quad (11)$$

Where the function $G: \Omega \times R^n \to R^n$ is determined as follows:

$$G(x,\lambda) = g(x, |\nabla\xi(x) + \lambda|) \frac{\nabla\xi(x) + \lambda}{|\nabla\xi(x) + \lambda|} - k_0 \nabla\xi(x),\ x \in \Omega,\ \lambda \in R^n$$

ITERATIVE METHOD CONVERGENCE STUDY

Let 's perform the study of the convergence process (Eq. 11) following the methods outlined in Koshelev's monograph (Koshelev, 1986; Koshelev and Chelkak, 2000). The study of the examined iterative method in the absence of a point source by other methods is performed by Lyashko and Karchevsky (1975) and Gaewsky *et al.* (1978). We can not use the results Koshelev (1986) concerning the successive approximation method convergence for the Eq. 10 because the function G(x, λ), generally speaking, does not meet the conditions (1-3). We have to use the following statements.

Lemma 1: Let the function $u \in W_p^{(1)}(\Omega)$, (p parameter from the term (5)), then the function $G(x, \nabla u(x))$ belongs to the area $L_p(\Omega)$.

Proof: According to Eq 5 and 9, the properties of Laplace fundamental solution, it follows that the function ξ satisfies the following terms:

$$|\nabla\xi(x)| \leq \frac{C_n}{|x|^{n-1}},\ x \in B_\varepsilon \subset \Omega,\quad C_n > 0;\ \xi \in W_p^{(1)}(\Omega \setminus B_\varepsilon) \quad (12)$$

Using this inequality and the inequality (Eq. 5), we obtain λuз R^n uxuз $B_{\varepsilon 0}$:

$$|G(x,\lambda)| = \left|\begin{pmatrix} g(x, |\nabla\xi(x)+\lambda|) \\ -k_0|\nabla\xi(x)+\lambda| \end{pmatrix} \frac{\nabla\xi(x)+\lambda}{|\nabla\xi(x)+\lambda|} + k_0\lambda\right| \le$$

$$\left|g(x, |\nabla\xi(x)+\lambda|) - k_0|\nabla\xi(x)+\lambda|\right| + \quad (13)$$

$$k_0|\lambda| \le C|x|^{\beta}|\nabla\xi(x)+\lambda| + d(x) + k_0|\lambda| \le$$

$$\frac{CC_n}{|x|^{n-1-\beta}} + d(x) + \left(C|x|^{\beta} + k_0\right)$$

$$|\lambda| \le \tilde{d}(x) + \left(C|\varepsilon|^{\beta} + k_0\right)|\lambda|$$

From the term (Eq. 5) to β, we have that p (1+β-n)>-n thus, the function $x \to |x|^{1+\beta-n}$ belongs to $L_p(\beta_{\varepsilon_0})$ and therefore, the function $\tilde{d} = |x|^{1+\beta-n}+d$ from the area $L_p(\beta_{\varepsilon_0})$.

Using the condition (Eq. 4), we obtain the following inequality for any λ из R^n and any $x \in \Omega / \beta_{\varepsilon_0}$:

$$|G(x,\lambda)| \le |g(x, |\nabla\xi(x)+\lambda|)| + k_0|\nabla\xi(x)| \le$$
$$L|\nabla\xi(x)+\lambda| + k_0|\nabla\xi(x)| = L|\lambda| + \tilde{d}(x)$$

Where the function $\tilde{d} = (L+k_0)|\nabla\xi|$, due to (Eq. 12), belongs to $L_p(\Omega/\beta_{\varepsilon_0})$.

Using the last inequality and (Eq. 13), we obtain the following estimation:

$$|G(x, \nabla u(x))| \le \tilde{k}|\nabla u(x)| + \tilde{d}(x),$$
$$x \in \Omega, \tilde{k} = \max\left\{\left(C|\varepsilon|^{\beta} + k_0\right), L\right\}$$

from which the lemma statement follows. The lemma is proved.

Lemma 2: Let the function g satisfies the terms Eq. 3, 4. Then the function $g_\tau(x, s) \equiv s - \tau g(x, s)$ satisfies the term of Lipschitz continuity:

$$|g_\tau(x,z) - g_\tau(x,s)| \le K_\tau|z-s| \quad (14)$$
$$\forall z \ge 0, s \ge s_* + \delta, \forall x \in \Omega$$

Where:

$$K_\tau = \max\{1-\tau\tilde{\mu}; \tau L - 1\}, \quad \tilde{\mu} = \mu\frac{\delta}{s_* + \delta}$$

Besides, K_τ at $\tau \in (0, 2\backslash L)$ will belong to the interval (0,1) and $K_* \equiv \inf_{\tau>0} K_\tau = L - \tilde{\mu}\backslash L + \tilde{\mu}$ is achieved at $\tau_* = 2\backslash L + \tilde{\mu}$.

Proof: Let's prove the following inequality for the function g(s):

$$L|z-s| \ge |g(x,z) - g(x,s)| \ge \tilde{\mu}|z-s|, \quad (15)$$
$$\forall z \ge 0, s \ge s_* + \delta, \forall x \in \Omega$$

The left side of the inequality is obvious, let's prove the right one. Let $z \ge s_*$ and from the term (Eq. 4)

$$g(x,z) - g(x,s) = \tilde{g}(x, z-s_*) - \tilde{g}(x, s-s_*) \ge \mu(z-s)$$

Let $z < s_*$ from the term (Eq. 4) one may prove that:

$$g(x,z) - g(x,s) = \tilde{g}(x, s-s_*) \ge \mu(s-s_*) \ge \tilde{\mu}s$$

at $\tilde{\mu} = \mu\delta / s_* + \delta$ and the inequality (Eq. 15) is performed. Let's consider the following difference:

$$I = |g_\tau(x,z) - g_\tau(x,s)| = |z - s - \tau(g(x,z) - g(x,s))|$$

Suppose for definiteness z>s, then from (Eq. 15) and the term τ>0, we obtain the following τ(g(x,z)-g(x,s))>0. Let's consider the first case z-s>τ(g(x,z)-g(x,s)) and use the term (Eq. 15):

$$I = (z-s) - \tau(g(x,z) - g(x,s)) \le \quad (16)$$
$$(z-s) - \tau\tilde{\mu}(z-s) = (1-\tau\tilde{\mu})(z-s)$$

Let's denote $K_{\tau,1} = 1 - \tau\tilde{\mu}$. Let's consider the second case z-s≤τ(g(x,z)-g(x,s)) and use the condition (Eq. 15):

$$I = \tau(g(x,z) - g(x,s)) - (z-s) \le \quad (17)$$
$$\tau L(z-s) - (z-s) = (\tau L - 1)(z-s)$$

Let $K_{\tau,2} = \tau L - 1$. Let $K_\tau = \max\{K_{\tau,1}, K_{\tau,2}\}$. Then, it is clear from (Eq. 16, 17) that we obtain (Eq. 14). Minimum K_τ for is achieved at $\tau_* = 2 \backslash L + \tilde{\mu}$ and equal to $L - \tilde{\mu} \backslash L + \tilde{\mu}$. The values K_τ will belong to the interval (0, 1) at τ∈(0, 2\L). The lemma is proved.

Then let's introduce the area $W_{2,\alpha}^{(1)}(\Omega)$ with a finite norm:

$$\|u\|_\alpha = \sup_{x_0 \in \Omega}\left(\int_\Omega \left(|\nabla u|^2 + u^2\right)|x - x_0|^{\alpha} dx\right)^{1/2}$$

And the function:

$$H(u, \eta; \tau) = \int_{\Omega_\delta} (\nabla u(x) - \tau G(x, \nabla u(x)), \nabla\eta(x))dx$$

where the area $\Omega_\delta \subset \Omega$ is determined as $\Omega_\delta = \{x : |\nabla u(x)| \ge s_* - \delta\}$.

Lemma 3: Let the terms (Eq. 3-5), the functions $u, v \in W^{(1)}_{2,\alpha}(\Omega)$, $\eta \in W^{(1)}_{2,-\alpha}(\Omega)$ are performed for g function, then for $x \in \Omega_\delta = \{x: |\nabla u(x)| \geq s_* + \delta\}$ the following inequality takes place:

$$\left|H(u,\eta;\tau) - H(v,\eta;\tau)\right| \leq K_\tau \left[\int_{\Omega_\delta} |\nabla(u-v)|^2 |x-x_0|^\alpha dx\right]^{1/2} \left[\int_{\Omega_\delta} |\nabla\eta|^2 |x-x_0|^{-\alpha} dx\right]^{1/2} \tag{18}$$

Where:

$$K_\tau = \max\{1-\tau\tilde{\mu}; \tau L - 1\},\ \tilde{\mu} = \mu\frac{\delta}{s_* + \delta}$$

Proof: We have:

$$\nabla u(x) - \tau G(x, \nabla u(x)) - \left(\nabla v(x) - \tau G(x, \nabla v(x))\right) = \nabla u(x) - \tau\frac{g(x, |\nabla(u+\phi)(x)|)}{|\nabla(u+\phi)(x)|}\nabla(u+\phi)(x) + \tau k_0 \nabla\phi(x) -$$
$$\left(\nabla v(x) - \tau\frac{g(x, |\nabla(v+\phi)(x)|)}{|\nabla(v+\phi)(x)|}\nabla(v+\phi)(x) + \tau k_0\nabla\phi(x)\right) = \frac{|\nabla(u+\phi)(x)| - \tau g(x, |\nabla(u+\phi)(x)|)}{|\nabla(u+\phi)(x)|}\nabla(u+\phi)(x) -$$
$$\frac{|\nabla(v+\phi)(x)| - \tau g(x, |\nabla(v+\phi)(x)|)}{|\nabla(v+\phi)(x)|}\nabla(v+\phi)(x) = G_\tau(x, \nabla(u+\phi)(x)) - G_\tau(x, \nabla(v+\phi)(x))$$

Here:

$$G_\tau(x,s) = \frac{|s| - \tau g(x,|s|)}{|s|}s = \frac{g_\tau(x,|s|)}{|s|}s$$

Then, applying Lemma 1,2 at a fixed x from Ω_δ we get:

$$\left|G_\tau(x,s) - G_\tau(x,z)\right| \leq K_\tau|s-z|$$

Using this inequality for all x from Ω_δ we obtain the following inequality:

$$\left|\nabla u(x) - \tau G(x,\nabla u(x)) - \left(\nabla v(x) - \tau G(x, \nabla v(x))\right)\right| =$$
$$\left|G_\tau(x, \nabla(u+\phi)(x)) - G_\tau(x, \nabla(v+\phi)(x))\right| \leq$$
$$K_\tau\left|\nabla u(x) - \nabla v(x)\right|$$

Using this inequality and Holder's inequality, we obtain the following:

$$\left|H(u,\eta;\tau) - H(v,\eta;\tau)\right| \leq \int_{\Omega_\delta}\left|\begin{matrix}\nabla u(x) - \tau G(x,\nabla u(x)) - \\ \left(\nabla v(x) - \tau G(x,\nabla v(x))\right)\end{matrix}\right||\nabla\eta| dx \leq$$
$$K_\tau \int_{\Omega_\delta} |\nabla u(x) - \nabla v(x)||\nabla\eta||x-x_0|^{\alpha/2}|x-x_0|^{-\alpha/2}dx \leq$$
$$K_\tau\left[\int_{\Omega_\delta}|\nabla(u-v)|^2|x-x_0|^\alpha dx\right]^{1/2}\left[\int_{\Omega_\delta}|\nabla\eta|^2|x-x_0|^{-\alpha}dx\right]^{1/2}$$

The lemma is proved. The main results of this research are the following statements.

Theorem 1: Let the terms (Eq. 3-6) and the iterationparameter $\tau \in (0, 2/L)$ are performed for the function g, then the following inequality is performed within the area $\Omega_\delta = \{x: |\nabla u(x)| \geq s_* - \delta\}$ for iteration sequence:

$$\|u_m - u\|_\delta \leq (K_\tau)^m \|u_0 - u\|_\delta,\ m = 0,1,2,\ldots,$$

Where:

$$\|v\|^2_\delta = \int_{\Omega_\delta}|\nabla v(x)|^2 dx,\ K_\tau = \max\{1-\tau\tilde{\mu}; \tau L - 1\},\ \tilde{\mu} = \mu\frac{\delta}{s_* + \delta}$$

it is obvious that $\inf_{\tau>0} K_\tau = L - \tilde{\mu} \backslash L + \tilde{\mu}$ is achieved at $\tau_* = 2 \backslash L + \tilde{\mu}$.

Proof: In order to solve u the problem (Eq. 10) the following equality is performed:

$$\int_{\Omega_\delta}\left(\nabla u(x), \nabla\eta(x)\right) = H(u,\eta;\tau)$$

Let subtract from this equality the equality (Eq. 11) and we obtain the following:

$$\int_{\Omega_\delta}\left(\nabla u(x), \nabla\eta(x)\right) - \int_{\Omega_\delta}\left(\nabla u_{m+1}(x), \nabla\eta(x)\right) =$$
$$H(u,\eta;\tau) - H(u_m,\eta;\tau)$$

Let in the previous equation $\eta = u - u_{m+1}$ and use Lemma 3 for the right part at $\alpha = 0$, let's determine:

$$\|u-u_{m+1}\|_\delta^2 \le K_\tau \|u-u_m\|_\delta \|u-u_{m+1}\|_\delta,\ m=0,1,2,\ldots$$

And thus, we obtain the necessary inequality. The theorem is proved.

Theorem 2: The conditions (Eq. 3-5) are performed for the function g with the constant p>n and the constants L and $\tilde{\mu} = \mu\delta\backslash s_* + \delta$, satisfying the following inequality:

$$A \equiv (L-\tilde{\mu})/(L+\tilde{\mu})\sqrt{(n-2)^2/(n-1)+1} < 1$$

Then there is such a small γ>0 that the solution of the problem (Eq. 10) in any internal subarea $\omega \subseteq \Omega_\delta$ Holder's one with the value γ. At τ = τ* and the initial approximation:

$$u_0 \in W_p^{(1)}(\tilde{\omega}) \cap \overset{0}{W}{}_2^{(1)}(\Omega_\delta),\ \omega \subset\subset \tilde{\omega} \subset\subset \Omega_\delta$$

the iteration process (Eq. 11) converges as a geometric progression with the value A_ε = A+ε (ε = ε(γ) is arbitrarily small for a sufficiently small γ) within the norm of the area $W_{2,\alpha}^{(1)}(\Omega_\delta)$ where α = 2-n-2γ (and consequently within the norm of the area $C^\gamma(\omega)$).

Proof: Similar to the proof 3.1 (Koshelev, 1986), demonstrating the peculiar moments in details. Let's introduce the sequence of strictly nested areas so that $\omega \subset\subset \cdots \subset\subset \omega_m \subset\subset \ldots \subset\subset \omega_0 \subset\subset \tilde{\omega}$. Assuming that $u_m \in W_p^{(1)}(\omega_m)$ and using the Lemma 1 (the term 3 is used to prove the Theorem 3.1) (Koshelev, 1986), we obtain that $\nabla u_m - \tau G(x,\nabla u_m) \in L_p(\omega_m)$ and then it follows from (Eq. 11) that $u_{m+1} \in W_p^{(1)}(\omega_{m+1})$. As $u_0 \in W_p^{(1)}(\tilde{\omega})$, we have $u_m \in W_p^{(1)}(\omega)$ for m = 0, 1, 2, etc.

Further we choose a sufficiently small γ>0 (it is possible due to the term p>n, see the proof of the Lemma 2.6 (Koshelev, 1986) that at α = 2-n-2γ the inclusion $W_p^{(1)}(\omega) \subset W_{2,\alpha}^{(1)}(\omega)$ is performed and thus $u_m \in W_{2,\alpha}^{(1)}(\omega)$ for m = 0, 1, 2, etc. Let's note $w_m = u - u_m$ and put down the difference of two sequential integral identities (Eq. 11):

$$\int_\omega (\nabla w_{m+1}, \nabla\eta(x))\,dx = \int_\omega (\nabla w_m, \nabla\eta)\,dx - \tau \int_\omega (G(x,\nabla u) - G(x,\nabla u_m), \nabla\eta)\,dx$$

Let's apply to the right side of the last equation the Lemma 3 (in the proof of the Theorem 3.1 the Lemma 2.4 is used (Koshelev, 1986) at α = 2-n-2γ and we obtain the following inequality:

$$\int_\omega (\nabla w_m, \nabla v)dx \le \frac{L-\tilde{\mu}}{L+\tilde{\mu}}\left[\int_\omega |\nabla w_m|^2 |x-x_0|^\alpha dx\right]^{1/2} \left[\int_\omega |\nabla v|^2 |x-x_0|^{-\alpha} dx\right]^{1/2}$$

The further course of the proof coincides with the proof of the Theorem 3.1 (Koshelev, 1986). We obtain the following inequality:

$$\|u-u_m\| \le CA^n$$

There is the norm in the space H_α at the left part of the inequality. From the inclusion of H_α into C^γ, the Holder decision follows and the convergence of the iterative process within the norm of the area C^γ for any subdomain ω. The theorem is proved.

CONCLUSION

Thus, in the field of filtration the Holder continuity of additives at inhomogeneous nonlinear stationary filtration problem with degeneration at the presence of a point source is proved. Regarding the iterative process of this problem, it is found out that the optimal value of the parameter converges in a continuous norm.

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