Research Journal of Applied Sciences 14 (2): 49-53, 2019
ISSN: 1815-932X

Hyperbolic Center of Mass for a System of Particles on the Poincare Upper Half-Plane

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Abstract: In the present study, we obtain using the lever law of the lever, an explicit formula that allows to calculate the center of mass of a system of n-particles with masses m_1, m_2, ..., m_n>0, located on the half plane superior of Lobachevsky H^2, endowed with a conforming metric which induces constant and negative Gaussian curvature.

Key words: The poincare upper half-plane model, hyperbolic center of mass, hyperbolic lever rule, Geodesic, particles, superior

INTRODUCTION

Hyperbolic geometry is usually presented as one of the paradigms of so-called non-Euclidean geometries, since their different models arise if the postulate of the parallels of Euclid is denied. Its origins date back to the beginning of the 19th century with the researches of Schweikart, Taurinus, Bolyai and Lobachevsky. Some mathematicians of great importance such as Euler, Gauss, Riemann and Poicare, among many others, made notable contributions in this area of mathematical knowledge (Cannon *et al*., 1997).

One of the models of hyperbolic geometry, known as Minkowski space is one of the fundamental pillars for the understanding of space-time in the theory of relativity. Currently, the hyperbolic geometry has multiple applications to electrical phenomena and microwave transmission in engineering. There are also some connections that allow us to find applications of some groups of transformations (matrices) associated with models of hyperbolic geometry, to the theory of numbers as evidenced in 1995 with the demonstration of Fermat's last theorem by Andrew Wiles (Terras, 2013).

The purpose of this study is to present explicit formulas for the calculating (hyperbolic) center of mass for a system formed for two or more particles with positive masses sited on the poincare upper half-plane. The concept of center of mass is of great relevance for the study of physical phenomenal, since, this has important geometric and mechanic properties. Its definition on Euclidean spaces (zero curvature) it is easy since the existence of a linear structure in this type of spaces. However, in curved spaces (spaces with non zero Gaussian curvature), this definition is not direct and this is the reason for the absent of formulas to calculate centres of mass in this cases.

Diacu (2012) refers to the difficulty of defining center of mass in curved spaces. He provides a class of orbits in the curved n-body problem for which "no point that could play the role of the center of mass is fixed or moves uniformly along a geodesic". This proves that the equations of motion lack center-of-mass and linear momentum integrals. But nevertheless, he is not provide a way to calculate or determinate this element. Finally, Borisov and Mamaev (2006) establish the non existence of frame for the center of mass in spaces with constant non zero Gaussian curvature but is not provides a proof of this affirmation.

In celestial mechanics the concept of center of mass is important because in the classical case, it is an integral of motion whereas in the context of curved spaces it is not and this is the reason why the curved 2-body problem and the curved Kepler problem are not equivalent. This study is organized as follow:

MATERIALS AND METHODS

Preliminary

Center of mass of a system of particles: Given two positive masses m_1, m_2 sited at the points x_1, x_2.R^n, the (Euclidean) center of mass of the system is defined by:

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad (1)$$

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The point x is sited on the segment (Geodesic) joining x_1 with x_2 and moreover, if d_1 and d_2 are the Euclidean distances from x_1-x and x_2-x, respectively, then the following equation is satisfied:

$$m_1 d_1 = m_2 d_2 \tag{2}$$

This relation is known as the Euclidean lever rule. The definition of center of mass can be easily extended to n particles sited on the Euclidean space R^n, due to the linear structure with which they are provided but the curved spaces, this is not so simple. Garca-Naranjo *et al.* (2016), it is presented a definition for the hyperbolic center of mass in H by means of:

$$m_1 \sinh 2d_1 = m_2 \sinh 2d_2 \tag{3}$$

This definition of hyperbolic center of mass is introduced to simplify the study of some configurations called relative equilibria, for the 2-body problem in H. In this study, we deduce the expression for the hyperbolic center of mass by assuming that holds of the hyperbolic lever rule: $m_1 d_1 = m_2 d_2$ which is the natural extension of the Euclidean lever rule to H.

A deeper discussion about the definition of the center of mass in curved spaces can be encountered in (Ungar *et al*., 2008).

The Poincare Upper Half-Plane Model: Is the set define by H = {z∈C:Im(z)>0} endowed with the conformal metric given by:

$$-ds^2 = \frac{4dzd\overline{z}}{(z-z)^2}$$

If we consider this as a subset of the complex plane C or equivalently, H = {(x, y):y>0} with the metric $ds^2 = y^{-2}(dx^2 + dy^2)$. It is a well know fact that H with this metric is a two-dimensional Riemannian manifold geodesically complete of Gaussian curvature k = -1.

This arc length is invariant under the action of g in SL(2, R) on zH defined by fractional linear transformation:

$$gz = g(z) = (az + b)/(cz + d)$$

If g is the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d∈R and ad-bc = 1 and g gives a conformal or angle-preserving mapping of H.

In H, there are two types of geodesic: vertical half-lines with initial point on the x-axis, defined by L_b = {(b, y):y>0} or half-circles with centres in the x-axis, $L_{c,r}$ = {(x, y):$(x-c)^2+y^2 = r^2$, y>0}. In this model of hyperbolic geometry, there are infinitely many geodesics through a given point which are parallel to a given geodesic. The following proposition is very important for the definition of hyperbolic center of mass on H.

Proposition: For each pair p_1 and p_2 of distinct points in H, there exists a unique hyperbolic geodesic passing through p_1 and p_2 (Anderson, 2006; Stahls, 2007).

The Upper Half-Plane is a model of hyperbolic geometry and is an example of absolute or neutral geometry, moreover is a intrinsic geometry. There are two ways to calculate distances in H as follow: given two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$.

If $x_1 = x_2$ and $y_1<y_2$ (two points in the same Euclidean vertical line), the length of segment joining p_1-p_2 is given by $d(p_1, p_2) = \ln y_2/y_1$. If $x_1 \neq x_2$, then there exist a Euclidean line non vertical containing both points and a line perpendicular to it, passing for the middle point of segment joining p_1 and p_2 and this passing for the x-axis, at a point $\hat{c}(c, 0)$ which is the center of the semicircle passing for p_1 and p_2. It is easy to prove that c is given by:

$$c = \frac{x_1^2 + y_1^2 - (x_2^2 + y_2^2)}{2(x_1 - x_2)} \tag{4}$$

Let a_1, a_2 be the angles determined for the segments $\hat{c}p_1$ and $\hat{c}p_2$, measured from positive x-axis and generality is not lost if we suppose that $\alpha_1<\alpha_2$, in this case the hyperbolic distance is given by:

$$d(p_1, p_2) = \ln \frac{scc\alpha_2 - \cot\alpha_2}{scc\alpha_1 - \cot\alpha_1} \tag{5}$$

Now by using trigonometric identities:

$$\csc\gamma - \cot\gamma = \frac{1-\cos\gamma}{\sin\gamma} = \frac{2\sin^2\gamma/2}{2\sin\gamma/2\cos\gamma/2} = \tan\frac{\gamma}{2} \tag{6}$$

we have the following relation:

$$d(p_1, p_2) = \ln \frac{\tan\alpha_2/2}{\tan\alpha_1/2} \tag{7}$$

or by using the Cartesian coordinates:

$$d(p_1, p_2) = \ln \frac{y_2(x_1 - c - r)}{y_1(x_2 - c - r)} \tag{8}$$

We have that $c \to \infty$ when $x_2 \to x_1^+$ and $c \to -\infty$ when $x_2 \to x_1^-$ and so, $d(p, q) \to \ln y_2/y_1$, namely, the definition of distance is consistent with the definitions of two type of geodesic where the geodesics of first type can be considered a half-circles with center either at $+\infty$ or $-\infty$. An unified expression for the of hyperbolic distances d is:

$$d(p_1, p_2) = \text{arc cosh}\left(1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2y_1y_2}\right) \quad (9)$$

which is not very useful for the purpose of the present research.

RESULTS AND DISCUSSION

Center of mass on the Poincare upper half-plane model
Definition: Let m_1, m_2 be two particles sited, respectively in the points $p_1, p_2 \in H$. The hyperbolic center of mass of the system is the the unique point q in the Geodesic of H joining p_1 with p_2 that satisfies the hyperbolic lever rule:

$$m_1d_1 = m_2d_2$$

Where:

$d_1 = d(p_1, q)$
$d_2 = d(p_2, q)$
$d(p_1, p_2) = d_1+d_2$

Theorem: Let $m_1, m_2, ..., m_n$ be a system of n particles sited at the points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in H$, then there exists a unique point with coordinates (u, v) which is the hyperbolic center of mass for the system.

Proof: It is enough to consider two particles with masses m_1, m_2 sited at points (x_1, y_1), (x_2, y_2), respectively and consider the following two cases.

Case 1: $x_1 = y_1$. Let m_1, m_2 be two point masses sited at (x_1, y_1), (x_1, y_2), respectively and let (u, v) be the unique point satisfying the "hyperbolic lever rule" and d_1, the distance from (x_1, y_1) to (u, v) and d_2 the distance from the (x_1, y_2) to (u, v), then $u = x_1$ and by using the hyperbolic lever rule $m_1d_1 = m_2d_2$ and substituting the correspond values we obtain:

$$m_1 \ln\frac{v}{y_1} = m_2 \ln\frac{y_2}{v}$$

or equivalently:

$$v^{m_1+m_2} = y_1^{m_1}y_2^{m_2}$$

Inductively, if $m_1, m_2, ..., m_n$ are n masses sited in the points on the same vertical line $(x, y_1), (x, y_2), ..., (x, y_n)$, then the hyperbolic center of mass is sited in a point of the form (u, v) with u = x (sited in the same vertical line) and where v satisfies:

$$v^m = y_1^{m_1}y_2^{m_2}, ..., y_n^{m_n} \quad (10)$$

where, $m = m_1+m_2+, ..., m_n$.

Case 2: $x_1 \neq x_2$. Let $L_{c,r}$ be the geodesic passing through $(x_1, y_1), (x_2, y_2) \in H$. First, we can suppose that c = 0, namely the center of half-circle is $\hat{c}(0, 0)$, so, $x_1^2+y_1^2 = x_2^2+y_2^2 = r^2$. Let q(u, v) be the unique point at the geodesic $L_{0,r}$ that satisfies the hyperbolic lever rule $m_1d_1 = m_2d_2$ where d1 = $d((x_1, y_1), (u, v))$ and $d_2 = d((x_2, y_2), (u, v))$. If α_1, α_2 and θ are the angles determined by (x_1, y_1), (x_2, y_2) and q(u, v), respectively with the x-axis then the substitution in Eq. 10 yields:

$$m_1 \ln\frac{\tan\theta/2}{\tan\alpha_1/2} = m_2 \ln\frac{\tan\alpha_2/2}{\tan\theta/2} \quad (11)$$

or equivalently:

$$\tan^m\frac{\theta}{2} = \tan^{m_1}\frac{\theta_1}{2}\tan^{m_2}\frac{\theta_2}{2} \quad (12)$$

where, $m = m_1+m_2$ is the total mass of system. Now, we obtain an expression for calculating the center of mass in terms of the Cartesian coordinates and the given masses. In Eq. 12, we can use the identity $\tan\gamma/2 = \sin\gamma/1+\cos\gamma$ and the relations $\sin\alpha_1 = y_1/r$, $\cos\alpha_1 = x_1/r$, $\sin\alpha_2 = y_2/r$, $\cos\alpha_2 = x_2/r$, $\sin\theta = v/r$, $\cos\theta = u/v$, to obtain:

$$\frac{v}{r+u} = \left(\frac{y_1}{r+x_1}\right)^{m_1/m}\left(\frac{y_2}{r+x_2}\right)^{m_2/m} \quad (13)$$

The substitution:

$$a = \left(\frac{y_1}{r+x_1}\right)^{\frac{m_1}{m}}\left(\frac{y_2}{r+x_2}\right)\frac{m_2}{m}$$

into Eq. 13 and the relation $u^2+v^2 = r^2$ becomes the quadratic equation:

$$(1+a^2)u^2 + 2a^2ru + (a^2-1)r^2 = 0 \quad (14)$$

with two distinct roots, u = -r and $u = r1\text{-}a^2/1+a^2$. The value u = -r implies v = 0, a point that does not belong to H, accordingly, the unique admissible solution is the second value. Substituting the values of r and a, we obtain the next relations for the hyperbolic center of mass in terms of given Euclidean coordinates and masses:

$$u = \sqrt{x_1^2+y_1^2}\,\frac{\left[\left(\sqrt{x_1^2+y_1^2}+x_1\right)^{m_1}\left(\sqrt{x_2^2+y_2^2}+x_2\right)^{m_2}\right]^{\frac{2}{m}} - \left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}}{\left[\left(\sqrt{x_1^2+y_1^2}+x_1\right)^{m_1}\left(\sqrt{x_2^2+y_2^2}+x_2\right)^{m_2}\right]^{\frac{2}{m}} + \left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}} \quad (15)$$

and:

$$v=\sqrt{x_1^2+y_1^2}\,\frac{2\sqrt{x_1^2+y_1^2}\left[\left(y_1\left(\sqrt{x_1^2+y_1^2}+x_1\right)\right)^{m_1}\left(y_2\sqrt{x_2^2+y_2^2}+x_2\right)^{m_2}\right]^{\frac{1}{m}}}{\left[\left(\sqrt{x_1^2+y_1^2}+x_1\right)^{m_1}\left(\sqrt{x_2^2+y_2^2}+x_2\right)^{m_2}\right]^{\frac{2}{m}}+\left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}} \quad (16)$$

or in more compact form:

$$u=r\frac{\left[\left(r+x_1\right)^{m_1}\left(r+x_2\right)^{m_2}\right]^{\frac{2}{m}}-\left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}}{\left[\left(r+x_1\right)^{m_1}\left(r+x_2\right)^{m_2}\right]^{\frac{2}{m}}-\left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}} \quad (17)$$

and:

$$v=\frac{2r\left[\left(y_1\left(r+x_1\right)\right)^{m_1}\left(\left(r+x_2\right)^{m_2}\right)\right]^{\frac{1}{m}}}{\left[\left(r+x_1\right)^{m_1}\left(r+x_2\right)^{m_2}\right]^{\frac{2}{m}}+\left(y_1^{m_1}y_2^{m_2}\right)^{\frac{2}{m}}} \quad (18)$$

with $r=\sqrt{x_1^2+y_1^2}=\sqrt{x_2^2+y_2^2}$ and $m = m_1+m_2$ is the total mass of system. Inductively, we can deduce a formula for a finite number of mass sited in the same Euclidean semicircle centred at (0, 0): let $m_1, m_2, ..., m_n$ be, n punctual masses sited in H, in the same Euclidean half-circle centred at (0, 0) and radio r, $L_{0,r}$ and let $\alpha_1, \alpha_2, ..., \alpha_n$ be, the angles formed by these with the positive x-axis, then the point (u, v) that satisfies the hyperbolic lever rule, forming an angle θ with the positive x-axis satisfies the relation:

$$\tan^m\frac{\theta}{2}=\tan^{m_1}\frac{\alpha_1}{2}\tan^{m_2}\frac{\alpha_2}{2}, ..., \tan m_n\frac{\alpha_n}{2} \quad (19)$$

where, $m = m_1+m_2+, ..., m_n$ is the total mass of system. Moreover, by using the identity $\tan\theta/2 = \sin\theta/1+\cos\theta$ and the relations $\sin\alpha_k = y_k/r$, $\cos\alpha_k = y_k/r$, for k = 1, 2, ..., n and $\sin\theta = v/r$, $\cos\theta = u/r$, we obtain that:

$$u=r\frac{\left[\prod\left(r+x_k\right)^{m_k}\right]^{\frac{2}{m}}-\left(\prod y_k^{m_k}\right)^{\frac{2}{m}}}{\left[\prod\left(r+x_k\right)^{m_k}\right]^{\frac{2}{m}}+\left(\prod y_k^{m_k}\right)^{\frac{2}{m}}} \quad (20)$$

and:

$$v=\frac{2r\left[\prod y_k\left(r+x_k\right)^{m_k}\right]^{\frac{1}{m}}}{\left[\prod\left(r+x_k\right)^{m_k}\right]^{\frac{2}{m}}+\left(\prod y_k^{m_k}\right)^{\frac{2}{m}}} \quad (21)$$

where, $r^2 = x_k^2+y_k^2$ for k = 1, 2, ..., n. For the general case where $x_1 \neq x_2$ and $c \neq 0$, we make the horizontal translation $X_1 = x_1-c$, $X_2 = x_2-c$, $Y_1 = y_1$, $Y_2 = y_2$ this correspond to the isometry of H, $z \to z-c$ and it is easy to verify that:

$$X_1^2+Y_1^2=X_2^2+Y_2^2$$

namely, the points (X_1, Y_1) and (X_2, Y_2) are sited in a circle centred at origin (0, 0) and so, we can to apply the last procedure, to obtain the center of mass of the system the masses m_1, m_2 sited in the points (X_1, Y_1), (X_2, Y_2), obtaining a translated center of mass of coordinates (U, V) and so, the coordinates of the center of mass of the original system (u, v) are given haves for u = U+c, v = V+c. Proceeding inductively and making considerations of two cases, it is possible to computer the coordinates of the center of mass, for a system the particles in H. This concludes the proof.

Relation with the hyperbolic center of mass on the Poincare disk model: Consider, the Poincare Disk $D_R = \{z\in C:|z|<R\}$ with the metric $ds^2 = 4R^2dzd\bar{z}/\left(R^2-|z|^2\right)^2$ and the Poincare Upper Half-Plane $H_R = \{w\in C:Im(w)>0\}$ with the metric $ds^2=-4R^2dwd\bar{w}/\left(w-\bar{w}\right)^2$, then these two spaces have the same Gaussian curvature $k = -1/R^2$ and by the Minding's theorem, they are isometric locally. Moreover, the mapping, $z:H_R\to D_R$ defined by:

$$z(w)=\frac{R(iR-w)}{iR+w}$$

called the Cayley transform, provides an isometric between the Poincare half-plane model and the Poincare Disk Model. The inverse of Cayley transform is $w: D_R\to H_R$ defined by:

$$w(z)=\frac{iR(R-z)}{R+z}$$

Now, if $m_1, m_2, ..., m_n$ are positive masses sited at the points $z_1, z_2, ..., z_n \in D_R$, respectively with hyperbolic center of mass z_c and we make $W_k = W(Z_k)$, k = 1, ..., n and substituting this values in some of Eq. 10 or 19, we obtain:

$$\left(\frac{R-z_c}{R+z_c}\right)^m=\prod_{k=1}^{n}\left(\frac{R-z_k}{R+z_k}\right)^{m_k}$$

namely, w send hyperbolic center of masses in H_R to hyperbolic center of masses in D_R which is direct consecuence of isometric nature of w (Palencia and Victoria, 2017). Last equation is equivalent to:

$$\left(\frac{1-z_c/R}{1+z_c/R}\right)^{Rm}=\prod_{k=1}^{n}\left(\frac{1-z_k/R}{1+z_k/R}\right)^{Rm_k}$$

If $R\to\infty$ then $k\to 0$ and this equation is reduced to:

$$Z_c = \frac{1}{m}\sum_{k=1}^{n} m_k z_k$$

which correspond to the center of mass for a system of particles on the plane with the usual Euclidean metric.

CONCLUSSION

In this study, we give the definition of center of mass of a system of particles on a Euclidean space, emphasizing the "lever rule", moreover we introduce the poincare upper half-plane, endowed with a conformal metric, its geodesic and study the formula for calculating distances between its points. In this study 3, we deduce formulas to calculate the center of mass of two or more particles with positive masses sited over the upper half plane. Finally, we establish a connection with the hyperbolic center of mass on the model of Poincare disk.

ACKNOWLEDGEMENT

This research was supported by the Universidad de Cartagena, Cartagena-Colombia (Grant No. 2013, 2017).

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