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Interval Shrinkage Estimators for Location Parameter of the Exponential Distribution

Parviz Nasiri
Department of Statistics, University of Payam Noor, 19395-4697 Tehran, I.R. Iran

Abstract: This study considers estimation of an unknown distribution parameter in situations where we believe that the parameter belongs to finite interval. We proposed shrinkage and interval shrinkage estimators the location parameter of exponential distribution. The properties of these estimators are illustrated in a simulation study.

Key words: Interval information, mean square error, shrinkage estimator, two parameters exponential distribution, Iran

INTRODUCTION

The two-parameter exponential distribution plays an important role in the field of life testing and reliability theory since it is the only continuous distribution with a constant hazard function (failure rate). The reciprocal of the scale parameter is the failure rate. The location (threshold) parameter can translate the distribution along the time axis, so it is also known as the minimum life or guarantee time parameter. The guarantee time parameter can be used to model warranty periods for some products.

Epstein and Sobel (1954) obtained the Minimum Variance Unbiased Estimator (MVUE) for its scale parameter and location parameter, respectively. The shrinkage estimators for the scale parameter have been proposed by Bhattacharya and Srivastava (1974) and Pandey (1983). Chiou and Han (1989) proposed a pre-test estimator and a pre-test shrinkage estimator for the location parameter.

A prominent approach which allows utilizing additional (non-sample) information is the shrinkage estimators introduced by Stein. The idea of shrinkage is to provide a balanced trade-off between a conventional estimator and a shrinkage target. The latter can be interpreted as the point around which parameter values are most likely or estimation accuracy seems to be most crucial.

Epstein (1960) has studied estimation of the parameters of two exponential distributions from censored samples. It has been found in many problems life testing that there are occasions when a two parameter exponential distribution is more appropriate for fitting life test data then is a one parameter distribution. By, a two parameter exponential distribution we mean a density function f (x, θ, μ) such that:

$$f(x,\mu,\theta)=\frac{1}{\theta}e^{-(\frac{x-\mu}{\theta})},x>\mu,\theta>0 \quad (1)$$

With the following c.d.f:

$$F(x)=p(X\le x)=1-e^{-(\frac{x-\mu}{\theta})} \quad (2)$$

For simplify, we consider θ = 1, hence

$$f(x,\mu)=e^{-(x-\mu)},x>\mu \quad (3)$$

Based on random sample with size of n, the maximum likelihood estimation of μ is given as:

$$\hat{\mu}=\mathrm{Min}(X_1,X_2,...,X_n)=X_{(1)} \quad (4)$$

It easy to show that:

$$E(X_{(1)})=\int_\mu^\infty x_{(1)}e^{-n(x_{(1)}-\mu)}dx_{(1)}=\mu+\frac{1}{n}\ \mathrm{Var}(X_{(1)})=\frac{1}{n^2}$$

SHRINKAGE ESTIMATION OF μ

Let, $\hat{\mu}_{sh}$ be the shrinkage estimation of location parameter. Then:

$$\hat{\mu}_{sh}=\mu_0+k(\hat{\mu}-\mu_0)$$

To find of k, we consider MSE of estimator as:

$$\begin{aligned}\mathrm{MSE}(\hat{\mu}_{sh})&=E\{(\hat{\mu}_{sh}-\mu)^\wedge 2\}=E[\mu_0+k(\hat{\mu}-\mu_0)-\mu]^2\\&=E[k(\hat{\mu}-\mu)+(k-1)(\mu-\mu_0)]^2\\&=k^2\mathrm{MSE}(\hat{\mu};\mu)+(k-1)^2(\mu-\mu_0)^2\\&\quad+2k(k-1)(\mu-\mu_0)E(\hat{\mu}-\mu)\\&=\frac{2k^2}{n^2}+(k-1)^2(\mu-\mu_0)^2-\frac{2k(k-1)(\mu-\mu_0)}{n}\\&=\frac{2k^2}{n^2}+k^2(\mu-\mu_0)^2-2k(\mu-\mu_0)^2+(\mu-\mu_0)^2\\&\quad+\frac{2k^2}{n}(\mu-\mu_0)-\frac{2k}{n}(\mu-\mu_0)\end{aligned}$$

Now, we have to minimize the MSE:

$$\frac{dMSE(\hat{\mu}_{sh};\mu)}{dk} = \frac{4k}{n^2} + 2k(\mu-\mu_0)^2 - 2(\mu-\mu_0)^2 - \frac{4k}{n}(\mu-\mu_0) + \frac{2}{n}(\mu-\mu_0)\mu = 0$$

$$2k[\frac{2}{n^2} + (\mu-\mu_0)^2 + 2(\mu-\mu_0)\frac{1}{n}] = 2(\mu-\mu_0)^2 + 2(\mu-\mu_0)\frac{1}{n}$$

$$k = \frac{(\mu-\mu_0)^2 + (\mu-\mu_0)\frac{1}{n}}{\frac{2}{n^2} + (\mu-\mu_0)^2 + 2(\mu-\mu_0)\frac{1}{n}} \tag{5}$$

By using of Eq. 5 the shrinkage of estimator is given by:

$$\hat{\mu}_{sh} = \mu_0 + \left[\frac{(\mu-\mu_0)^2 + (\mu-\mu_0)\frac{1}{n}}{\frac{2}{n^2} + (\mu-\mu_0)^2 + 2(\mu-\mu_0)\frac{1}{n}}\right](\hat{\mu}-\mu_0) \tag{6}$$

FEASIBLE INTERVAL SHRINKAGE ESTIMATORS

Let, {X; t = 1, 2,...,T} be a stochastic process, where X_t is an i.i.d random variable from a known parametric class of distributions indexed by an unknown parameter $\theta \in \Theta \subset \mathbb{R}^1$. Our aim is to estimate θ. Assume the existence of an unbiased conventional estimator which is based only on the sample information. It is denoted by $\hat{\theta} = f(X_1, X_2, ..., X_T)$ and has the expectation $E(\hat{\theta}) = \theta$ and the variance denoted by $V(\hat{\theta})$. Moreover, it is extant presumed that the true parameter belongs to the guessed interval $\theta \in [\theta_0, \theta_1] \subset \Theta$ where θ_0 and θ_1 are some finite bounds. In order to incorporate this additional prior information into the estimation procedure, we suggest using a shrinkage approach. The conventional linear point shrinkage estimator is given by:

$$\hat{\theta}(\tilde{\theta}) = \omega\hat{\theta} + (1-\omega)\tilde{\theta} \tag{7}$$

where $\omega \in [0,1]$ is the shrinkage factor? Selecting ω based on the MSE criterion requires the solution of the following minimization problem:

$$\begin{aligned}\omega^* &= \underset{\omega}{\text{argmin}} MSE[\hat{\theta}(\tilde{\theta}) = \arg\min_{\omega} E[(\hat{\theta}(\tilde{\theta}) - \theta)^2]\\ &= E[\omega\hat{\theta} + (1-\omega)\tilde{\theta} - \theta]^2 = E[\omega(\hat{\theta}-\theta) + (1-\omega)(\tilde{\theta}-\theta)]^2\\ &= E[\omega^2(\hat{\theta}-\theta)^2 + (1-\omega)^2(\tilde{\theta}-\theta)^2 + 2\omega(1-\omega)(\hat{\theta}-\theta)(\tilde{\theta}-\theta)]\\ &= \omega^2 V(\hat{\theta}) + (1-\omega)^2(\tilde{\theta}-\theta)^2\end{aligned}$$

$$\frac{dMSE(\hat{\theta}(\tilde{\theta}))}{d\omega} = 2\omega V(\hat{\theta}) - 2(1-\omega)(\tilde{\theta}-\theta)^2 = 0$$

The resulting MSE optimal shrinkage factor is given by:

$$\omega^* = \frac{\left(\tilde{\theta}-\theta\right)^2}{\left(\tilde{\theta}-\theta\right)^2 + V\left(\hat{\theta}\right)} \text{with } \omega \in [0,1] \tag{8}$$

where $(\tilde{\theta}-\theta)^2$ is a squared bias of the target $\tilde{\theta}$. Golosnoy and Liesenfeld (2011) for unbiased conventional sample estimator of $\tilde{\theta}$ with $E(\hat{\theta}) = \theta$, shrinkage estimator towards the interval $\theta \in [\theta_0, \theta_1] \subset \Theta$ given by:

$$\tilde{\theta}(\theta) = \hat{\theta} + \sqrt{V(\hat{\theta})}.\frac{\theta-\hat{\theta}}{\theta_1-\theta_0}[\arctan\left(\frac{\theta_1-\theta}{\sqrt{V(\hat{\theta})}}\right) - \arctan\left(\frac{\theta_0-\theta}{\sqrt{V(\hat{\theta})}}\right) + \frac{V(\hat{\theta})}{2(\theta_1-\theta_0)}\ln[\frac{V(\hat{\theta}) + (\theta_1-\theta)^2}{V(\hat{\theta}) + (\theta_0-\theta)^2}] \tag{9}$$

For $E(\hat{\theta}) = \theta$, the Eq. 9 is given by:

$$E[\tilde{\theta}(\theta)] = \theta + \frac{V(\hat{\theta})}{2(\theta_1-\theta_0)}\ln[\frac{V(\hat{\theta}) + (\theta_1-\theta)^2}{V(\hat{\theta}) + (\theta_0-\theta)^2}]$$

Let:

$$\begin{aligned}\tilde{\theta}(\hat{\theta}) &= \hat{E}[\tilde{\theta}(\theta)]\\ &= E[\tilde{\theta}(\theta)] = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1-\theta_0)}\ln[\frac{V(\hat{\theta}) + (\theta_1-\hat{\theta})^2}{V(\hat{\theta}) + (\theta_0-\hat{\theta})^2}]\end{aligned} \tag{10}$$

In last equation With replace of μ as θ, we have:

$$\tilde{\mu}(\hat{\mu}) = \hat{\mu} + \frac{V(\hat{\mu})}{2(\mu_1-\mu_0)}\ln[\frac{V(\hat{\mu}) + (\mu_1-\hat{\mu})^2}{V(\hat{\mu}) + (\mu_0-\hat{\mu})^2}] \tag{11}$$

But, we know that:

$$V(\hat{\mu}) = V(X_{(1)}) = \frac{1}{n^2}$$

$$\tilde{\mu}(\hat{\mu}) = \hat{\mu} + \frac{1}{2n^2(\mu_1-\mu_0)}\ln[\frac{1 + n^2(\mu_1-\hat{\mu})^2}{1 + n^2(\mu_0-\hat{\mu})^2}] \tag{12}$$

Let:

$$A = 1 - \frac{1 + n^2(\mu_1-\hat{\mu})^2}{1 + n^2(\mu_0-\hat{\mu})^2}$$

then we have:

$$\tilde{\mu}(\hat{\mu}) = \hat{\mu} + \frac{1}{2n^2(\mu_1-\mu_0)}\ln(1-A) \tag{13}$$

Table 1: Bias and MSE of Shrinkage (μ_{sh}) and Interval shrinkage ($\tilde{\mu}(\hat{\mu})$) estimation for $\mu_0 = 1$, $\mu_1 = 2$, $\mu_2 = 3$

n	Bias of μ_{sh}	MSE μ_{sh}	Bias $\tilde{\mu}(\hat{\mu})$	Bias $\tilde{\mu}(\hat{\mu})$
10	-0.0036	0.0089	0.1040	0.1040
20	-0.0023	0.0022	0.0498	0.0498
30	-0.0002	0.0011	0.0342	0.0342
40	0.0007	0.0006	0.0263	0.0263
50	0.0003	0.0004	0.0207	0.0207
60	0.0003	0.0003	0.0173	0.0173
70	0.0001	0.0002	0.0146	0.0146
80	0.0001	0.0001	0.0128	0.0128
90	0.0002	0.0001	0.0114	0.0114
100	0.0001	0.0001	0.0102	0.0102

Table 2: Bias and MSE of Shrinkage (μ_{sh}) and Interval shrinkage ($\tilde{\mu}(\hat{\mu})$) Estimation for $\mu_0 = 2$, $\mu_1 = 4$, $\mu_2 = 6$

n	Bias μ_{sh}	MSE μ_{sh}	Bias $\tilde{\mu}(\hat{\mu})$	MSE $\tilde{\mu}(\hat{\mu})$
10	-0.0048	0.0090	0.0994	0.0099
20	0.0016	0.0024	0.0528	0.0026
30	0.0000	0.0011	0.0339	0.0011
40	0.0003	0.0006	0.0256	0.0006
50	0.0004	0.0004	0.0206	0.0004
60	-0.0002	0.0002	0.0166	0.0002
70	-0.0001	0.0002	0.0143	0.0002
80	0.0002	0.0001	0.0128	0.0001
90	0.0005	0.0001	0.0117	0.0001
100	0.0003	0.0001	0.0104	0.0001

For $0 \le A \le 1$, the Eq. 13 is given by:

$$\tilde{\mu}(\hat{\mu}) = \hat{\mu} - \frac{1}{2n^2(\mu_1 - \mu_0)} \sum_{t=1}^{\infty} \frac{A^t}{t} \qquad (14)$$

From the Eq. 14, it is easy to compute the all moment of $\tilde{\mu}(\hat{\mu})$.

NUMERICAL STUDY

In order to have some idea about Bias and Mean Square Error (MSE) of Shrinkage (μ_{sh}) and Interval shrinkage ($\tilde{\mu}(\hat{\mu})$) Estimation. We perform sampling experiments using a R software. The result of bias and MSE of the estimators are given in Table 1-3 for different value of μ_0, μ_1, μ_2. According to Table 1 and 3, the bias of shrinkage estimation less than the interval shrinkage estimation. The MSE of both estimator decrease, when sample size increase. With comparing of MSE two estimators, it is finding that both estimators work same.

Table 3: Bias and MSE of Shrinkage (μ_{sh}) and Interval shrinkage ($\tilde{\mu}(\hat{\mu})$) Estimation for $\mu_0 = 3$, $\mu_1 = 5$, $\mu_2 = 7$

n	Bias μ_{sh}	MSE μ_{sh}	Bias $\tilde{\mu}(\hat{\mu})$	MSE $\tilde{\mu}(\hat{\mu})$
10	-0.0041	0.0093	0.1002	0.0103
20	-0.0002	0.0024	0.0509	0.0025
30	0.0012	0.0011	0.0351	0.0011
40	-0.0003	0.0006	0.025	0.0006
50	0.0004	0.0004	0.0206	0.0004
60	0.0004	0.0003	0.0172	0.0003
70	0.0003	0.0002	0.0147	0.0002
80	0.0002	0.0001	0.0128	0.0001
90	0.0007	0.0001	0.0118	0.0001
100	-0.0001	0.0001	0.0099	0.0001

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