Research Journal of Applied Sciences 13 (11): 675-680, 2018
ISSN: 1815-932X

Related Reflections to the Axioms of Separation in Semigroups with Topologies and Some Applications

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Abstract: In this study, we study the reflections of the category of topological and semitopological semigroups on the category of the class of T_0-T_3 and regular topological spaces and we apply its properties to find conditions under which a topological semigroup has the Souslin property.

Key words: Colombia, semitopological, category, topological spaces, properties, conditions

INTRODUCTION

Let C be an epireflective subcategory of all topological spaces (in what follows top (Herrlich and Strecker, 1997) and abusing terminology, let C be the functor associated with C. By Hernandez-Arzusa and Hernandez (2017) it is proved that each semitopological (separately continuous) algebraic structure in X∈Top is reflected in C(X), also it is proved that if the functor C respects finite products then the topological (jointly continouos) algebraic structures are reflected. By Tkachenko (2014), we can find similar results but only in semitopological groups. In this research, we studied the functors related to axioms of separation in the category of topological and semitopological semigroups. Not having inverse in the semigroup operation gives us certain problems, for example, one of them is that not always the quotient mappings induced by congruences are open what happens in semitopological groups. For this reason, we have considered in ours results topological semigroups with open shifts these are studied in by Ravsky and Banach (2016).

The fact that the classes of the T_0-T_3 regular and Tychonoff spaces, constitute an epiriflective subcategories of Top is proved by Herrlich (1967) and Tkachenko (2014, 2015a, b) are studied the related functors to these reflections in the category of the semitopological and paratopological groups.

We give the basic facts of the theory and the basic notation. Additionally, we give examples of topological (semitopological) semigroups with open shifts in order to prove that this class contains the class of the semitopological (resp. paratopological) groups as a proper subclass.

We initially construct the reflection on T_2 spaces of semitopological monoids with open shifts (Theorem 2.5). In relation to T_2 semigroup we gived a partial answer (Corollary 2.4) to the following question: if S is a topological semigroup and ~ is a closed congruence on S is S/~ a topological semigroup. This problem is addressed by Gonzalez (2001) and (Khosravi, 2012) but unlike these researches in our results the condition being T_2 is not assumed for semitopological semigroups. Moreover, we give a characterization of the reflection on T_3 spaces for topological monoids with open shifts (Corollary 2.7) where we proved the topological monoid with open shifts are quasiregular spaces, obtaining a more general result than Theorem 2.4 by Tkachenko (2014). Finally, we construct the reflection on regular spaces for topological monoids with open shifts (Theorem 2.8).

We find results about epireflections that preserve products in the category of semigroups. Analogues results are found by Tkachenko (2015a, b) but in the category of groups.

We use the C-reflections to study the cellularity of cancellative topological monoids with open shifts. Given C_r-reflection preserves the cellularity (Theorem 4.5), we can obtain some results releted to the Souslin property without using separation axioms. By Tkachenko (1983) proved each σ-compact topological group has countable cellularity, this result was generalized by Uspenskij (1982). Later, Arhalhensky and Reznychenko (2008) extended it for Hausdorff σ-compact paratopological groups by Arhangel'skii and Tkachenko (2008).

MATERIALS AND METHODS

Preliminaries: A semigroup is a set S≠ø, endowed with an associative operation. If also S has neutral element, we say that S is a monoid. A mapping f: S→H between

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semigroups is a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in S$. A semitopological semigroup (monoid) consists of a semigroup (resp. monoid) S and a topology τ on S such that for all $a \in S$, the shifts $x \to ax$ and $x \to xa$ (noted by l_a and r_a, respectively) are continuous mapping of the S to itself. We say that a semitopological semigroup has open shifts if for each $a \in S$ and for each open set U in S, we have $l_a(U)$ and $r_a(U)$ are open sets in S. A topological semigroup (monoid) (paratopological group) consists of a semigruop (resp. monoid) (resp. group) S and a topology τ such that the operation of S as a mapping of $S \times S$-S is continuous when $S \times S$ is endowed with the product topology. An congruence on a semigrouop S is an equivalence relation on S, ~ such that if x~y and a~b then xa~yb. If S is a semitopological semigroup then we say that ~ is a closed congruence if ~ is closed in $S \times S$. If ~ is an equivalence relation in a semigroup (monoid) S and π: $S \to S/\sim$ is the respective quotient mapping, then S/~ is a semigroup (monoid) and π an homomorphism if and only if ~ is a congruence (Gonzalez, 2001), (Theorem 1).

A class C of topological spaces is called closed under super topologies if $(X, \tau) \in C$ implies $(X, p) \in C$ for each topology p on X finer than τ.

Let X be a topological space a cellular family in X is a non empty family of non empty open sets in X and pairwise disjoint. The cellularity of a space X, noted by c(X) is defined by:

$$c(X) = \sup\{| U |: U \text{ is celluler family in } X\} + N_0$$

If $c(X) = N_0$, we say that X has countable cellularity or X has the Souslin property. If X is a topological space and $A \subseteq X$. We will note by Int_X and $Cl_X(A)$, the interior and the closure of A in X or simply Int(A) and $\bar{A}$, respectively, when the space X is understood. An open set U in X is called regular open in X if $Int\,\bar{U} = U$. It is easy to prove the regular open ones form a base for a topology which we will call semiregularitation of X, X endowed with this topology, we will note by X_{Sr}. X is called quasiregular if X_{Sr} is a T_3 space.

If C is an epireflective class of TOP, X is a topological space and the morphism r: $X \to B$ is the C-reflection of X, then, given the reflections are essentially unique in order to agree with the notation, we will note r by $\varphi_{(C, X)}$ and B by C(X). The functor induced by the C-reflection, we will note it by C, therefore, if f: $X \to Y$ is a continuous mapping there is an unique continuous mapping C(f): $C(X) \to C(Y)$ such that $C(f) \circ \varphi_{(C,X)} = \varphi_{(C, Y)} \circ f$. C_0-C_3, C_r will note the class of the spaces, T_0-T_3, regular and Tychonoff, respectively.

A topological space X is called σ-compact if it is countable union of compact subsets. X is called sequentially compact if each sequence in X has a subsequence converging in X.

The following examples, guarantee the class of cancellative topological monoids with open shifts is non empty and it is bigger than the class of paratopological groups.

Example 1.1: Let $\mathbb{R}^+ = [0, \infty)$, together with the usual sum in $\mathbb{R}$, endowed with the generated topology by the sets $[a, \infty)$ being $a \in \mathbb{R}^+$. Then $\mathbb{R}^+$ is a compact cancellative topological monoid with open shifts and it is not a group.

Example 1.2: Let G be a paratopological group that is not a topological group (for example, the Sorgenfrey line) and let U be an open non symmetric ($U^{-1} \neq U$) neighborhood in G of the neutral element e of G. If $S = U_{n \in \mathbb{N}}$, then S is a cancellative topological monoid with open shifts that is not a group.

Example 1.3: It is possible to obtain open shifts from semitopological (topological) monoids. Indeed let S be a semitopological (topological) monoid and let N_e be an open local base of the neutral element e of S. The set γ = $\{aU: U \in N_e, a \in S\} U \{Ua: U \in N_e, a \in S\}$ generates a topology of semitopological (resp. topological) monoid with open shifts.

Example 1.4: Let $S = \{(x, y) \in \mathbb{R}^2: y \geq 0\}$ endowed with the usual sum and with the generated topology by the family $\{V_x\}_{x \in \mathbb{R}}$ where $V_x = \{(x, y): y \geq 0\}$. Then S is a cancellative topological monoid with open shifts. Moreover, S is not T_0 space.

Example 1.5: Let S be an infinite cancellative semigroup, if S is endowed with the cofinite topology, then S is a cancellative semitopological semigroup with open shifts. The following result clarifies the action the epi-reflection functor for subcategories closed under supertopologies. By Herrlich and Strecker (1997) for the proof which is straightforward anyway).

Proposition 1.6: Let C be an epireflective class of top. Then C(X) is a quotient of X if and only if C is closed under super topologies. The theorem 3.4 by Tkachenko (2014) allows us to obtain an inner characterization of the C_1-reflection in the category of semitopological groups, the following corollary is more general.

Proposition 1.7: Let C be an epireflective subcategory of top closed under super topologies. Then $C(X) = X/R_C$.

Where:

$$RC = \cap\{R : R \text{ is an equivalence relation and } X / R \in C\}$$

Proof: From proposition 1.6, C(X) = X/R_1, being R_1 an equivalence relation in X. If π: X→X/R_C is the, respective, quotient mapping, since, $R_C \subseteq R_1$, the mapping f: X/R_C → X/R_1, defined by f (π(x)) = $\varphi_{(C, X)}$ is well defined, bijective and quotient, therefore, it is homeomorphism, so that, $R_1 = R_C$ this completes the proof. We can find the proof of following proposition by Acosta and Rubio (2006) where the reflection on the T_0 spaces is called the T_0-identification.

Proposition 1.8: Let X be a topological space. Then $\varphi^{-1}(c_0, x)(\varphi(c_0, x)(U)) = U$ for each closed or open set in x, therefore, $\varphi(c_0, x)$ is an open and closed mapping. Also, $\varphi(c_0, x)(x) \neq \varphi(c_0, x)(y)$ if and only there exist U, open in x such that U∩ {x, y} is a singleton. The following theorem appears by Hernandez-Arzusa and Hernandez (2017) for more general algebraic structures, we give a similar proof for semigroups.

Theorem 1.9: Let S be a semitopological semigroup (monoid) and C an epireflective class of top. Then C (S) is a semitopolgical semigroup (resp. monoid) and φ_S is a homomorphism.

Proof: For each a∈S, the continuous mappings l_a and r_a allow to define continuous mappings C(l_a) and C(r_a) from C(S) to itself for $C(l_a)((\varphi_S(x)) = \varphi_S(ax)$ and $C(r_a)((\varphi_S(x)) = \varphi_S(xa)$. Therefore, the operation on C(S) defined by $\varphi_S(x)\,\varphi_S(y) = \varphi_S(xy)$ is well defined and also φ_S is a homomorphism.

Proposition 1.10: Let S be a semitopological semigroup (monoid) and let ~ be a congruence in S and let π: S→S/~ the respective quotient mapping. Then S~ is a semitopological semigroup (resp. monoid) and π is a homomorphism. Also, if S is a topological semigroup and π×π is quotient mapping, then S/~ is a topological semigroup (resp. monoid). In particular if π is open and S is a topological semigroup, then the same is true for S/~.

Proof: Let S be a semitopological semigroup, if ~ is a congruence, obviously the operation defined by π(x)*π(y) = π(xy) for each x, y∈S is well defined and associative on S/~, therefore, (S/~, *) is a semigroup and π is a homomorphism. If S is monoid and e is its neutral element, then π(e) is the neutral element in S/~, therefore, S/~ is a monoid. Since, π is quotient mapping, we have * is separately continuous and in consequence we have S/~ is a semitopological semigroup. If S is a topological semigroup and π×π is a quotient mapping, then the continuity of the operation on S implies that * is continuous, therefore, S/~ would be a topological semigroup.

Since, if X es T_3 es, then $C_0(X)$ is T_3 (From Proposition 1.8 $\varphi(c_0, x)$ is open and closed mapping), the proof of the following proposition is trivial.

Proposition 1.11: For each topological space X, we have $C_r(X) = C_0(C_3(X))$.

RESULTS AND DISCUSSION

Related functors to axioms of separations in semigroups

Proposition 2.1: Let S be a semitopological monoid where right shifts or left shifts are open and let~a congruence on S. Then, the respective quotient mapping π: S→S/~ is open.

Proof: Only we will prove the statement when the left shifts are open, the right case is analogue. Proving that π is open, we will proof that $\pi^{-1}(\pi(U))$ is open in X for each U open in X. Indeed let x be in $\pi^{-1}(\pi(U))$ where U is open in X. Hence, there is u∈U such that π(x) = π(u). Since, $l_u(e) = u$, we can find a neighborhood of e, V, such that uV⊆U. We will prove that $xV \subseteq \pi^{-1}(\pi(U))$, this would prove that $x \in \text{Int}\, \pi^{-1}(\pi(U))$ and therefore $\pi^{-1}(\pi(U))$ would be open. Let t∈xV, then t = xv where v∈V. Since, ~ is a congruence, we have that π(t) = π(xv) = x(uv)⊆π(U), therefore, $t = xv \in \pi^{-1}(\pi(U))$, this completes the proof.

Since, the class of the T_2, T_1 and T_0 space are closed for supertopologies, then from propositions 1.6, 1.9, 1.10 and 2.1, we have the following corollary.

Corollary 2.2: If S is a topological monoid with open shifts, then $C_i(S)$ is a monoid for each i∈{0, 1, 2}.

Proposition 2.3: Let X be a topological space and ~ an equivalence relation on X. Then, if X/~ is T_2, ~ is closed X×X. The reciprocal holds if the quotient mapping π: X→X/~ is open.

Proof: Let us suppose that (x, y)∉~, then π(x)≠π(y), if X/~ is T_2, there are open disjoint neighborhoods $U_{\pi(x)}$ and $U_{\pi(y)}$ of π(x) and π(y) in X/~, respectively. The continuity of π guarantees that there are neighborhood V_x, V_y, of x and y in X, respectively, such that $\pi(V_x) \subseteq U_{\pi(x)}$ and $\pi(V_y) \subseteq U_{\pi(y)}$. It follows that $(V_x \times V_y) \cap \sim\, = \emptyset$. This proves that ~ is closed in X×X. Reciprocally, let us suppose that π is open and ~ is closed and let us prove X/~ is T_2. Indeed, let us π(x)≠π(y) where x, y∈X. Since, π(x)≠π(y) and ~ is closed, we can find open sets in X V_x and V_y, containing to x and y, respectively, such that $(V_x \times V_y) \cap \sim = \emptyset$. It follows that. But $\pi(V_x)$ and $\pi(V_y)$ are neighborhoods of $\pi(x)$ and $\pi(y)$, respectively, this proves that X/~ is T_2. The next corollary easily follows from propositions 2.1 and 2.3.

Corollary 2.4: Let S be a semitopological monoid with left shift s or right open shifts and let ~ be a congruence on S. Then S/~ is T_2 if and only if ~ is a closed congruence in S.

Theorem 2.5: Let S be a semitopological monoid with left shifts or right open shifts, then $C_2(S)$ = S/~ where ~ is the smallest closed congruence on S.

Proof: Let ~ be the smallest closed congruence on S and let π: S→S be the respective quotient mapping. Corollary 2.4 guarantees that S/~ is T_2, therefore, there is a continuous mapping g: $C_2(S)$ such that g°φ(S, C_2). By proposition 4.3, theorem 1.9 and given that the T_2 spaces class is closed under super topologies, we have $C_2(S) = S/\simeq$ being ≃ a congruence on S which is closed by Proposition 2.3. Therefore, $\sim \subseteq \simeq$, so, we can define a continuous mapping h: $S/\sim \to S/\simeq$ such that $h \circ \pi = \varphi(S, C_2)$. It easily follows that h is the inverse of g, this completes the proof.

Theorem 2.6: Let S be a topological semigroup with open shifts, then S_{Sr} is a topological semigroup. Also, if S is a monoid, then S is a quasiregular monoid.

Proof: Let S be a topological semigroup with open shifts. Let U be a regular open in S such that ab∈U being a, b∈S. Given that the operation on S is continuous, we can find open sets in S, V and W, containing a and b, respectively, holding VW⊆U. The continuity of the operation and the fact that Int(Cl(V))Cl(W) is open in S, imply:

$$\begin{aligned} \mathrm{Int}(\bar{V})\mathrm{Int}(\bar{W}) \subseteq \mathrm{Int}(\overline{VW}) &= \mathrm{Int}(\mathrm{Int}(\bar{V}\mathrm{Cl}(W))) \\ &\subseteq \mathrm{Int}(\overline{VW}) \\ &\subseteq \mathrm{Int}(\overline{VW}) \\ &\subseteq \mathrm{Int}(\bar{U}) = U \end{aligned} \quad (1)$$

Therefore, S_{Sr} is a topological semigroup. Let us suppose S is a monoid and let us prove that S_{Sr} is a topological monoid T_3. Obviously S_{Sr} is a monoid and we have already proved S_{Sr} is a topological semigroup it remains to prove that it is T_3. Indeed let U a regular open in and x∈U. Since, the operation in S is continuous and ex = x we can find an open neighborhood of e, V and a open neighborhood of x,W such that VW⊆U. Therefore:

$$\begin{aligned} x \in \mathrm{Int}(\bar{W}) \subseteq \bar{W} \subseteq V\bar{W} = \mathrm{Int}(\overline{V\bar{W}}) &\subseteq \mathrm{Int}(\overline{VW}) \\ &\subseteq \mathrm{Int}(\overline{VW}) \\ &\subseteq \mathrm{Int}(\bar{U}) = U \end{aligned} \quad (2)$$

This proves that S_{Sr} is T_3. From similar argument to the proof of theorem 2.6 of Tkachenko (2015a, b) and theorem 2.6, we obtain the following result.

Corollary 2.7: If S is a topological monoid with open shifts, then $C_3(S) = S_{Sr}$. From proposition 1.11, theorem 2.6 and corollary 2.7, we have the following theorem.

Theorem 2.8: Let S be a topological monoid with open shifts, then $C_r(S) = C_0(S_{sr})$.

Epireflections preserving products: Given an epireflective subcategory of top, C, we say that the epireflection induced by C preserves products in a subcategory D of top, if $C(\prod_{i\in} X_i) = \prod_{i\in I} C(X_i)$ for each family $\{X_i\}_{i\in I}$ of spaces in D.

Theorem 3.1: Let C an epireflective subcategory of top closed under super topologies satisfying $\prod_{j\in J} X_j \in C$ if and only if $X_j \in C$ for each j∈J. Then, the C-epireflection preserves products in the category of semitopological monoids with open shifts.

Proof: Let $\{S_i\}_{i\in I}$ be a family of semitopological semigroups with open shifts and let C an epireflective subcategory of top closed under super topologies. Proposition 1.6 and theorem 1.9 imply $C(\prod_{i\in I} S_i) = (\prod_{i\in I} S_i)/R$, being R a congruence in $\prod_{1\in I} S_i$. Given k∈1, let us define the following relation in X_k: $x \sim_k y$ if there exists x, y∈ $\prod_{i\in I} S_i$ such that $\varphi(C,\prod_{i\in I} S_i)(x) = \varphi(C,\prod_{i\in I} S_i)(x)$ and $p_k(x) = p_i(y)$, being p_k: $\prod_{i\in I} S_i \to S_k$, the k-th projection. It is easy to prove that $\sim_k$ is a congrujence on X_k. Let π_k: $X_k \to X_k/\sim_k$ the respective quotient mapping. From definition of ~k for each k∈K, we have f: $\prod_{I\in I}(S_i/\sim i) \to (\prod_{i\in I} S_i)/R$, given by $f((\pi_i(x_i))_{i\in I}) = \varphi(C,\prod_{i\in I} S_i)((x_i)_{i\in I}))$ is a bijection. Proposition 2.1 implies π_i is open for each i∈1, therefore, f is a homemorphism, the hypothesis over C guarantees that $S_i/\sim_i \in C$ for each i∈l, therefore, for each i∈l we can define a continuous mapping g_i: $C(S_i) \to S_i/\sim i$ by $g_i(\varphi(C, S_i)(x)) = \pi_i(x)$, for each x∈$S_i$. Therefore, $\prod_{i\in I} g_i$: $\prod_{i\in I} C(S_i) \to \prod_{i\in I}(S_i/\sim_i)$ is continuous, so that, $f^\circ\prod_{i\in I} g_i$: $\prod_{I\in I} C(S_i) \to C(\prod_{i\in I} S_i)$ is continuous. On the other hand, since, $\prod_{i\in I} C(S_i) \in C$, we can find a continuous mapping k: $C(\prod_{i\in I} S_i) \to \prod_{i\in I} C(S_i)$ such that $k^\circ\varphi(C, \prod_{i\in I} S_i) = \prod_{I\in I} \varphi(C, S_i)$. It is easy to prove that k and $f^\circ\prod_{i\in I} g_i$ are inverses of each other. This completes the proof.

It is known that the C_i-reflection does not preserve products (introduction of Husek and Vries (1987), $I\in\{1, 2, 3, t\}$. However, C_i, $I\in\{0, 1, 2\}$ preserves products in the category of semitopological groups (propositions 3.3, 3.4 and 3.5 of Tkachenko (2015a, b)) while C_3 and C_r preserve products in the category of paratopological groups (Proposition 3.6 of Tkachenko (2015a, b). In the two following results we give similar results in the semitopological and topological monoids with open shifts. Given C_i, $i\in\{0, 1, 2\}$, satisfies the hypothesis of theorem 3.1, we have the following corollary.

Corollary 3.2: The C_i-reflection preserves products in the category of semitopological semigroups with open shifts, $i\in\{0, 1, 2\}$.

According to lemma 3 of Mrsevic *et al.* (1985), we have the semiregularization respects products, therefore, from corollary 2.7, theorem 2.8 and corollary 3.2, we have the following theorem.

Theorem 3.3: The C_i-reflection preserves products in the category of topological monoids with open shifts, $i\in\{3, r\}$.

The cellularity of topological monoids

Proposition 4.1: $c(X) = c(C_0(X))$ for each topological space X.

Proof: Given that $C_0(X)$ is a continuous image of X, we have $c\ (C_0(X))\leq c(X)$. Let x a cellular family in X, proposition 1.8 guarantees that $\varphi^{-1}(c_0, x)\ \varphi(c_0, x)\ (U)) = U$ for each $U\in u$, therefore, $\{\varphi(c_0, x)\ (U)\colon U\in u\}$ is a cellular family in $C_0(X)$, so that, $c(X)\leq c(C_0(X))$ this completes the proof.

Proposition 4.2: $c\ (X) = c(X_{Sr})$ for each topological space X.

Proof: Given that X_{Sr} is a continuous image of X, we have $c(X_{Sr})\leq X$. Let u a cellular family in X and let U and V be in u, therefore, $U\cap V = \varnothing$. Since, U and V are open sets in X, we have $\mathrm{int}\,\bar{U}\cap \mathrm{int}\,\bar{V} = \varnothing$. Also, since, for all $U\in u$ it holds $U\subseteq \mathrm{int}\bar{U}$, we have $\mathrm{int}\,\bar{U}\neq\varnothing$ for all $U\in u$. This proves that the $\{\mathrm{int}\,\bar{U} : U\in u\}$ is a cellular family in X_{Sr}, therefore, $c(X_{Sr})$, this completes the proof. From Propositions 4.1 and 4.2 theorem 2.8, we have the following corollary.

Corollary 4.3: Let X be a quasiregular space, then $c(X) = c(C_r(X))$.

Corollary 4.4: If X is quasiregular space, we have $c(X) = c(C_i(X))$ for each $i\in\{0, 1, 2, 3, r\}$.

Proof: Let X be a quasiregular space. By proposition 4.1 and the definition of cellularity it is clear that $c(C_r(X))\leq c\ (C_3(X))$ and $c(C_r(X))\leq c(C_2X))\leq c(C_1(X))\leq c(C_0(X)) = c(X)$. Corollary 4.3 guarantees that $c(X) = c(C_r(X))$, this completes the proof. From theorem 2.6 and the corollary 4.4, we have the following theorem.

Theorem 4.5: Let S be a topological monoid with open shifts. Then the following statements are equivalent:

- S has cellularity countable
- $C_0(S)$ has cellularity countable
- $C_1(S)$ has cellularity countable
- $C_2(S)$ has cellularity countable
- $C_3(S)$ has cellularity countable
- $C_r(S)$ has cellularity countable

Lemma 4.6: If S is a cancellative topological semigroup with open shifts, then $C_0(S_{Sr})$ is cancellative.

Proof: Let S be a cancellative topological semigroup with open shifts and let us see what $C_0(S_{Sr})$ is cancellative. Indeed, let us suppose $\varphi_{(C0SSr)}(cx) = \varphi(C_0S_{Sr})(cy)$ but $\varphi(C_0,S_{Sr})(x)\neq\varphi\ (C_0S_{Sr})\ (y)$. From proposition 1.8 we can find an open regular in S_{sr}, U that without loss of generality, we can assume that $x\in U$ and $y\notin U$. Since, $l_c\colon S\to cS$ is a homemorphism, we have cU is open regular in cS. There exists an open set V in S such that $cU = V\cap cS$. Given that cS is open in S, we have $cx\in cU = \mathrm{int}_{cS}(\mathrm{Cl}_{cS}(cU)) = cS\cap \mathrm{int}_S(\mathrm{Cl}_S(V))$. Since, S is cancellative, $cy\notin cU$ and in consequence $cy\notin \mathrm{int}_S(\mathrm{Cl}_S(V))$ but $\mathrm{int}_S(\mathrm{Cl}_S(V))$ is open in S_{Sr} from proposition 1.11, we can say $\varphi(C_0S_{Sr})\ (cx)\neq\varphi(C_0S_{Sr})(cy)$ obtaining a contradiction, this implies that $\varphi(C_0S_{Sr})(x)\neq\varphi(C_0S_{Sr})(y)$ and therefore, $C_0(S_{Sr})$ is cancellative to the left, proving that is cancellative to the right is analogues.

Tkachenko (2015a, b) proved that the σ-compact paratopological groups have countable cellularity. In the following two theorems we give analogues results for topological semigroups but given that we do not have group operation, we have changed the σ -compactness for compactness in the first theorem and in the second theorem, in addition to the σ-compactness we have added the sequential compactness.

Theorem 4.7: Let S be a compact topological monoid cancelative with open shifts, then S has countable cellularity.

Proof: Let S be a compact topological monoid cancelative with open shifts from proposition 1.11, corollary 2.2, theorem 2.6 and lemma 4.6, we have $C_r(S) = C_0(S_{Sr})$ is a cancellative topological monoid which is compact. Since, S is compact, theorem 2.5.2 of Arhangel'skii and Reznichenko (2005) implies that $C_r(S)$ is a compact topological group and from corollary 2.3 of Tkachenko (2015a, b), we have $C_r(S)$ has countable cellularity. Finally, applying Theorem 4.5, we have S has countable cellularity.

Theorem 4.8: If S is a σ-compact and sequentially compact cancellative topological monoid with open shifts, then S has countable celullarity.

Proof: Let S be a σ-compact and sequentially compact cancellative topological monoid with open shifts. Lemma 4.6 guarantees that $C_r(S)$ is cancellative, also being continuous image of S, we have $C_r(S)$ is σ-compact and sequentially compact. Theorem 6 of Bokalo and Guran (1996) implies that $C_r(S)$ is a σ-compact topological group and from corollary 2.3 of Tkachenko (2015a, b), we have $C_r(S)$ has countable celullarity. If we apply theorem 4.5, we have $C_r(S)$ has countable celullarity.

CONCLUSION

Finally, Tkachenko proved that the σ-compact paratopological groups have countable cellularity. We present similar results for compact cancellative topological monoids with open shifts (theorem 4.7) and σ-compact and sequentially compact cancellative topological monoids with open shifts (theorem 4.8). The T_0-T_3, regular and Tychonoff spaces are defined according to Tkachenko (2015a, b).

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