

Asymptotic Analysis of the Boundary Layer by Matching the WKB Solutions of the Inner and Outer Layers of a Neo-Hookean Cylindrical Shell

¹Taherh Shokuhi, ²Majid Pazand and ³Nasrin Hamidi

¹Department of Applied Mathematics,

²Department of Agricultural Economics, Higher Educational Complex of Saravan, Saravan, Iran

³Ministry of Education, Zahedan, Iran

Abstract: We analyzed and compared the asymptotic outer, inner and the matching solutions with the numerical counterpart results of the eigen-value problem of a neo-Hookean elastic cylindrical shell of arbitrary thicknesses subjected to an external hydrostatic pressure. In order to study thin-walled shells (i.e., a thin layer between the two regions $A_1^{-1} = O(1)$ and $A_1^{-1} = O(1/n)$, where A_1 and a_1 are the inner radii of the shell before and after deformation respectively on $0 < A_1 < 1$) and for the purpose of matching the two regions, it is necessary to reconsider the asymptotic solutions obtained previously and offer the summarized relations of the relevant eigenvalues, i.e., $\mu = a_1/A_1$. For analyzing thin-walled shells, the theory of boundary layer and also Van Dyke's matching rule has been employed.

Key words: WKB method, boundary layer theory, Van Dyke's matching rule, finite elasticity, thin-walled shells

INTRODUCTION

WKB theory is a powerful tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter like, it contains boundary layer theory as a special case. The boundary layer theory shown how to construct an approximate solution to a differential equation containing a small parameter. This construction requires one to match slowly varying outer solutions to rapidly varying inner solutions. We will consider the method of matched asymptotic expansions applied to the fourth-order differential equation related to an incompressible isotropic homogeneous elastic shell. The purpose of this study is to introduce the notion of matched asymptotic expansions. Asymptotic matching is an important perturbation method which is used often in both boundary layer and WKB theories to determine analytically the approximate global properties of the solution to a differential equation. Asymptotic matching is usually used to determine a uniform approximation to the solution of a differential equation and to find other global properties of differential equations such as the eigen values. The procedure of finding the leading-order boundary layer approximation applied to the solution of a differential equation of an elastic shell may be formulated and one can obtain and asymptotically match the layers in an overlap region. The self-consistency of

the boundary layer theory depends on the success of the asymptotic matching. Ordinarily, if the inner and outer solutions match to all thicknesses, then boundary layer theory gives an asymptotic approximation to the exact solution of the differential equation. The possibility of matching the outer solution with the inner solution to form a composite solution is discussed. It should be noticed that since the outer solution is independent of A_1 , the composite solution is simply the inner solution according to Van Dyke's matching rule (Bush, 1992). Therefore, if Van Dyke's matching rule works for this problem, the inner solution should also be valid in the outer region where $A_1^{-1} = O(1)$. It is verified here that the inner and outer solutions can be matched at leading order but not at the second order. The failure of Van Dyke's matching rule at second order is discussed in details on (Fu, 1998). We can show and indeed provide a uniformly valid curve for μ_1 against A_1 for $A_1^{-1} = O(1)$ and $A_1^{-1} = O(1/n)$ by solving numerically Eq. 19 with the use of compound matrix method (Fig. 1) (Sanjaranipour, 2010). Since, the numerical solution corresponding to Eq. 19 fails for $A_1^{-1} < O(1/n)$ and in order to obtain a proper thickness, we have to continue our asymptotic analysis for the case of two term inner solution. Fu and Lin (2002) performed successfully matching the outer and inner solutions concerning the analysis of the buckling of an spherical shell made of an elastic Neo-Hookean material which perfectly

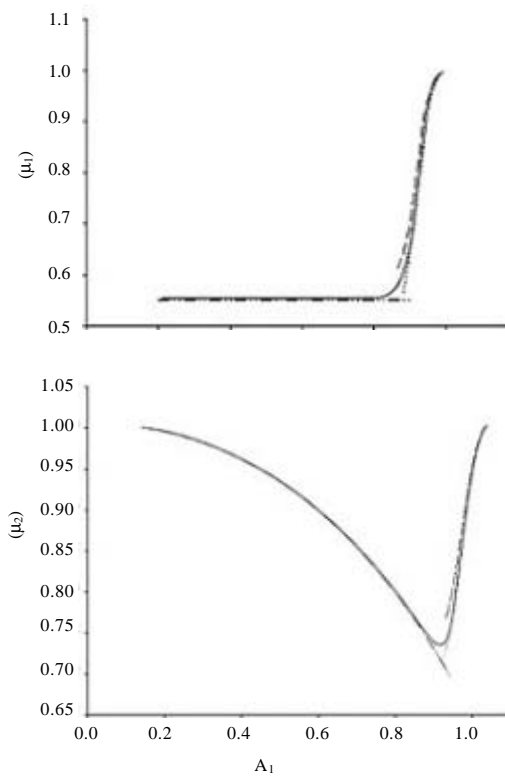


Fig. 1: Neutral stability curves for $n = 20$. Solid lines: numerical result; dashed lines: one-term asymptotic result; dotted lines: two-term asymptotic results

confirms the results obtained of thin shells with the numerical results but this conformation has not been done in cylindrical tube which is going to be study here.

Definition of the problem: We consider an incompressible isotropic homogeneous elastic shell and assume that the outer surface of the shell is subjected to a hydrostatic pressure and that the tube is in a state of plain strain. The co-ordinates of a representative material particle in the un-deformed and deformed states are (R, Θ, Z) and (r, θ, z) , respectively. Thus, the hollow cylinder in its un-deformed configuration is defined by $0 < A_1 = R = A_2, 0 = \Theta = 2\pi, 0 = Z = L$, the plane-strain deformation is denoted by:

$$r = r(R) \tag{1}$$

Where:

- r = The radial coordinate of a material point which is at a distance
- R = The radial coordinate) from the center of the shell before the pressure is applied
- A_1 and A_2 = are the inner and outer radii, respectively and the deformed state is:

$$0 < a_1 \leq r \leq a_2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq L, \tag{2}$$

where, a_1 and a_2 are the deformed inner and outer radii, respectively. We assume that:

$$a_1 = \mu_1 A_1 \text{ and } a_2 = \mu_2 A_2 \tag{3}$$

where μ_1 and μ_2 are controlling parameters which for the compression problem satisfy $0 < \mu_1, \mu_2 < 1$. Incompressibility implies no volume change and so μ_1 and μ_2 are related by:

$$\begin{aligned} (\mu_1^2 - 1)A_1^2 &= (\mu_2^2 - 1)A_2^2 \\ \mu_2^2 &= 1 - (1 - \mu_1^2) \frac{A_1^2}{A_2^2}, 0 < A_1 < A_2 \end{aligned} \tag{4}$$

Either μ_1 or μ_2 can be taken as the controlling parameters and the aim of a linear stability analysis is to find the value of μ_1 or μ_2 at which the shell buckles. We shall assume that all variables and parameters which have adimension of length have been scaled by A_2 . Thus, we have:

$$\begin{aligned} a_1 &= \mu_1 A_1, a_2 = \mu_2 A_2 \\ \mu_2^2 &= 1 - (1 - \mu_1^2) A_1^2, 0 < A_1 < 1 \end{aligned} \tag{5}$$

The eigen value problem resulting from a linear stability analysis can be found in Haughton and Ogden (1979) and also in refs (Sanjaranipour, 2010) and consists of solving the following fourth-order differential equation for a neo-Hookean material:

$$\begin{aligned} V'''' + \left(\frac{6}{r} + \frac{2k}{r^3} \right) V'' + QV'''' + \left[\frac{-k^2(3+n^3)}{r^6} + \frac{2k(1-n^2)}{r^4} + \frac{(5-2n^2)}{r^2} \right] V'' \\ + Q^2 V'' + \left[\frac{k^3(3+n^3)}{r^9} + \frac{k^2(5+n^2)}{r^7} \right] V' + \left[\frac{k(1-4n^2)}{r^5} + \frac{(1+2n^3)}{r^3} \right] V \\ + Q^3 V'' + (n^2 - 1) \left[\frac{3k^2}{r^3} + \frac{2k}{r^6} + \frac{(n^2 - 1)}{r^4} \right] Q^2 V = 0 \end{aligned} \tag{6}$$

Subject to the boundary conditions:

$$V'' + \frac{1}{r} V' + \frac{n^2 - 1}{r^2} V = 0 \tag{7}$$

$$V'''Q^{-1} + \left[\frac{2k}{r^3} + \frac{4}{r} \right] V'' - \left[Q \left(\frac{k^2}{r^6} + \frac{n^2 - 1}{r^2} \right) + \frac{2n^2}{r^2} Q^{-1} \right] V' + \left[\frac{n^2 - 1}{r^3} - \frac{k(n^2 - 1)}{r^6} \right] V = 0 \tag{8}$$

onr = a1, a2, where V = V (r), a prime denotes differentiation with respect to r, n is the longitudinal mode number and:

$$Q = \frac{r^2}{k + r^2}, k = A_1^2 (1 - \mu_1)^2 \tag{9}$$

It is necessary to mention that this problem has been solved by the use of the asymptotic WKB method and the results are related to the outer i.e., A1-1 = O(1) and inner, i.e., A1-1 = O(1/n) solutions separately and are compared with the counter part numerical results obtained throughout the whole region, i.e., 0 < A1 < 1. Our main aim on this study is to solve this logarithmic layer which appears in between the two mentioned layers with the help of the Van Dyke matching rule and also the boundary layer theory. In order to be able to analysis this thin-walled shells, first a brief description about the previously studied inner and outer solutions (Sanjaranipour, 2010) should be written down here.

MATERIALS AND METHODS

Asymptotic analysis for thick shells in order to find the Boundary layer. The linear eigen-value problem is to solve Eq. 3 subject to Eq. 8-9 onr = a1, a2. In order to find the eigenvalue μ1 and in large n limit, we look for a WKB solution e.g., Bender and Orszag (1978) and (Sanjaranipour, 2010) of the following form:

$$V = \exp \left(n \int_{a1}^r s(u) du \right) \tag{10}$$

$$S = S_0 + \frac{1}{n} S_1 + \frac{1}{n^2} S_2 + \dots = n \sum_{m=0}^{\infty} \frac{1}{n^m} S_m \tag{11}$$

By substituting relations Eq. 10-11 in Eq. 6-8 and by equating the coefficients of the like powers of n with the aid of Mathematica, we obtain a number of differential equations for S0, S1,.... There are four independent solutions for S. We denote the ith solution by S(i) and write:

$$S^{(i)} = n S_0^{(i)} + S_1^{(i)} + \frac{1}{n^2} S_2^{(i)} + \dots \tag{12}$$

Where the details of S0⁽ⁱ⁾, S1⁽ⁱ⁾, S3⁽ⁱ⁾, ..., (i = 1, 2, 3, 4) are given on (Sanjaranipour, 2010). All the above functions are analytic in the region of interest, i.e., 0 < a1 = r = a2 = 1. Now, we focus on the general solution which for V is given by:

$$V = \sum_{m=1}^4 k_i \exp \left(n \int_{a1}^r s^{(i)}(u) du \right) \tag{13}$$

where, Ki are constants and by substituting Eq. 13 and the relevant derivatives into the equations of the boundary conditions Eq. 7 and 8 we obtain a matrix equation of the form:

$$\sum_{j=1}^4 C_{ij} K_j = 0, (i = 1, 2, 3, 4) \tag{14}$$

where, (Cij) is given by:

$$C_{ij} = \begin{bmatrix} F_1(a_1)F_2(a_1)F_3(a_1)F_4(a_1) \\ G_1(a_1)G_2(a_1)G_3(a_1)G_4(a_1) \\ E_1F_1(a_1)E_2F_2(a_1)E_3F_3(a_1)E_4F_4(a_1) \\ E_1G_1(a_1)E_2G_2(a_1)E_3G_3(a_1)E_4G_4(a_1) \end{bmatrix} \tag{15}$$

The constants E_j and function F_j(r) and G_j(r) in the above matrix are defined by:

$$E_j = \exp \left(n \int_{a1}^{a2} s^{(i)}(u) du \right) \tag{16}$$

$$F_j(r) = \left(S^{(i)} \right)^2 + \frac{1}{n} \left(S^{(i)} \right)' + \frac{1}{nr} S^{(i)} + \frac{n^2 - 1}{n^2 r^2} \tag{17}$$

$$G_j(r) = Q^{-1} \left[\left(S^{(i)} \right)^3 + \frac{3}{n} \left(S^{(i)} \right)' + \frac{1}{n^2} S^{(i)*} \right] + \left[\frac{2k}{r^3} + \frac{4}{r} \right] \left[\frac{1}{n^2} S^{(i)} \right]' + \frac{1}{n} \left(S^{(i)} \right)^2 - \left[Q \left(\frac{k^2}{r^6} + \frac{n^2 - 1}{r^2} \right) + \frac{2n^2}{r^2} Q \right] \frac{1}{n^2} S^{(i)} + \frac{1}{n^3} \left[\frac{n^2 - 1}{r^3} + \frac{k(n^2 - 1)}{r^5} \right] \tag{18}$$

Where:

- S₀ = Denotes differentiation with respect to r
- a2-a1 = O(1) = The constants
- E₁ and E₃ = Are exponentially large whereas
- E₂ and E₄ = Are exponentially small. A none trivial solution for (K_j) requires:

$$\det (C_{ij}) = 0 \tag{19}$$

Which is the required bifurcation condition in the large n limit. In order to show the dependence of μ_1 on A_1 , this algebraic bifurcation condition should be solved numerically. Because of the appearance of E_1 and E_3 in C_{ij} which are exponentially large, we first find more explicit results and then in order to be able to compare this asymptotic results with the counter part numerical ones, the numerical determinant method (Houghton and Orr, 1997) is used and presented.

Asymptotic results for $A_1-1 = O(1)$: For $A_1-1 = O(1)$, we have $a_2 - a_1 = O(1)$. It means that $a_2 \gg a_1$ and clearly the term proportional to $E_1 E_3$ is exponentially dominant and hence we may write Eq. 19 as:

$$\frac{\det C_{ij}}{E_1 E_3} = \begin{bmatrix} F_2(a_1) & F_4(a_1) \\ G_2(a_1) & G_4(a_1) \end{bmatrix} \begin{bmatrix} F_1(a_1) & F_3(a_1) \\ G_1(a_1) & G_3(a_1) \end{bmatrix} + E.S.T = 0 \quad (20)$$

where, E.S.T. stands for exponentially small terms. By solving the matrix equation Eq. 14 and finally by expanding μ_1 in terms of $1/n$, for the case of $A_1-1 = O(1)$, we obtain:

$$\mu_1^{out} = 0.5436 + \frac{0.3522}{n} - \frac{3.7132}{n^2} + \dots \quad (21)$$

as remarked earlier, this expression is independent of A_1 . This fact is confirmed by our numerical results shown in Fig. 1.

Asymptotic results for $A_1-1 = O(1/n)$: It is clear from the Fig.1 that, by increasing A_1 , μ_1 is constant on $A_1-1 = O(1)$. Since for $A_1-1 = O(1/n)$, $a_2 a_1$ is small, the exponentials E_1 and E_3 are no longer exponentially large and Eq. 20 fails to approximate Eq. 19. It can be deduced that E_1 and E_3 become $O(1)$ and hence all the C_{ij} 's become $O(1)$ when $A_1-1 = O(1/n)$. In the latter regime we write:

$$A_1 = 1 + \frac{1}{n} \xi \quad (22)$$

Where ξ is an $O(1)$ constant and look for an asymptotic solution for μ_1 of the form:

$$\mu_1 = 1 + \frac{1}{n_2} + \dots \quad (23)$$

where n_1 and n_2 are to be determined. On substituting Eq. 22 and 23 in Eq. 5 and 9 we have:

$$a_1 = 1 + \frac{1}{n} (n_2 + \xi) + \frac{1}{n^2} 2\xi + a_2 = 1 + \frac{1}{n} \left(2 + 1\xi - \frac{\xi}{1} \right) + \dots$$

$$k = 1 + \frac{1}{n_1^2} + \frac{2}{n} (\xi - \xi_1^2 + n_2) \quad (24)$$

By substituting the new a_1 , a_2 and k from Eq. 24 into Eq. 15 and by collecting the coefficients of the leading order terms of $\det(C_{ij}) = 0$ with the aid of Mathematica, we obtain the following simplified expression:

$$16z(1+z^2)^2 - (1+4z+2z^2+z^4)^2 \cosh\left(\xi - \frac{\xi}{z}\right) + (1-4z+2z^2+z^4)^2 \cosh\left(\xi + \frac{\xi}{z}\right) = 0 \quad (25)$$

where we used $z = n_1$ and multiplied the second and fourth rows by 1^{-1} . We may extract 1 by expanding:

$$\cosh\left(\xi - \frac{\xi}{z}\right)$$

and:

$$\cosh\left(\xi + \frac{\xi}{z}\right)$$

Eq. 25 and finally get:

$$1 = 1 - \frac{\xi^2}{18} - \frac{\xi^4}{1080} + \dots \quad (26)$$

Equation 26 satisfied by 2 which is linear in 2 is obtained by collecting the coefficients of the next order terms in $\det(C_{ij}) = 0$ with the aid of Mathematica and is given by:

$$n_2 \xi_1 + \xi_2 = 0 \quad (27)$$

Since, the relations of ξ_1 and ξ_2 are too long to be written out here, we refer the reader to Eq. 27 in (Sanjaranipour, 2010). For any given mode number and any value of A_1 close to unity, the corresponding value of ξ is determined by Eq. 22. Now by substituting ξ from Eq. 26 into Eq. 27 and by expanding we may find 2 as follows:

$$n_2 = -\frac{5\xi^2}{18} + \frac{5\xi^3}{36} + \frac{\xi^4}{135} + \dots \quad (28)$$

The value of μ_1 is then calculated by using 1 and 2 from Eq. 26 and 28 respectively in Eq. 23. In Fig. 1, we have shown the comparison between the numerical results and the 1-term (corresponding to $\mu_1 = 1$) and 2-term (corresponding to $\mu_1 = 1 + 2/n$) asymptotic results for $n = 20$. We see that the 2-term asymptotic results approximate the numerical results extremely well over the region of their validity.

In Fig. 1 we find that for $n \gg 1$, μ_1 is constant over almost the entire region of $0 < A_1 < 1$ and decreases sharply from this constant value to unity as A_1 tends to unity (the thin shell limit). Comparison between WKB and the numerical results shows that agreement is not good over a thin region between $A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n)$ where this region has boundary layer structure.

Solving boundary layer: We now arrive to the case of forming a composite solution by matching the outer solution Eq. 21 with the inner solution Eq. 23. As we mentioned earlier the outer solution Eq. 21 is independent of A_1 and because of that the composite solution is simply the inner solution according to Van Dyke's matching rule (Bush, 1992). It is evident from the study has been done by Fu in the case of spherical shell (Van Dyke, 1975) that Van Dyke rule works for this problem also, i.e., the inner solution also valid in the outer region where $A_1 - 1 = O(1)$. By focusing on the Fig. 1 ab we see that, this is indeed the case for the leading order inner solution but not for the case of two-term inner solution. As mentioned on Eq. 3, Van Dyke (1975) matching rule fails at second order. According to the results obtained by Fu for the spherical shell on Fu, 1998, it is seen that Eq. 19 does indeed provide a uniformly valid solution for $A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n)$. However, the numerical solution corresponding to Eq. 19 fails for $A_1 - 1 < O(1/n)$ which will be studied in the following section.

Asymptotic analysis for thin-walled shells: The logarithmic layer is a thin layer between the two inner and outer regions which a rapid change occurs in the value of the dependent variable. For solving this logarithmic layer it is necessary to reconsider the layer on a thin regions like $A_1 - 1 = O(\epsilon^p)$, $0 \ll p = 2$ and for matching the two solutions, we can use the theory of boundary layer and also Van Dyke's matching rule. If the boundary layer is located in the region, the appropriate stretching transformation is $x = x - x_0 / \partial(\epsilon)$ the choice of the power p , indeed more generally the choice of function $\partial(\epsilon)$ to use in the stretching transformation $x = x - x_0 / \partial(\epsilon)$ is determined by the need to represent the region of rapid change correctly. We must ensure that the boundary layer solution contains rapidly varying functions. In order to choose the boundary layer thickness, it is necessary to seek a stretching transformation which retains the largest number of terms in the dominant equation governing μ_1 . This is referred to as the principle of least degeneracy by Van Dyke (1975). It can be deduced from Eq. 25 and the expression for η_2 that, as $\xi \rightarrow 0$, we have:

$$\mu_1^i = 1 - \frac{\xi^2}{18} - \frac{\xi^4}{1080} + \dots + \frac{1}{n} \left(-\frac{5\xi^2}{18} + \frac{5\xi^3}{36} + \frac{\xi^4}{135} + \dots \right) + \dots \tag{29}$$

The inner expansion Eq. 29 is not uniformly valid for all n , because this asymptotic expansion becomes disordered and hence invalid when ξ_1 and ξ_2/n are of the same order, i.e., when $A_1 - 1 = O(1/3/n^2)$. For matching outer and inner solutions and in order to find a uniform solution, we must find a thickness which includes the region of rapid changes. This is the regime we consider in the next part.

Asymptotic results for $A_1 - 1 = O(1/3/n^2)$: For $A_1 - 1$ as small as $O(1/3/n^2)$, a_1 is close to a_2 and it is convenient to define a new stretched variable x through:

$$x = \frac{r - a_1}{a_2 - a_1}, 0 < a_1 \leq r \leq a_2 \tag{30}$$

In terms of x , the inner and outer surfaces of the shell correspond to $x = 0, 1$ and the governing Eq. 6 and the boundary conditions Eq. 7 and 8 becomes:

$$\begin{aligned} & \frac{d^4 V}{dx^4} + Q \left[\frac{6}{r} + \frac{2k}{r^3} \right] \epsilon \delta \frac{d^3 V}{dx^3} + \\ & Q^2 \left[\frac{-k(1 + 3\epsilon^2)}{r^6} + \frac{2k(\epsilon^2 - 1)}{r^4} + \frac{(5\epsilon^2 - 2)}{r^2} \right] \delta^2 \\ & \frac{d^2 V}{dx^2} + Q^3 \left[\frac{k^3(3\epsilon^2 + 1)}{r^9} + \frac{k^2(5\epsilon^2 + 1)}{r^7} + \frac{k(\epsilon^2 - 4)}{r^5} \right] \\ & \left[\frac{-(\epsilon^2 + 2)}{r^3} \right] \\ & \epsilon \delta^3 \frac{dV}{dx} + Q^2 \left[(1 - \epsilon^2) \left(\frac{3k^2 \epsilon^2}{r^8} + \frac{2k\epsilon^2}{r^6} + \frac{(1 - \epsilon^2)}{r^4} \right) \right] \delta^4 V = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} & \frac{d^2 V}{dx^2} + \frac{\epsilon \delta dV}{r dx} + \frac{(1 - \epsilon^2)}{r^2} V = 0, \frac{d^3 V}{dx^3} + Q \left[\frac{4}{r} + \frac{2k}{r^3} \right] \\ & \epsilon \delta \frac{d^2 V}{dx^2} - \left[Q^2 \frac{(k^2 m^2)}{r^6} + \frac{(1 - \epsilon^2)}{r^2} \right] \delta \frac{dV}{dx} + \\ & \left[\frac{(m + m^3)}{r^3} - \frac{k(m + m^3)}{r^5} \right] \delta^3 V = 0 \end{aligned} \tag{32}$$

These two boundary conditions hold on $x = 0, 1$. For convenience we have introduced $\epsilon = 1/n$, $\delta = (a_2 - a_1) n$. Now we write:

$$A_1 = 1 + \epsilon^{\frac{3}{2}} \xi \tag{33}$$

and look for a solution for μ_1 of the form:

$$\mu_1 = 1 + \epsilon_1^{(1)} + \epsilon^{\frac{3}{2}} V_2^{(1)} + \epsilon^2 V_3^{(1)} + \dots \tag{34}$$

For V we assume:

$$V = V_0^{(1)} + \epsilon V_1^{(1)} + \epsilon^{\frac{3}{2}} V_2^{(1)} + \epsilon^2 V_3^{(1)} + \dots \tag{35}$$

By substituting Eq. 34-36 into Eq. 31-33 and equating coefficients of similar powers of ϵ , we obtain a couple of differential equations and boundary conditions which can be derived and solved successively with the aid of Mathematica. By solving these equations up to and including $O(\epsilon)$, we find:

$$\mu_1^{ii} = 1 - \epsilon \frac{5\xi^2}{18} - \epsilon^2 \left(\frac{1}{2} + \frac{5\xi^2}{18} + \frac{37\xi^4}{1080} \right) + \dots \tag{36}$$

This solution for $A_1 - 1 = O(\epsilon^{3/2})$ can be matched with the solution Eq. 23 valid for $A_1 - 1 = O(\epsilon)$ by using Van Dyke (1975) matching rule, i.e., by substituting $\epsilon = 1/n$ we have $\mu_1^{ii} = \mu_1^i$ which indicates that on the leading order μ_1^{ii} is matched with μ_1^i . But it cannot be matched for two-term, therefore, the mentioned thickness does not include all the rapid changes anticipated and hence, we should continue our study further to reach to the proper thickness. In terms of the original variables A_1 and n , (Eq. 37) becomes:

$$\mu_1^{ii} = 1 - \frac{1}{18} n^2 (A_1 - 1)^2 - \frac{1}{2n^2} - \frac{5}{18} n (A_1 - 1)^2 - \frac{37}{1080} n^4 (A_1 - 1)^4 + O\left(\frac{1}{n^3}\right) \dots \tag{38}$$

To determine the critical mode number at which μ_1^{ii} attains its minimum, we consider:

$$\frac{\partial \mu_1^{ii}}{\partial n} = -\frac{1}{9} n (A_1 - 1)^2 - \frac{1}{n^3} - \frac{5}{18} (A_1 - 1)^2 - \frac{37}{270} n^3 (A_1 - 1)^3 + O\left(\frac{1}{n^4}\right) \tag{39}$$

For $A_1 - 1 = O(1/3n^2)$, the first term on the right-handed side of Eq. 39 is the only leading order term and is negative. Thus, μ_1^{ii} is a decreasing function of n . It is clear that the critical mode number cannot be of order $(A_1 - 1)3/2$. It should be noticed that $\partial \mu_1^{ii} / \partial \mu_1^i$ may become zero if the first term can be balanced by the higher-order terms. This occurs when $A_1 - 1 = O(1/n^2)$ which is the regime we consider in the next subsection.

RESULTS AND DISCUSSION

Asymptotic results for $A_1 - 1 = O(1/n^2)$: An inspection of Eq. 37 shows that this asymptotic expansion becomes invalid when $\epsilon \xi^2 = O(\epsilon^2)$. In terms of A_1 this is $A_1 - 1 = O(\epsilon^2)$. In the new regime we write:

$$A_1 = 1 + \epsilon^2 \tag{40}$$

where is an $O(1)$ constant and we look for the following form of solutions for μ_1 and V:

$$\mu_1 = 1 + \epsilon^2 V_1^{(2)} + \epsilon^3 V_2^{(2)} + \epsilon^4 V_3^{(2)} + \dots \tag{41}$$

$$V = V_0^{(2)} + \epsilon^2 V_1^{(2)} + \epsilon^3 V_2^{(2)} + \epsilon^4 V_3^{(2)} + \dots \tag{42}$$

Where the expansions for μ_1 and V are deduced from Eq. 36 and 37 by replacing ξ by $\xi^{3/2} \Phi$. By substituting Eq. 40-42 into Eq. 31-33 and equating coefficients of similar powers of ϵ , we obtain an infinite number of differential equations and boundary conditions. As in the previous subsection, these equations and boundary conditions can again be derived and solved successively with the aid of Mathematica. By solving these equations up to and including $O(\epsilon^8)$ we find:

$$\mu_1^{iii} = 1 - \epsilon^2 \left(\frac{1}{2} + \frac{5}{18} F^2 \right) + \epsilon^3 \left(\frac{1}{2} - \frac{5}{18} F^2 \right) + \epsilon^4 \left(-\frac{1}{135} F^4 + \frac{5}{36} F^3 - \frac{1}{18} F^2 + \frac{1}{9} F - \frac{1}{8} \right) + \dots \tag{43}$$

We note that with Φ replaced by ζ/ϵ $\zeta/1/\epsilon_2$, μ_1^{iii} and reduces to equations μ_1^{ii} and μ_1^{iii} , respectively. Thus equation μ_1^{iii} include all the regions of rapid change and can be matched by equation μ_1^{out} (i.e., (21s) directly. With the use of Van Dyke's matching rule by following form:

$$\lim_{x \rightarrow \infty} \lim_{x \rightarrow \infty} \mu_1^{iii} = \lim_{\tau \rightarrow 1} \lim_{\tau \rightarrow 1} \mu_1^{out} \tag{44}$$

where X and $(x - x_0)/\delta(\epsilon) = O(1)$, when $x \rightarrow \infty \epsilon^2$, $\delta \rightarrow 0$, we obtain μ_1^{match} by replacing $\Phi = \zeta/\epsilon$ into expansion Eq. 43 as follows $\mu_1^{match} = \epsilon^{2 - \lim} \mu_1^{iii}$.

$$1 - \frac{5\xi^2}{18} + \dots + \epsilon \left(-\frac{5\xi^2}{18} + \frac{5\xi^3}{36} + \dots \right) \quad (45)$$

where, μ_1^{match} is the expansion of either the inner or outer approximations in the matching region. We find the following composite solution which is uniformly valid for $A_1 - 1 = O(\epsilon^p)$, $0 \leq m \leq 2$:

$$\begin{aligned} \mu_1^{comp} &= \mu_1^{out} - \frac{\epsilon^{2\mu_1}}{2} s^{comp} + \frac{\epsilon^3}{2} + \\ &\epsilon^{4\mu_1^{out}} \left(-\frac{1}{8} + \mu_1^{iii} + \frac{1}{9}\Phi + \frac{1}{18}\Phi^2 \right) + \dots \end{aligned} \quad (46)$$

$A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n^2)$ by using μ_1^{match} . We have shown the composite solution is the cause of the complete matching of the two regions $A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n^2)$. We compared neutral stability curves of asymptotic matching solution (uniformly valid curve for μ_1 and μ_2 against A_1 for $A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n^2)$) by solving numerically Eq. 19 with the use of Compound matrix method. Matching two regions $A_1 - 1 = O(1)$ and $A_1 - 1 = O(1/n^2)$ (Fig. 2 and 3) solid line: Numerical result. The numerical solution, μ_1 and μ_{12} for $n = 20$ are smaller than those for $n = 15$. We find a critical neutral curve of Eq. 43 which is the envelope of the neutral curves corresponding to different mode numbers. In terms of the original variables A_1 and n , Eq. 43 takes the form:

$$\begin{aligned} \mu_1^{iii} &= 1 - \frac{1}{2n^2} - \frac{5}{18}n^2(A_1 - 1)^2 + \\ &\frac{1}{2n^3} - \frac{5}{18}n(A_1 - 1)^2 + O\left(\frac{1}{n^4}\right) \\ \frac{\partial \mu_1^{iii}}{\partial n} &= -\frac{5}{9}n(A_1 - 1)^2 - \frac{5}{18} \\ &(A_1 - 1)^2 + \frac{1}{n^3} - \frac{3}{4n^4} + O\left(\frac{1}{n^5}\right) \end{aligned} \quad (47)$$

By solving $\partial \mu_1^{iii} / \partial n = 0$ we obtain the critical mode number:

$$n = \frac{1}{2} + \frac{5}{9\sqrt{14}}(A_1 - 1)^2 \quad (48)$$

On substituting Eq. 48 into Eq. 47, we obtain the following expression for the critical neutral curve which is the envelope of the neutral curves corresponding to different mode numbers:

$$\mu_1 = 1 + \frac{1}{9}(A_1 - 1) - \frac{13}{14}(A_1 - 1)^2 + \dots$$

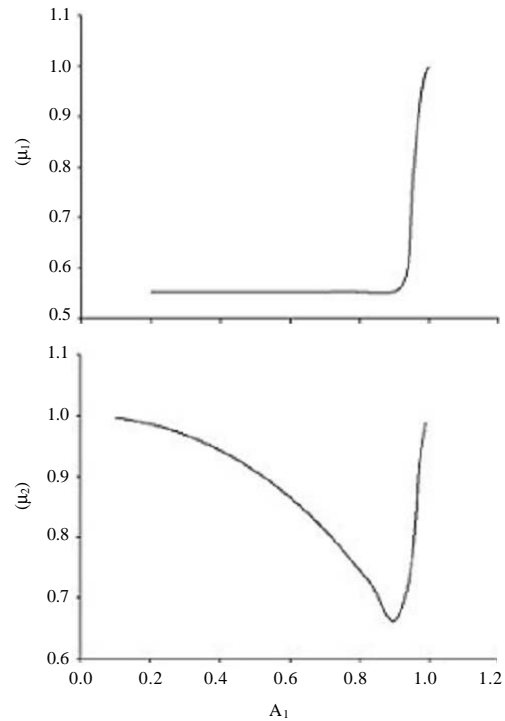


Fig. 2: Neutral stability curve for $n = 20$ of the asymptotic result obtained from matching two regions

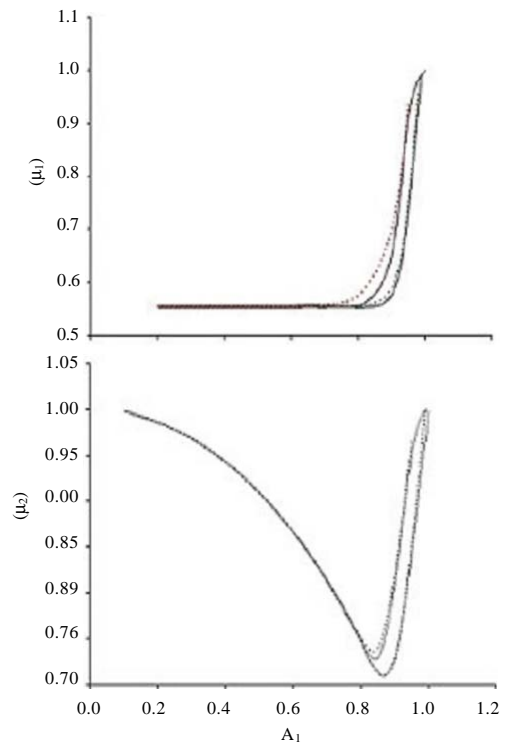


Fig. 3: Neutral stability curves for $n = 15, 20$. Dotted line: the asymptotic result obtained from

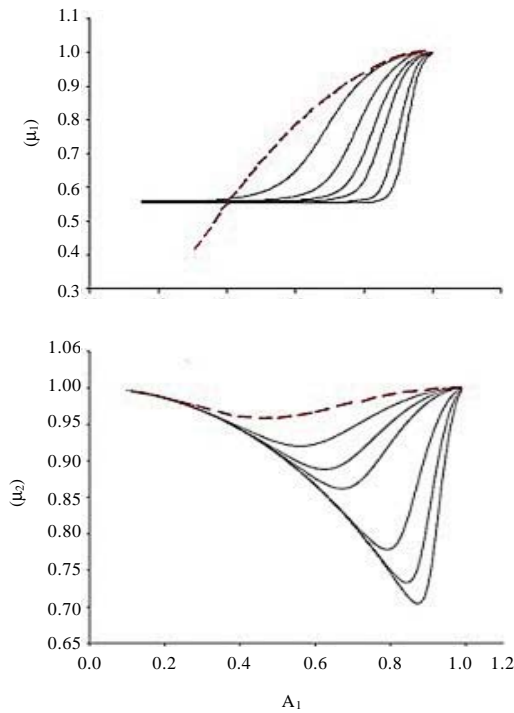


Fig. 4: The solid lines represent the numerical results for $n = 5, 8, 10, 15, 20, 40$. The dotted lines are the asymptotic critical neutral curves corresponding to Eq. 48

We have shown the asymptotic results Eq. 48 together with the results from a fully numerical integration. The validity of Eq. 48 is obvious (Fig. 4).

CONCLUSION

On solving the fourth order eigenvalue problem of the neo-Hookean cylindrical shell by WKB method which has been done in two layers (i.e., outer $A_1-1 = O(1)$ and inner $A_1-1 = O(1/n)$ on $0 < A_1 < 1$), we observe that a thin layer exists between the two solutions. In this study, our main concern is to find and solve the asymptotic solution of this logarithmic layer and finally compare this solution

with the counterpart numerical one. In order to be able to study this layer, we are writing down a brief description of the asymptotic inner and outer solutions previously done by M. Sanjaranipour (2010). In order to obtain the proper thickness, the boundary layer theory and Van Dyke’s matching rule were employed. For A_1-1 , we checked different thickness (i.e., $O(1/n)$, $O(1/3n^2)$ and $O(1/n^2)$. Finally by obtaining a composite solution which is uniformly valid for $A_1-1 = O(\epsilon^m)$ $0 \leq m \leq 2$, we could match the outer region ($A_1-1 = O(1)$) with the inner region $A_1-1 = O(1/n)$. The asymptotic matching solution is coincident with the numerical solution. A_1 critical neutral curve which enveloped the neutral curves corresponding to different mode numbers is also verified.

REFERENCES

Bender, C.M. and S.A. Orszag, 1978. Advanced Mathematical Methods for Scientists and Engineers. McGraw-Hill, New York, USA., Pages: 593.
 Bush, A.W., 1992. Perturbation Methods for Engineers and Scientists. S.C. Press Association, USA.
 Fu, Y., 1998. Some asymptotic results concerning the buckling of a spherical shell of arbitrary thickness. Int. J. Non Linear Mech., 33: 1111-1122.
 Fu, Y.B. and Y.P. Lin, 2002. A WKB analysis of the buckling of an everted neo-Hookean cylindrical tube. Math. Mech. Solids, 7: 483-501.
 Houghton, D.M. and A. Orr, 1997. On the eversion of compressible elastic cylinders. Int. J. Solids Struct., 34: 1893-1914.
 Houghton, D.M. and R.W. Ogden, 1979. Bifurcation of inflated circular cylinders of elastic material under axial loading -II: Exact theory for thick-walled tubes. J. Mech. Physics Solids, 27: 489-512.
 Sanjaranipour, M., 2010. WKB analysis of the buckling of a neo-hookean cylindrical shell of arbitrary thickness subject to an external pressure. Int. J. Appl. Mech., 2: 857-870.
 Van Dyke, M., 1975. Perturbation Methods in Fluid Mechanics. Annotated Edn., Parabolic Press, Stanford, CA., ISBN: 13: 9780915760015.