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Generalized-Quasi-Maximum Likelihood Estimator for Conditional Value at Risk (CVAR): An Analysis in the Iranian Stock Market Data

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Abstract: Value at risk is a statistical risk management technique that monitors and quantifies the risk level associated with an investment portfolio. The maximum amount of loss over a specified time horizon with a given confidence level is usually measured by this technique. The presented study aimed at estimating the conditional value at risk for Tehran stock market data through two steps. First the volatility parameter is estimated with a generalized-Quasi-Maximum Likelihood Estimator (gQMLE) and then empirical quantile of the residuals is estimated using the estimated rescaled innovations.

Key words: GARCH, generalized quasi maximum likelihood estimator, GJR, value at risk, estimateor

INTRODUCTION

Value at Risk (VaR) is a method of presenting the market risk which basically combines the sensitivity of the portfolio to market changes and the probability of a given market change. It was the 1970's and the 1980's when the concept was presented and financial institutions have begun to discover the internal models in measuring the risk as a whole. The credibility of this concept received a boost in 1996 when the basel committee on banking supervision proposed that banks could apply VaR to calculate the regulatory capital required for general market risk provided that they met certain standards (Choudhry, 2006).

Value at risk measures the maximum potential loss of a given portfolio over a specific period at a given confidence level which is normally chosen to be 1 or 5%. It is preferable to take into account all the available information by reasoning on the conditional distribution of the returns (McNeil *et al*., 2005; Kuester *et al*., 2006). Formally the (conditional VaR) CVaR of a sequence of returns (r_i) is actually the opposite of the α-quantile of the conditional distribution, defined as:

$$CV\alpha R_t(\alpha) = -\inf\left\{x : p(r_{t+1} \le x | r_u, u \le t) \ge \alpha\right\}$$

One can find the alternative names of CVAR in literatures as average value at risk, expected shortfall or Tailed conditional expectation. Richt-arik and Wolf for more details. CVaR is easily derived from the generalized α-tail distribution of a random variable X (which represents loss). For more detailed discussion see (Rockafellar and Uryasev, 2000). Rockafellar and Uryasev (2000) proposed a linear program in Rockafellar and Uryasev (2002) to optimize the CvaR of a portfolio. Norm properties of CVaR were identified by Pavlikov and Uryasev (2014), enriched by some ideas of the researcher. Assume the general conditionally heteroscedastic model for log-returns:

$$r_t = \sigma_t(\theta_0)\eta_t \quad (1)$$

where, η_t is a sequence of independent and identically distributed (iid) random variables which is independent of r_i, i<t. $\theta_0 \epsilon R^m$ is a parameter which belongs to a compact parameter space Θ and σ^2 is generally referred to as the volatility of returns which is a positive measurable function of the past log-returns defined as:

$$\sigma_t = \sigma(\theta_0) = (r_{t-1}, r_{t-2}, ..; \theta_0)$$

Simple example of above model is ARCH which has been defined by Bollerslev (1986). Accordingly the volatility model is calculated as:

$$VaR_t(\alpha) = -\sigma_t(\theta_0)\varepsilon_\alpha,$$

where, ε_a is the α-quantile of the $p_{\hat{\eta}}$ the distribution of the innovations. Needless to say, the Gaussian QMLE model is the most widely used estimator of ARCH-type models. Bollerslev (1986) also presented GARCH (p, q) Model as a widely used example of Eq. 1 which is defined as follows:

$$\begin{cases} r_t = \eta_t \sigma_t \\ \sigma_t^2 = \omega_0 + \sum_{i=1}^{q} \alpha_{0i} r_{t-i}^2 + \sum_{j=1}^{q} \beta_{0j} \sigma_{t-j}^2 \end{cases} \quad (2)$$

where, $\omega_0>0$, $\alpha_{0i}\geq 0$, $\beta_{0j}\geq 0$. Note that r_t is a strictly stationary and there exists s>0 such that $E|r_1|^s<\infty$. Setting few regularity assumptions, Consistency and Asymptotic Normality (CAN) of this estimator is achieved. For example, Francq and Zakoian (2004) for the case of standard GARCH and ARMA-GARCH Models, Straumann and Mikosch (2006), Bardet and Wintenberger (2009) for more general models. Horv and Kokoszka (2003) also defined generalized non-Gaussian QMLE (gQMLE) and established their CAN under alternative identifiability conditions. For the general model Eq. 1 (Francq and Zakoian, 2013) showed that particular gQMLE lead to convenient one-step predictions. Francq and Zakoian (2015) proposed a gQMLE which allows estimating a conditional VaR in one step and compared this method with the more standard two-step method which consists of estimating the volatility parameter by Gaussian QMLE and the quantile of the innovations by the empirical quantile of the residuals.

Asymmetric Power GARCH (APARCH) models had been introduced by Ding *et al*. (1993) which included the standard GARCH, the TARCH and GJR models. Glosten *et al*. (1993). Let, $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$ the model is defined as follows:

$$\begin{cases} r_t = \sigma_t \eta_t \\ \sigma_t^\delta = \omega_0 + \sum_{i=1}^{q} \alpha_{0i} + (r_{t-i}^+)^\delta + \alpha_{i0} - (-r_{t-i})^\delta + \sum_{j=1}^{q} \beta_{j0} \sigma_{t-j}^\delta \end{cases}$$

where, $\delta>0$, $\omega>0$, $\alpha_{0i0}\geq 0$, $\alpha_{0i-}\geq 0$, $\beta_{0j}\geq 0$. When α_{i0}->α_{i0+}, negative return has a higher impact on the future volatility than a positive return of the same magnitude which is a well-documented as "leverage effect". Note that $\delta = 2$, $\alpha_{0i+} = \alpha_{0i-}$ leads to standard GARCH $\delta = 1$ will get to TARCH Model and $\delta = 2$ leads to GIR model. For more detailed information (Hamadeh and Zakoian, 2011).

MATERIALS AND METHODS

In this study, the conditional VaR is estimated by investigating the use of gQMLE's based on a generic instrumental density h. Under mild regularity conditions by Ghourabi *et al*. (2016) the standard Gaussian QMLE which is based on the instrumental density, $\varphi(x) = (1/\sqrt{2\pi} e^{\frac{-x^2}{2}})$ converges to the volatility parameter θ_0 assume a continuous function H such that for any $\epsilon\Theta$, K>0 and any sequence X_i we have:

$$K\sigma(x_1, x_2, ..., \theta) = \sigma(x_1, x_2, ..., H(\theta, K)) \quad (3)$$

which indicates that the parametric form of the volatility is stable by scaling. Given the observations r_1, r_2, ..., r_n and arbitrary initial values $\tilde{r}_i$ for $i\leq 0$, we define:

$$\tilde{\sigma}_t(\theta) = \sigma(r_{t-1}, r_{t-2}, ..., \tilde{r}_1, \tilde{r}_0, \tilde{r}_{-1}, ..., \theta)$$

Now consider the QML criterion with the instrumental density, h>0 as follows:

$$\tilde{Q}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} g(r_t, \tilde{\sigma}_t(\theta))$$

with $g(x, \sigma) = \log 1/\sigma h(x/\sigma)$. Then the (generalized) QMLE is given by:

$$\hat{\theta}_n^* = \arg\max_{\theta\in\Theta} \tilde{Q}_n(\theta)$$

Note that the function $g\rightarrow Eg(\eta_0, \sigma)$, takes its values in $[-\infty, \infty]$ and has a unique maximum at some point $\sigma\epsilon(0, \infty)$. Now we define the parameter $\theta_0 = H(\theta_0, \sigma)$ which belongs to the compact parameter space Θ. Under mild conditions presented in Ghourabi *et al*. (2016) almost surly convergence of g-QMLE is achieved as follows:

$$\hat{\theta}_n^* \rightarrow \theta_0^* \quad \text{as}$$

Now for the next step we define:

$$\eta_0^* = \frac{\eta_t}{\sigma_*} \quad (4)$$

then the general volatility model is presented as:

$$VaR_t(\alpha) = -\sigma_{t+1} \theta_0^* \zeta_\alpha^* \quad (5)$$

where, $\zeta^*\alpha$ is the α-quantile of η^*_0 and:

$$\theta_{0,\alpha}^* = H(\theta_0^* - \xi_0^*) \quad (6)$$

which is called the VaR parameter by Francq and Zakoian (2015). Contrary to the one-step estimator, the resulting two step estimator of the VaR does not take advantage of the (hypothesized) symmetry of the errors distribution. For detailed discussion in asymptotic distribution of two-step estimators of the VaR parameter we refer to Bardet and Wintenberger (2009) and Ghourabi *et al.* (2016). Under mild condition, if H is differentiable at $(\theta_0\text{-}\zeta_\alpha)$ one may have:

$$0, G_* \sum{}^* G_* \tag{7}$$

where, $G = [\theta H(\theta, K/\theta(\theta, K)]$ is the covariance matrix defined in Francq and Zakoian (2013). The simplest but often very useful GARCH process is of course the GARCH Eq. 1 process defined as:

$$\sigma_t^2 = \omega_0 + \alpha_0 r_{t-1}^2 + \beta_0 \sigma_{t-1}^2$$

where, $\theta_0 = \omega_0, \alpha, \beta_0()\epsilon(0, \infty)\times[0, \infty)\times[0, 1]$. For the mentioned model we have:

$$\sigma_t^2 = \sum_{i=1}^{\infty} \beta_0^{i-1}(\omega_0 + \alpha_0 r_{t-1}^2)$$

Now assuming (Eq. 5 and 6), the model can be reparametrized as:

$$\begin{cases} r_t = \sigma_t^* \eta_t^* \\ \{\sigma_t^* = \sigma(r_{t-1,} r_{t-2}, \ldots, \theta_0^*) \end{cases}$$

In such a case, the matrix G defined in Eq. 7, can be written as:

$$G_* = \begin{bmatrix} (\xi_\alpha^*)^2 & 0 & 0 & -2\xi_\alpha^* \omega_0^* \\ 0 & (\xi_\alpha^*)^2 & 0 & -2\xi_\alpha^* \omega_0^* \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now for any $\theta^*_0 = (\omega^*_0, \alpha^*_0, \beta^*_0)$ we have:

$$(\omega_0^*, \alpha_0^*, 0)\frac{\partial \sigma_t^2(\theta_0^*)}{\partial \theta} = \omega_0^* + \alpha_0 r_{t-1}^2 + \beta_0^*$$

$$\left\{(\omega_0^*, \alpha_0^*, \beta_0^*)\frac{\partial \sigma_{t-1}^2(\theta_0^*)}{\partial \theta}\right\} =$$

$$\sum_{i=0}^{\infty} \beta_0^* \left\{\omega_0^* + \alpha_0^* r_{t-1}^2\right\} = \sigma_t^2(\theta_0^*)$$

Using again the delta method, confidence intervals for $V\alpha R_t(\alpha) = -\sigma_{t+1}\theta^*_0, \alpha$ at a given estimation-risk level can be deduced, exactly as Francq and Zakoian (2015) did for the VaR estimation method based on the Gaussian QMLE.

RESULTS AND DISCUSSION

Numerical studies: In this study, we are going to fit the above mentioned models and methods to calculate VaR on real datasets and evaluate their performance. The data used in the study is the daily value of 3 stock market indices, K-SHARQ (Shomal Sharq Shahroud Industrial, http://www.shomalshargh.com.) NIORD (National Iranian oil refining and Distribution of Tehran, http://www.niordc.ir/index.aspx?siteid=77andpageid=536), SORC (Shiraz oil refining company, http://www.sorc.ir/sorc.ir.) obtained from the data stream database services of Tehran Over-the-Counter Market (OTC) (http://www.ifb.ir.). Since 2009, Iran has been developing an over-the-counter market for bonds and equities. OTC provides a complete available achieve of data, based on different sectors and dates. Our sample covers daily log-return data from early March 2013 to the end of May 2014, when the historical data exist. To the best our knowledge, this has not been addressed earlier via QMLE Method.

We report parameters of GARCH 1 and GJR 1 based on for two different instrumental densities h, namely the Gaussian and Student (v) distributions for log-returns in Table 1 and 2. The estimated standard deviations are given into brackets. Then for $\eta t \sim \lceil(1, 2, d)$, d = 1.7, 2, for Gaussian and Student (i) distributions the parameters of models are estimated in Table 3 and 4. Finally in Table 5 and 6 we report the VaR parameters at levels $\alpha = 0.05$and $\alpha = 0.01$. All the programs are done in Matlab (MATLAB emulation package). Figure 1 we can see log-returns of all indices versus time.

Generally our empirical results reveal that according to the tables, the more volatile return environment leads to significantly wider VaR distributions, i.e., VaR point estimates are associated with higher uncertainty. This effect is most pronounced in (more complex) models that react faster to volatility changes as it is the case for our GARCH and GJR Models in SORC data.

Another important finding is that over the 3 indices, it is clear to note that in GARCH (1, 1) Model, for K-SHARQ and NIORD the estimated parameters based on the Gaussian and Student distributions are quite similar, while there is significant difference between estimated parameters based on mentioned distributions for SORC. Table 2 reports the same result for GJR (1, 1) Model. Based on $\eta t \sim \lceil(1, 2, d)$ and levels $\alpha = 0.05$ and $\alpha = 0.01$ for both GARCH (1, 1) and GJR (1, 1). Table 3-6 in line with the evidence presented below, show significate difference between SORC estimated parameters based on Gaussian and Student distribution for d = 1.72.

Table 1: Parameter Estimation for GARCH (1, 1) based on Gaussian and Student (v) distributions. The estimated standard deviation are displayed in brackets

Index	w_0	α_0	β_0
NIORD			
Gaussian-QMLE	$3.7759e^{-5}$ ($5.0046e^{-5}$)	0.12198 (0.11104)	0.840131 (0.3246)
Student-QMLE	$3.7893e^{-5}$ ($8.6713e^{-5}$)	0.12373 (0.12216)	0.83922 (0.15074)
SORC			
Gaussian-QMLE	$4.4462e^{-5}$ ($2.5428e^{-5}$)	0.27393 (0.10434)	0.69038 (0.091014)
Student-QMLE	$6.3097e^{-5}$ ($3.0962e^{-5}$)	0.47407 (0.19059)	0.52509 (0.1144)
K-SHARQ			
Gaussian-QMLE	$4.1868e^{-5}$ ($3.3712e^{-5}$)	0.11386 (0.05592)	0.87598 (0.08858)
Student-QMLE	$4.1665e^{-5}$ ($3.7409e^{-7}$)	0.11505 (0.06514)	0.8314 (0.0889)

Table 2: Parameter Estimation for GJR(1,1) based on Gaussian and Student (v) distributions. The estimated standard deviation are displayed in brackets

Index	w_0	α_0	β_0	$Levrage_0$
NIORD				
Gaussian-QMLE	$3.2597e^{-5}$ ($4.6374e^{-5}$)	0.085761 (0.12531)	0.85552 (0.12309)	0.053279 (0.13574)
Student-QMLE	$3.27953e^{-5}$ ($5.9808e^{-5}$)	0.08779 (0.15906)	0.85414 (0.15801)	0.053026 (0.1478)
SORC				
Gaussian-QMLE	$5.2658e^{-5}$ ($2.9152e^{-5}$)	0.31918 (0.13467)	0.67628 (0.09908)	-0.08999 (0.1787)
Student-QMLE	$6.4838e^{-5}$ ($3.134e^{-5}$)	0.51116 (0.21912)	0.52002 (0.11996)	-0.06236 (0.3102)
K-SHARQ				
Gaussian-QMLE	$3.2526e^{-5}$ ($2.433e^{-5}$)	0.12322 (0.06587)	0.88425 (0.0637)	-0.1081 (0.07894)
Student-QMLE	$3.2554e^{-5}$ ($2.449e^{-7}$)	0.12499 (0.07483)	0.8837 (0.06405)	-0.1081 (0.08385)

Table 3: VaR Estimation for GARCH(1,1) based on Gaussian and Student (v) distributions when $\eta_t \sim \Gamma$ (1, 2, d), d = 1.7, 2 and level $\alpha = 0.05$

Index	d	$w_{0.05}$	$\alpha_{0.05}$	$\beta_{0.05}$
NIORD				
Gaussian-QMLE	1.7	$1.0445e^{-4}$	0.3374	0.8403
	2	$1.0813e^{-4}$	0.3949	0.8403
Student-QMLE	1.7	$1.0361e^{-4}$	0.3383	0.8393
	2	$1.0538e^{-4}$	0.3441	0.8392
SORC				
Gaussian-QMLE	1.7	$1.3073e^{-4}$	0.8058	0.6902
	2	$1.2971e^{-4}$	0.7997	0.6904
Student-QMLE	1.7	$1.7863e^{-4}$	1.3446	0.5251
	2	$1.8433e^{-4}$	1.3875	0.5251
K-SHARQ				
Gaussian-QMLE	1.7	$1.0943e^{-4}$	0.2975	0.8319
	2	$1.1100e^{-4}$	0.3015	0.8317
Student	1.7	$1.1172e^{-4}$	0.3097	0.8315
	2	$1.1926e^{-4}$	0.3305	0.8314

Table 4: VaR Estimation for GJR(1,1) based on Gaussian and Student (v) distributions when $\eta_t \sim \Gamma$ (1, 2, d), d = 1.7, 2 and level $\alpha = 0.05$

Index	d	$w_{0.05}$	$\alpha_{0.05}$	$\beta_{0.05}$	$Levrage_{0.05}$
NIORD					
Gaussian-QMLE	1.7	$9.5763e^{-5}$	0.252	0.8555	0.1565
	2	$9.1544e^{-5}$	0.2408	0.8555	0.1495
Student	1.7	$9.6194e^{-5}$	0.2575	0.8541	0.1555
	2	$9.5494e^{-5}$	0.2513	0.8555	0.1561
SORC					
Gaussian-QMLE	1.7	$1.3860e^{-4}$	0.8402	0.6763	-0.2366
	2	$1.4907e^{-4}$	0.9035	0.6762	-0.2545
Student	1.7	$2.0623e^{-4}$	1.6259	0.5201	-0.1982
	2	$2.0457e^{-4}$	1.6127	0.5200	-0.1966
K-SHARQ					
Gaussian-QMLE	1.7	$9.5188e^{-5}$	0.3605	0.8841	-0.3159
	2	$7.5879e^{-5}$	0.2874	0.8842	-0.2520
Student	1.7	$7.8588e^{-5}$	0.3018	0.8834	-0.2630
	2	$6.7569e^{-5}$	0.2560	0.8842	-0.2244

Table 5: VaR Estimation for GARCH(1,1) based on Gaussian and Student (v) distributions when $\eta t \sim \Gamma$(1, 2, d), d = 1.72 and level $\alpha = 0.01$

Inde	d	$w_{0.01}$	$\alpha_{0.01}$	$\beta_{0.01}$
NIORD				
Gaussian-QMLE	1.7	$2.1416e^{-4}$	0.6918	0.8403
	2	$2.4038e^{-4}$	0.7766	0.8403
Student	1.7	$2.0655e^{-4}$	0.6744	0.8393
	2	$2.5038e^{-4}$	0.8176	0.8392

Table 5: Continue

Inde	d	$w_{0.01}$	$\alpha_{0.01}$	$\beta_{0.01}$
SORC				
Gaussian-QMLE	1.7	$2.6394e^{-4}$	1.6269	0.6902
	2	$2.9443e^{-4}$	1.8152	0.6904
Student	1.7	$3.6213e^{-4}$	2.7258	0.5251
	2	$4.566e^{-4}$	3.4375	0.5251
K-SHARQ				
Gaussian-QMLE	1.7	$2.2299e^{-4}$	0.6063	0.8319
	2	$2.6268e^{-4}$	0.7134	0.8317
Student	1.7	$2.3104e^{-4}$	0.6405	0.8315
	2	$2.5773e^{-4}$	0.7141	0.8314

Table 6: VaR Estimation for GJR(1,1) based on Gaussian and Student (v) distributions when $\eta t \sim \Gamma(1, 2, d)$, d = 1.72 and level $\alpha = 0.01$ and investigated how

Index	d	$w_{0.01}$	$\alpha_{0.01}$	$\beta_{0.01}$	$Lverage_{0.01}$
NIORD					
Gaussian-QMLE	1.7	$2.0215e^{-4}$	0.5319	0.8555	0.3305
	2	$1.6839e^{-4}$	0.4430	0.8555	0.2750
Student	1.7	$1.6683e^{-4}$	0.4466	0.8541	0.2697
	2	$1.9065e^{-4}$	0.5016	0.8555	0.3117
SORC					
Gaussian-QMLE	1.7	$2.6474e^{-4}$	1.6049	0.6763	-0.4520
	2	$2.9225e^{-4}$	1.7712	0.6762	-0.4990
Student	1.7	$4.2933e^{-4}$	3.2849	0.5201	-0.4126
	2	$4.2247e^{-4}$	3.3305	0.5200	-0.4059
K-SHARQ					
Gaussian-QMLE	1.7	$1.5590e^{-4}$	0.5905	0.8841	-0.5175
	2	$1.6229e^{-4}$	0.6147	0.8842	-0.539
Student	1.7	$1.4975e^{-4}$	0.5751	0.8834	-0.5011
	2	$1.7167e^{-4}$	0.6503	0.8842	-0.5702

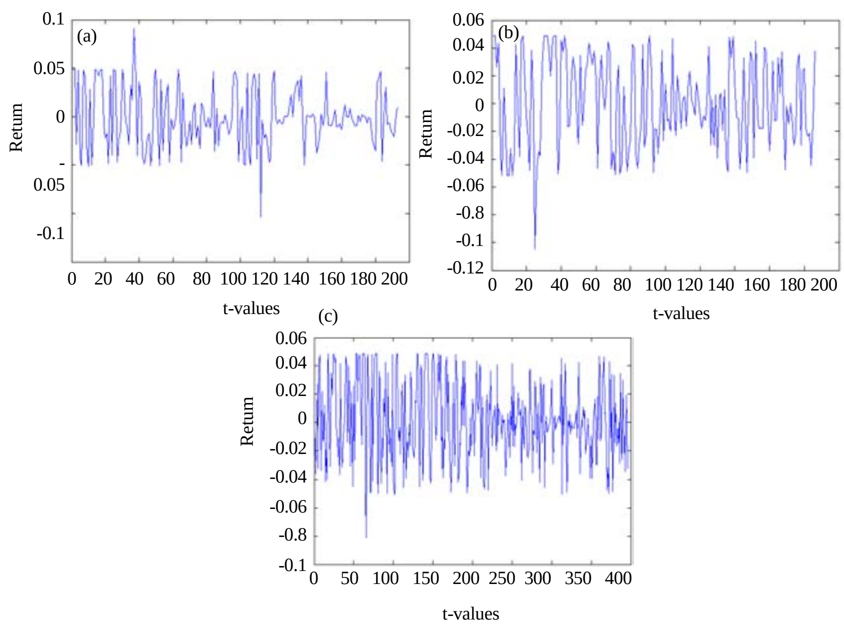

Fig. 1: Log-return of indices: SORC, NIORD and K-SHARQ

CONCLUSION

Value at Risk (VaR) is the standard measure that financial analysts use to quantify risk. It is defined as the maximum potential loss in value of a portfolio of financial instruments with a given probability over a certain horizon. There are two contributions in this study. We discussed the concept of CVaR two-step evaluation method by investigating the use of gQMLE's based on a generic instrumental density h and investigated how VaR distributions could be used in market risk management and how to account for VaR uncertainty in choosing traditional VaR point estimates used to calculate capital requirements for financial institutions. Second, the empirical part of this study is based on 3 different financial assets with daily data from the period between early March 2013 and end of May 2014. Our parametric VaR modeling is based on a GARCH framework for modeling volatility. In a first step we analyze the effect of the return volatility on the uncertainty of VaR estimates. Our empirical results reveal that the uncertainty in VaR estimates highly depends on the volatility level in the market. Then the natural two-step conditional VaR at the level α has been obtained which leads to a consistent estimation of the VaR.

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