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Series Solutions of Mathematical Problems of Quantum Mechanics

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Abstract: Quantum mechanics has played a major role in photonics, quantum electronics, and microelectronics. A series method is a powerful tool for solving quantum mechanical problems. In this study, we obtained the approximate solutions of operators using Harmonic Oscillator in a linear combination of the energy eigenstates. Also, the commutator of monomials of operators obeying constant commutation relations is expressed in terms of anti-commutators. We obtained the angular momentum operators in an eigenfunction with the use of matrices. Finally, we determined some exact solutions of eigenvalues and eigenvectors in a matrix representation of the operator to some set of orthonormal basis vectors.

INTRODUCTION

In theoretical physics and other physics-related elds, much research interest has been focused on finding and discussing the mathematical problems of Quantum mechanics. Quantum mechanics is a vital rational accomplishment of the 20th century. It is one of the more erudite field in physics that has affected our considerate of nano-meter length scale systems vital for materials, optics, chemistry and electronics[1]. The reality of orbitals and energy levels in atoms can only be described by quantum mechanics. Quantum mechanics can clarify the actions of insulators, conductors, semiconductors and giant magneto-resistance. It can enlighten the quantization of light and its particle nature in addition to its wave nature. Quantum mechanics can also explain the radiation of a hot body and its change of color with esteem to temperature. It explains the presence of holes and the transport of holes and electrons in electronic devices. Quantum mechanics has played an imperative part in photonics, quantum electronics and microelectronics[2]. Nevertheless, many more developing technologies need the understanding of quantum mechanics and hereafter, scientists and engineers must understand quantum mechanics well. One area is nano-technologies due to the new advent of nanofabrication methods. Subsequently, nano-meter size systems are an extra mutual place. In electronics as transistor devices become lesser, how the electrons move through the device is fairly different from when the devices are bigger: nano-electronic transport is quite different from micro-electronic transport[3]. The quantization of electromagnetic field is important in the area of nano-optics and quantum optics. It clarifies how photons relate to atomic systems or materials. It also allows the use of electromagnetic or optical field to carry quantum information. Furthermore, quantum mechanics is also needed to understand the interaction of photons with materials in solar cells as well as many topics in material science. When two objects are positioned close together, they experience a force called the Casimir force that can only be explained by quantum mechanics[4, 5]. This is important for the understanding of Micro/Nano-Electromechanical Sensor systems (M/NEMS). Additionally, the understanding of spins is important in spintronics, another developing technology where giant magneto-resistance, tunneling magneto-resistance and spin-transfer torque are being used. Quantum mechanics is also giving rise to the areas of quantum cryptography, quantum information, quantum computing and quantum communication[5]. It is seen that the productivity of quantum physics will significantly affect the imminent generation technologies in many aspects.

Harmonic oscillator solutions with operators: Operator methods are very valuable equally for solving the Harmonic Oscillator problem and for any type of computation for the Harmonic Oscillator (HO) potential. The operators, we develop will also be useful in quantizing the electromagnetic field. The Hamiltonian for the 1D Harmonic Oscillator will be:

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2y^2 \tag{1}$$

It could be written as the square of an operator. It can be rewritten in terms of the operator B:

$$B=\left(\sqrt{\frac{m\omega}{2\hbar}}y+i\frac{p}{\sqrt{2m\hbar\omega}}\right) \tag{2}$$

And its Hermitian conjugate:

$$B^{\dagger}\left(\sqrt{\frac{m\omega}{2\hbar}}y-i\frac{p}{\sqrt{2m\hbar\omega}}\right) \tag{3}$$

The commutator of Eq. 2 and 3 is:

$$\begin{gathered}\left[B,B^{\dagger}\right]=\frac{m\omega}{2\hbar}[y,p]+\frac{1}{2m\hbar\omega}[p,q]-\frac{i}{2\hbar}[p,y]=\\ \frac{i}{2\hbar}\langle-[y,p]\rangle+[p,\ y])=\frac{i}{\hbar}[p,y]=1\end{gathered} \tag{4}$$

Let use this modest commutator:

$$[B,\ B^{\dagger}]=1$$

to compute commutators with the Hamiltonian. This is easy if H is written in terms of B and $B^{\dagger}$:

$$[H,B]=\hbar\omega[B^{\dagger}B,B]=\hbar\omega[B^{\dagger},B]B=-\hbar\omega B$$

$$[H,B^{\dagger}]=\hbar\omega[B^{\dagger}B,B^{\dagger}]=\hbar\omega B^{\dagger}[B,\ B^{\dagger}]=-\hbar\omega B^{\dagger}$$

Both terms in the Harmonic Oscillator Hamiltonian are squares of operators in Eq. 2 and 3. Note that B is chosen, so that, $B+B^{\dagger}$ is close to the Hamiltonian. First, let just compute the quantity:

$$B^{\dagger}B=\frac{m\omega}{2\hbar}y^2+\frac{p^2}{2m\hbar\omega}+\frac{i}{2\hbar}(yp-py)$$

$$B^{\dagger}B=\frac{m\omega}{2\hbar}y^2+\frac{p^2}{2m\hbar\omega}+\frac{i}{2\hbar}(p-y)$$

$$\hbar\omega(B^{\dagger}B)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2-\frac{1}{2}\hbar\omega$$

From this, we can see that the Hamiltonian can be written in terms of $B+B^{\dagger}$ and some constants:

$$H=\hbar\omega\left(B^{\dagger}B+\frac{1}{2}\right) \tag{5}$$

For instance, let find the expected value of $1/2\ m\omega^2y^2$ in eigenstate. The expectation of y^2 will have some nonzero terms:

$$\begin{gathered}\left\langle u_n\middle|y^2\middle|u_n\right\rangle=\frac{\hbar}{2m\omega}\left\langle u_n\middle|BB+BB^{\dagger}+B^{\dagger}B+B^{\dagger}B^{\dagger}\,|\,u_n\right\rangle=\\ \frac{\hbar}{2m\omega}\left\langle u_n\middle|BB^{\dagger}+B^{\dagger}B\,|\,u_n\right\rangle\end{gathered} \tag{6}$$

We could drop the BB term and the $B^{\dagger}B^{\dagger}$ term, since, they will produce 0 when the dot product is taken:

$$\begin{gathered}\left\langle u_n\middle|y^2\middle|u_n\right\rangle=\frac{\hbar}{2m\omega}\left(\left\langle u_n\middle|\sqrt{n+1}Bu_{n+1}\right\rangle+\left\langle u_n\middle|\sqrt{n}B^{\dagger}u_{n-1}\right\rangle\right)=\\ \frac{\hbar}{2m\omega}\left(\left\langle u_n\middle|\sqrt{n+1}\sqrt{n+1}u_n\right\rangle+\left\langle u_n\sqrt{n}\sqrt{n}u_n\right\rangle\right)=\\ \frac{\hbar}{2m\omega}\left((n+1)+n\right)=\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}\end{gathered} \tag{7}$$

With this, we can calculate the expected value of the potential energy as:

$$\begin{gathered}\left\langle u_n\middle|\frac{1}{2}m\omega^2y^2\middle|u_n\right\rangle=\frac{1}{2}m\omega^2\left(n+\frac{1}{2}\right)\\ \frac{h}{m\omega}=\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega=\frac{1}{2}E_n\end{gathered} \tag{8}$$

For example, let consider a 1D harmonic oscillator in a linear combination of the energy eigenstates and find the expected value of p:

$$\psi = \sqrt{\frac{2}{3}}u_o - i\sqrt{\frac{1}{3}}u_1$$

$$\langle\psi|\psi\|\psi\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\left\langle\sqrt{\frac{2}{3}}u_0 - i\sqrt{\frac{1}{3}}u_1\left|B-B^\dagger\right|\sqrt{\frac{2}{3}}u_0 - i\sqrt{\frac{1}{3}}u_1\right\rangle =$$

$$-i\sqrt{\frac{m\hbar\omega}{2}}\left\langle\sqrt{\frac{2}{3}}u_0 - i\sqrt{\frac{1}{3}}u_1 - |-\sqrt{\frac{2}{3}}u_1 - i\sqrt{\frac{1}{3}}u_0\right\rangle = \quad (9)$$

$$-i\sqrt{\frac{m\hbar\omega}{2}}\left(-i\sqrt{\frac{2}{9}} - \sqrt{\frac{2}{9}}\right) =$$

$$-\sqrt{\frac{8}{9}}\sqrt{\frac{m\hbar\omega}{2}} = \frac{2}{3}\sqrt{m\hbar\omega}$$

Let consider an operator $\hat{B}$ defined by $\hat{B}|\pm\rangle = \pm 1/2i\hbar|\mp$, we can write it out in a matrix form as:

$$\hat{B} = \begin{pmatrix} \langle +|\hat{B}|\rangle & \langle +|\hat{B}|-\rangle \\ \langle -|\hat{B}|+\rangle & \langle -|\hat{B}|-\rangle \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix} \quad (10)$$

From Eq. 10, we have:

$$\hat{B}|1\rangle = \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix}\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\frac{1}{2}i\hbar\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \quad (11)$$

$$\frac{1}{2}\hbar\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Consequently, we have $\hat{B}|1\rangle = 1/2\,\hbar||1\rangle$ which incidentally express that $|1\rangle$ is an eigenstate of $\hat{B}$.

Quantization, monomials and ordering problems: Anti-commutators do not follow similar algebraic properties as commutators. For example, commutators satisfy the Jacobi identity:

$$\left[[\hat{X}, \hat{Y}], \hat{Z}\right] + \left[\hat{X}, [\hat{Y}, \hat{Z}]\right] + \left[\hat{Y}, [\hat{X}, \hat{Z}]\right] = 0 \quad (12)$$

Whereas for anti-commutators:

$$\left\{\{\hat{X}, \hat{Y}\}\hat{Z}\right\} + \left\{\hat{X}\{\hat{Y}, \hat{Z}\}\right\} + \left\{\hat{Y}, \{\hat{X}, \hat{Z}\}\right\} \neq 0 \quad (13)$$

Mendas, Milutinovic and Popovic discussed in detail the Baker-Hausdorff lemma (also known as the Lie series or group):

$$\exp[\hat{X}]\,\hat{Y}\exp[-\hat{X}] = \hat{Y} + [\hat{X}, \hat{Y}] + \left[\hat{X}, [\hat{X}, \hat{Y}]\right] + \frac{1}{2}\left[\hat{X}, [\hat{X}, \hat{Y}]\right] + \frac{1}{2}\left[\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]\right] +, ..., \quad (14)$$

which is needed for the proof of the Baker-Campbell-Hausdorff theorem and has also many applications[6]. The authors found a similar relation to anti-commutators[7-9]:

$$\exp[\hat{X}]\,\hat{Y}\exp[-\hat{X}] = \hat{Y}\left\{\{\hat{X}, \hat{Y}\} + \frac{1}{2}\left\{\hat{X}, \{[\hat{X}, \hat{Y}\}\right\} + \frac{1}{6}\left\{\hat{X}, \{\hat{X}, \{\hat{X}, \{\hat{X}, \hat{Y}\}\right\}\right\} \quad (15)$$

which is more appropriate for determining comparison transformations whenever operators $\hat{X}$ and $\hat{Y}$ are such that the frequent anti-commutators are easier to estimate than the equivalent frequent commutators. Multiplying by $\exp[-2\hat{X}]$ on the right leads to:

$$\exp[\hat{X}]\,\hat{Y}\exp[-\hat{X}] = \left(\hat{Y} + \{\hat{X}, \hat{Y}\} + \frac{1}{2}\left\{\{\hat{X}, \{\hat{X}, \hat{Y}\}\right\}\right) + \frac{1}{6}\left\{\hat{X}, \{\hat{X}, \{\hat{X}\,\hat{Y}\}\}\right\} +, ..., \times\exp[-2\hat{X}] \quad (16)$$

In the present study, we study the situation where the commutator is corresponding to a constant c multiplied by the identity operator $\hat{I}$:

$$[\hat{X}, \hat{Y}] = c\hat{I} \quad (17)$$

Which means that $[\hat{X}, \hat{Y}]$ commutes with $\hat{X}$ and $\hat{Y}$. This is the situation, for instance, the canonical relation $[\hat{x}, \hat{p}_x] = i\hbar\,\hat{I}$ where $\hat{x}$ and $\hat{p}_x$ are the first coordinates of position and impulsion, respectively or for the bosonic annihilation $(\hat{b})$ and creation $(\hat{b}^\dagger)$ operators which are such that $(\hat{b}, \hat{b}^\dagger) = 1$. Though, relations for commutators following different commutation relations can also be obtained[6] for the case $(\hat{X}, \hat{Y}) = \lambda\hat{Y}$ where λ is a function of $\hat{X}$.

In the quantization of classical systems, one encounters an infinite number of quantum operators corresponding to a particular classical expression. This is due to the numerous possible orderings of the non-commuting quantum operators. Any analytical function of two non-commuting operators $\hat{X}$ and $\hat{Y}$ is defined by its power series expansion in terms of these operators:

$$f\,(\hat{X}, \hat{Y}) = \sum_k\sum_l\sum_m\,, ..., \sum_n f_{k,l,m,...,n}\hat{X}^k\hat{Y}^l\hat{X}^m, ..., \hat{Y}^n \quad (18)$$

A relation among commutators and anti-commutators can help to reorder products of operators, drawing connections ("intertwinnings") among normal, anti-normal and Weyl orderings (see also the representation of Glauber-Suddarshan and Husimi in quantum optics). Sinha *et al*.[7] proposed a specific ordering of the quantum mechanical momentum and position operators $\hat{x}$ and $\hat{p}_x$, to define a Hermitian Hamiltonian. They proposed to replace the multinomial quantity $p^m{}_x x^n$ in classical mechanics by:

$$p_x^m x^m \rightarrow \frac{1}{m+1}\sum_{k=0}^{m} \hat{P}_x^{\,m-k}\, \hat{x}^n\, \hat{p}_x^k \tag{19}$$

Two years later, Selsto and Forre[8] specified another order, symmetric under the interchange of $\hat{x}$ and $\hat{p}_x$ a defined the multinomial operator $\hat{T}_{m\,n}$ as the Weyl-ordered form of the classical function $\hat{Y}^m\hat{X}^n$:

$$\hat{T}_{m,n} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{X}^k\hat{Y}^m\hat{X}^{n-k} \tag{20}$$

The polynomials $\hat{T}_{m,n}$ can be expressed as the symmetrized form containing m factors of $\hat{X}$ and n factors of $\hat{Y}$ normalized by the number of terms in the expression. One has, for instance:

$$\begin{aligned}
&\hat{T}_{1,1} = \frac{1}{2}\left(\hat{X}\hat{Y}+\hat{Y}\hat{X}\right) = \frac{1}{2}\left\{\hat{X},\hat{Y}\right\}\\
&\tilde{T}_{1,2} = \frac{1}{3}\left(\hat{X}^2\hat{Y}+\hat{X}\hat{Y}\hat{X}+\hat{Y}\hat{X}^2\right)\\
&\tilde{T}_{2,2} = \frac{1}{6}\left(\hat{X}^2\hat{Y}^2+\hat{Y}^2\hat{X}^2+\hat{X}\hat{Y}\hat{X}\hat{Y}+\hat{Y}\hat{X}\hat{Y}\hat{X}+\hat{X}\hat{Y}^2\hat{X}+\hat{Y}\hat{X}^2\hat{Y}\right)\\
&\tilde{T}_{0,4} = \tilde{X}^4
\end{aligned} \tag{21}$$

The following relation[9]:

$$\begin{aligned}\tilde{T}_{m,n}\tilde{T}_{T,S} = \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)}{j!}\sum_{k=0}^{j}(-1)^{j-k}\ k!^2\,(j-k)^{!2}\binom{j}{k}\times\\
\binom{m}{j-k}\binom{n}{k}\binom{r}{k}\binom{s}{j-k}\times\tilde{T}_{m+r-j,n+s-j}\end{aligned} \tag{22}$$

allows one to find the terms of the commutator:

$$\begin{aligned}\left[\tilde{T}_{m,n},\tilde{T}_{r,s}\right] = 2\sum_{j=0}^{\infty}\frac{(1/2)^{2j+1}}{(2j+1)!}\sum_{k=0}^{2j+1}(-1)^k\,K!^2\,(2j+1-k)\\
!^2\binom{2j+1}{k}\times\binom{m}{k}\binom{n}{2j+1-k}\binom{s}{k}\times\tilde{T}_{m+r-2j,n+s-2j}\end{aligned} \tag{23}$$

then the anti-commutator will be:

$$\begin{aligned}\left[\tilde{T}_{m,n},\tilde{T}_{r,s}\right] = 2\sum_{j=0}^{\infty}\frac{(i/2)^{2j}}{(2j)!}\sum_{k=0}^{2j}(-1)^k\,K!^2\,(2j-k)\\
!^2\binom{2j}{k}\times\binom{m}{k}\binom{n}{2j-k}\binom{r}{2j-k}\binom{s}{k}\times\tilde{T}_{m+r-2j,n+s-2j}\end{aligned} \tag{24}$$

Also, precise Bender-Dunne polynomials can be articulated as anti-commutators. For instance, one has:

$$\tilde{T}_{m,m+k} = \frac{(2m+k)!m!}{2(2m)!(m+k)!}\left\{\tilde{T}_{m,m}\,\tilde{X}^k\right\} \tag{25}$$

And:

$$\tilde{T}_{m+k,m} = \frac{(2m+k)!m!}{2(2m)!(m+k)!}\left\{\tilde{T}_{m,m}\,\tilde{Y}^k\right\} \tag{26}$$

Cahill and Glauber introduced the concept of "s-ordered displacement operator" by:

$$D(\tilde{\alpha},s) = eb\hat{b}^{\dagger\text{-}b^*\tilde{\alpha}e^{s|b|^2/2}} \tag{27}$$

Where:
α = A complex number
b^* = It's conjugate

By the Baker-Campbell-Hausdorff formula for the three discrete values of s = 1, 0 and -1, the operator D(b, s) can be written as→normal order:

$$D(b,1) = e^{b\hat{b}^\dagger}\ e^{-b^*\hat{b}} \tag{28}$$

Weyl order:

$$D(b,0) = e^{b\hat{b}^\dagger\text{-}b^*\hat{b}} \tag{29}$$

and anti-normal order:

$$D(b,-1) = e^{-b^*\hat{b}^\dagger}e^{b\hat{b}^\dagger} \tag{30}$$

The authors de ned the s-ordered product $\vdots(\hat{b}^\dagger)^n\hat{b}^m\vdots(s)$ as:

$$\vdots(\hat{b}^\dagger)^n\hat{\alpha}^m\vdots(s) \equiv \frac{\partial^{n+m}D(b,s)}{\partial b^n\partial(-b^*)^m}\,|\,\alpha = 0 \tag{31}$$

We can write:

$$\vdots(\hat{b}^\dagger)^n\hat{b}_m\vdots_{(s)} = \sum_{k=0}^{\min(m,n)}k!\binom{n}{k}\binom{m}{k}\left(\frac{t-s}{2}\right)^k\times\vdots(\hat{b}^\dagger)^{n-k}\hat{b}^{m-k}\vdots_{(t)} \tag{32}$$

In the second-quantization concept of atomic spectroscopy, articulating $\left[\hat{b}^n,(\hat{b}^\dagger)^m\right]$ in terms of $\left\{\hat{b}^i,(\hat{b}^\dagger)^j\right\}$ can be supportive. Certainly, to know the requirement of the operators with esteem to the number of particles, a matrix element is written as a product of annihilation and creation operators and the creation operators must be stimulated to the left (the annihilation operators being moved to the right) with the help of anti-commutation relations.

A study of l = 1 operators and eigenfunctions: The set of states with the same total angular momentum and the angular momentum operators which act on them are often signified by vectors and matrices. For instance, the different m states for l = 1 will be represented by component vectors like

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_0 \\ \psi_- \end{pmatrix}$$

and the angular momentum operators will be $\langle L'_m|L_z|L_m\rangle = m\hbar\delta_{m'm}$:

$$L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\langle l'_m \,|\, L_\pm \,|\, L_m\rangle = \sqrt{l(l+1)\text{-}m(m\pm 1)\hbar\delta'_{mm\pm 1}}$$

$$L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

$$L_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

The matrices representing the angular momentum operators for l = 1 are as follows:

$$L_x = \frac{1}{2}(L_+ + L_-)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{34}$$

$$L_y = \frac{1}{2i}(L_+ + L_-)\frac{\hbar}{2i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{35}$$

The same matrices also represent spin 1, s = 1 but of course, would act on a different vector space. For instance, let compute $[L_z\text{-}L_y]$ using matrices:

$$[L_z L_y = \frac{\hbar^2}{2i}\left(\begin{matrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\end{matrix}\right) \tag{36}$$

$$\begin{gathered}\frac{\hbar^2}{2i}\left(\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}\right) = \\ \frac{\hbar^2}{2i}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} = i\hbar\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar L_z\end{gathered} \tag{37}$$

$$\begin{gathered}\frac{\hbar^2}{2i}\left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right) = \\ \frac{\hbar^2}{2i}\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} = i\hbar\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i\hbar L_y\end{gathered} \tag{38}$$

The rotation operators transform an angular momentum state vector into an angular momentum state vector in the rotating system as:

$$R_z(\theta_z) = e^{i\theta_z L_z/\hbar}$$
$$\psi' = R_z(\theta_z)\psi$$

Since, there is nothing special about the z-axis, rotations about the other axes follow the same form:

$$R_x(\theta_x) = e^{i\theta_x L_x/\hbar}$$
$$R_x(\theta_x) = e^{i\theta_y L_y/\hbar}$$

The above formulas for the rotation operators must apply in both the matrix representation and in the differential operator representation. Rede ning the coordinate axes cannot change any scalars, like dot products of state vectors. Operators that preserve dot products are called unitary. We proved that operators of the above form (with Hermitian matrices in the exponent) are unitary[10]. For instance, let a computation of the operator for rotations about the z-axis gives:

$$e^{i\theta L_z/\hbar} = \sum_{n=0}^{\infty}\frac{\left(\dfrac{i\theta l_z}{\hbar}\right)^n}{n!}$$

$$\left(\frac{L_z}{\hbar}\right)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\frac{L_z}{\hbar}\right)^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\left(\frac{L_z}{\hbar}\right)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\frac{L_z}{\hbar}\right)^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = L_z/\hbar$$

All the odd powers are the same. All the nonzero even powers are the same. The ħs all cancel out. We now must look at the sums for each term in the matrix and identify the function it represents. If we look at the sum for the upper left term of the matrix, we get a 1 times $(i\theta)^n/n!$. This is just $e^{i\theta}$. There is only one contribution to the middle term, that is a one from n = 0. The lower right term is like the upper left except the odd terms have a minus sign. We get $(i\theta)^n/n!$ Term n. This is just $e^{-i\theta}$ The rest of the terms are zero:

$$R_z(\theta_z) = \begin{pmatrix} e^{i\theta_z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta_z} \end{pmatrix} \tag{39}$$

For instance, let a computation of the operator for rotations about the y-axis gives:

$$e^{i\theta L_y/\hbar} = \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta l_y}{\hbar}\right)^n}{n!}$$

$$\left(\frac{L_y}{\hbar}\right)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\frac{L_y}{\hbar}\right)^1 = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left(\frac{L_y}{\hbar}\right)^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\left(\frac{L_y}{\hbar}\right)^3 = \frac{1}{\sqrt{2}i}\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} = \left(\frac{L_y}{\hbar}\right)$$

All the odd powers are the same. All the nonzero even powers are the same. The ħs all cancel out. We now must look at the sums for each term in the matrix and identify the function it represents:

- The n = 0 term contributes 1 on the diagonals
- The n = 1, 3, 5, ..., terms sum to $\sin(\theta)\ (iL_y/\hbar)$
- They n = 2, 4, 6, ..., terms (with a -1 in the matrix) are nearly the series for $1/2\cos(\theta)$. The n = 0 term is missing so subtract 1. The middle matrix element is twice the other even terms

$$e^{i\theta L_y/\hbar} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin(\theta)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \frac{1}{2}(\cos(\theta)-1)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{40}$$

Putting this all together, we have:

$$R_y(\theta_y)\begin{pmatrix} \frac{1}{2}\left(1+\cos(\theta_y)\right) & \frac{1}{\sqrt{2}}\sin(\theta_y) & \frac{1}{2}\left(1-\cos(\theta_y)\right) \\ -\frac{1}{\sqrt{2}}\sin(\theta_y) & \cos(\theta_y) & \frac{1}{\sqrt{2}}\sin(\theta_y) \\ \frac{1}{2}\left(1-\cos(\theta_y)\right) & -\frac{1}{\sqrt{2}}\sin(\theta_y) & \frac{1}{2}\left(1+\cos(\theta_y)\right) \end{pmatrix} \tag{41}$$

For instance, let a computation of the operator for rotations about the x-axis gives:

$$e^{i\theta L_x/\hbar} = \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta l_x}{\hbar}\right)^n}{n!}$$

$$\left(\frac{L_x}{\hbar}\right)^0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\frac{L_x}{\hbar}\right)^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\frac{L_x}{\hbar}\right)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\frac{L_x}{\hbar}\right)^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = L_x/\hbar$$

All the odd powers are the same. All the nonzero even powers are the same. The ħs all cancel out. We now must look at the sums for each term in the matrix and identify the function it signifies. If we look at the sum for the upper-middle term of the matrix, we get 1 times $(i\theta)^n/n!$. This is just $e^{i\theta}$. There is only one contribution to the middle term, that is one from n = 0. The lower right term is like the upper-middle except for the odd terms. We get $(i\theta)^n/n!$. Term n. This is just $e^{i\theta}$. The rest of the terms are zero:

$$R_z(\theta_z) = \begin{pmatrix} 0 & e^{i\theta_x} & 0 \\ e^{i\theta_x} & 0 & e^{i\theta_x} \\ 0 & e^{i\theta_x} & 0 \end{pmatrix} \tag{42}$$

let's calculate an expected value of L_x in the matrix representation for the general state ψ:

$$\langle\psi|L_x|\psi\rangle = \left(\psi_1^* \ \psi_2^* \ \psi_3^*\right)\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} =$$

$$\frac{\hbar}{\sqrt{2}}\left(\psi_1^* \ \psi_2^* \ \psi_3^*\right)\begin{pmatrix}\psi_2\\ \psi_1 + \psi_3\\ \psi_2\end{pmatrix} = \quad (43)$$

$$\frac{\hbar}{\sqrt{2}}\left(\psi_1^*\psi_2+\psi_2^*\left(\psi_1+\psi_3\right)+\psi_3^*\psi_2\right)$$

For instance, let's assume that, we select $\varphi_1 = \sin(kx)$ and $\varphi_2 = \cos(kx)$ as the degenerate eigenfunctions of H with the same eigenvalue $E_k = h^2k^2/2$ m. We now want to find with this technique the common eigenfunctions of $\hat{p}$. We first need to find the matrix ć (here a 2×2 matrix) by applying $\hat{p}$ to the eigenfunctions:

$$\hat{p}\varphi_1 = i\hbar\frac{d\varphi_1}{dx} = i\hbar k\cos(kx) = -i\hbar k\varphi_2 \quad (44)$$

And $\hat{p}\varphi_2 = i\hbar\varphi_1$. Then the matrix ć is:

$$\overline{c} = \begin{pmatrix}0 & i\hbar k\\ -i\hbar k & 0\end{pmatrix} \quad (45)$$

With eigenvalues $b^j = \pm\hbar k$ and eigenvectors (not normalized), we have:

$$v^1 = \begin{bmatrix}-i\\ 1\end{bmatrix},\ v^2 = \begin{bmatrix}i\\ 1\end{bmatrix}$$

We then write the ψ eigenfunctions as:

$$\psi^1 = v_1^1\varphi_1+v_2^1\varphi_1 = -i\sin\left(kx\right)+\cos\left(kx\right)\alpha\ e^{ikx},\ \psi^2 = v_1^2\varphi_1+v_2^2\varphi_1 = i\sin\left(kx\right)+\cos\left(kx\right)\alpha\ e^{ikx} \quad (46)$$

Eigenvectors and eigenvalues: Operators act on states to map them into other states. Between the possible results of the action of an operator on a state is to map the state into a multiple of itself:

$$\hat{B}|\phi\rangle = b_\phi|\phi\rangle \quad (47)$$

where $|\varphi\rangle$ is, in general, a complex number. The state $|\varphi\rangle$ is then said to be an eigenstate or eigenket of the operator $\hat{B}$ with b_φ the related eigenvalue. The fact that operators can possess eigenstates might be believed of as a mathematical fact related to the physical content of quantum mechanics but it turns out that the opposite is the case: the eigenstates and eigenvalues of numerous types of operators are essential parts of the physical interpretation of the quantum theory and hence warrant close study. Notationally, it is often useful to use the eigenvalue related to an eigenstate to label the eigenvector, i.e., the notation:

$$\hat{B}|b\rangle = b|b\rangle \quad (48)$$

This notation, or minor variations of it, will be used almost exclusively here. Determining the eigenvalues and eigenvectors of a given operator $\hat{B}$ infrequently referred to as solving the eigenvalue problem for the operator, amounts to finding solutions to the eigenvalue equation $\hat{B}|\phi\rangle = b_\phi|\phi\rangle$. Written out in terms of the matrix representations of the operator to some set of orthonormal basis vectors $\{|\varphi\rangle;\ n = 1, 2, ...\}$, this eigenvalue equation is:

$$\begin{pmatrix}B_{11} & B_{12} & \cdots\\ B_{21} & B_{22} & \cdots\\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\end{pmatrix} = b\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\end{pmatrix} \quad (49)$$

This expression is equivalent to a set of simultaneous, homogeneous, linear equations:

$$\begin{pmatrix}B_{11}\text{-}b & B_{12} & \cdots\\ B_{21} & B_{22}\text{-}b & \cdots\\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\end{pmatrix} = 0 \quad (50)$$

Which have to be solved for the possible values for a and the associated values for the components Φ_1, Φ_2, ..., of the eigenvectors. The method is standard. The determinant of coefficients must disappear to get non-trivial solutions for the components Φ_1, Φ_2, ...

$$\begin{vmatrix}B_{11}\text{-}a & B_{12} & \cdots\\ B_{21} & B_{22} & \cdots\\ \vdots & \vdots & \end{vmatrix} = 0 \quad (51)$$

which yields an equation identified as the secular equation, or characteristic equation, that has to be solved to give the possible values of the eigenvalues b. Once these are known, they have to be re-substituted into Eq. 49 and the components Φ_1, Φ_2, ..., of the related eigenvectors determined. The details of how this is done properly belong to a text on linear algebra and will not be considered any further here, but to say that the eigenvectors are typically determined up to an unknown multiplicative constant. This constant is usually fixed by the requirement that these eigenvectors be normalized to unity. In the situation of frequent eigenvalues, i.e., when the characteristic polynomial has multiple roots (otherwise known as degenerate eigenvalues), the determination of the eigenvectors is made more complex still.

In general, for a state space of finite dimension, it is found that the operator $\hat{A}$ will have one or more discrete eigenvalues b_1, b_2, ... and related eigenvectors $|b_1\rangle$, $|b_2\rangle$, ... The collection of all the eigenvalues of an operator is called the eigenvalue spectrum of the operator. Note also that more than one eigenvector can have the same eigenvalue. Such an eigenvalue is said to be degenerate.

CONCLUSION

In this study, we obtained a relation for the commutator of two monomial operators $\hat{X}^n$ and $\hat{Y}^m$ in the case where the commutator $[\hat{X}, \hat{Y}]$ is a constant. The formula is an expression in terms of anti-commutators (with lower powers of $(\hat{X}$ and $\hat{Y})$. The generalization to the commutator of products of an arbitrary number of monomials is in progress as well as to nests of commutators fitted into each other. We obtained the estimated solutions of operators using Harmonic Oscillator in a linear combination of the energy eigenstates. We determined in detail the angular momentum operators in an eigenfunction with the use of matrices. Finally, we obtained some exact solutions of eigenvalues and eigenvectors in a matrix representation of the operator to some set of orthonormal basis vectors. Our results could find usefulness in both molecular and atomic physics.

Data availability statement: The data that support the findings of this study are available from the corresponding author[5], upon reasonable request.

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