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Space Time Physics with Geometry and the use of Four-Vectors: A Review

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Abstract: Spacetime geometric algebra is a unified mathematical language for physics. The geometric representation of spacetime and the use of four-vectors are vital to the successful findings of three-dimensional, four-dimensional non-Euclidean geometry in Lorentz and Galilean transformations. Thus, the usual opinion that there is a sole set of events presents now in a three-dimensional or four-dimensional spacetime cannot be continual. The geometric representations discussed in this study include the following: Minkowski's path to spacetime Galilean transformation as a geometrical representation of motion Lorentz transformation as a geometrical representation of inertial frames and worldline of a particle. The expansion of the special relativity theory using four-vectors space-time four-vectors.

INTRODUCTION

In Physics, spacetime is any mathematical model which fuses the three dimensions of space and the one dimension of time into a single four dimensional manifold. Spacetime diagrams can be used to visualize relativistic effects such as why different observers perceive differently where and when events occur. The idea of spacetime with an opinion concerning accepting its instant foundations and modification was suggested and articulated by Harmann Minkowski in 1908. To narrate the stages of Minkowski's discovery, we begin with an account of Poincaré's principle of gravitation, where Minkowski initiates some of the origins of space-time. The Poincaré's geometric explanation of the Lorentz transformation is scrutinized, along with Minkowski's details for not following a four-dimensional or three-dimensional vector calculus. The description of the effects of special relativity is carried out within the four-dimensional Minkowski space R(3, 1). We differentiate the Minkowski idea from the recent ideas as discovered from our findings that the Euclidean geometry of a four-dimensional non-Euclidean manifold is incredible in the structure of spacetime from a mathematical viewpoint. The impression of a four-velocity vector, i.e., w_1-w_4 can be expressed as a hyperbolic velocity space that is always a point on the surface of a four-dimensional space in Lorentz and Galilean Transformations. An expression of two events can be used to explain the phenomenon behind the four-vector of three or four-dimensional displacement in spacetime[1-4]. However, strangely enough an important pedagogical work still remains to be done if one retains from that adventure one of its most striking aspects, namely the existence of a united geometrical representation of space and time, called space time and the logical necessity of its introduction on the basis of the special properties of the velocity of light. In fact, we think it worthwhile and possible to communicate this geometrical representation not only to learned scientists, but also to any scientifically-curious and/or philosophically-minded student. One can then say that as a geometry of description, Euclidean geometry appears as the oldest manifestation of the spirit of mathematical physics. As it may be already familiar to pupils at the terminal level of high-school, this implies a relationship between algebra and geometry whose interest is two-fold. On the one-hand, the properties of geometrical curves can be equivalently represented by algebraic equations relating the coordinates of their points. This representation is unique, once the choice of a system of coordinates has been specified. For example, in orthogonal coordinates, the equation of the unit circle $x^2+y^2-1 = 0$ makes use of the standard Pythagoras theorem for characterizing the points M = (x, y) of the curve[5-7]. The geometrical constructions which may be associated with the pictorial representation of a physical phenomenon in a plane or in a three-dimensional space equipped with coordinates pertain to what we shall call a geometry of representation.

Minkowski's path to spacetime: To begin with Minkowski discussed neither spacetime, manifolds or non-Euclidean geometry but vectors. Borrowing Poincaré's de nitions of radius and force density ϱ and adding (like Marcolongo before him) expressions for four-current density e and four-potential, ψ Minkowski expressed Maxwell's vacuum equations in the compact form:

$$\psi_j = -\varrho_j (j = 1,2,3,4,5) \quad (1)$$

where ϱ_j is the Dalembertian operator. According to Minkowski, no one had comprehended before that the equations of electrodynamics could be written so concisely, "not even Poincaré''. Seemingly, Minkowski had not observed Marcolongo's paper, mentioned above. The next mathematical object that Minkowski introduced was a real step forward and soon recognized as such by physicists. This is what Minkowski called a "Traktor" a six-component object later called a "six-vector", and more newly, an antisymmetric rank-2 tensor. Minkowski de ned the Traktor's six mechanisms through his four-vector potential using a two-index notation: $\psi_{ik} = \partial\psi_k/\partial x_j, -\partial\psi_j/\partial x_k$ nothing the antisymmetry relation $\psi_{kj} = -\psi_{ik}$ and zeros along the diagonal $\psi_{jj}= 0$ such that the components ψ_{14}, ψ_{24}, ψ_{34}, ψ_{23}, ψ_{31}, ψ_{12} match the field quantities $-iE_x$, iE_y, iE_z, iE_x B_x, B_y, B_z. To express the source equations, Minkowski introduced a "polarisationstrakor", p:

$$\frac{\partial p_{1j}}{\partial x_1}+\frac{\partial p_{2j}}{\partial x_2}+\frac{\partial p_{3j}}{\partial x_3}+\frac{\partial p_{4j}}{\partial x_4} = \sigma_j = \varrho_j \quad (2)$$

Where σ is the four-current density for the matter. Minkowski is yet to reveal the sense in which the world is a "four-dimensional non-Euclidean manifold" [8]. His argument continued as follows. The tip of a four-dimensional velocity vector w_1, w_2, w_3, w_4 is constantly a point on the surface:

$$w_1^2+w_2^2+w_3^2+w_4^2 = -1 \quad (3)$$

Or if we prefer:

$$t^2-x^2-y^2-z^2 = 1 \quad (4)$$

and represents both the four-dimensional vector from the origin to this point and null velocity or rest, being a genuine vector of this sort. Non-Euclidean geometry, of which we spoke earlier in an inexact fashion, now unfolds for these velocity vectors.

These two surfaces, a pseudo-hypersphere of the unit imaginary radius in Eq. 3 and its real counterpart, the two-sheeted unit hyperboloid in Eq. 4, give rise to well-known models of hyperbolic space, propagated by Helmholtz in the late nineteenth century. The upper sheet (t>0) of the unit hyperboloid in Eq. 4 models hyperbolic geometry. The conjugate diameters of the hyperboloid in Eq. 4 give rise to a geometric image of the Lorentz transformation. Any point in Eq. 4 can be considered to be at rest, i.e., it may be taken to lie on a t-diameter as shown in Fig. 1. This change of axes links to an orthogonal transformation of the time and space coordinates which is a Lorentz transformation (letting c =1). Although, Minkowski did not spell out his geometric explanation, he possibly documented that a displacement on the hypersurface in Eq. 4 corresponds to a rotation ψ about the origin such that frame velocity ν is designated by a hyperbolic function, ν = tanhψ. Though, we did not yet realize that hypersurfaces represent the set of events occurring at the coordinate time t' = 1 of all inertial observers, the worldlines of whom pass via. the basis of coordinates (with a common origin of time). According to Eq. 4, this time is imaginary, a fact which may have obscured the latter interpretation. When we de ned four-velocity, we took over the mechanisms of the normal velocity vector w for the spatial part of four-velocity and added an imaginary fourth component, $i\sqrt{1-i\nu^2}$. This resulted in four components of four-velocity, w_1, w_2, w_3, w_4:

$$w_x, w_y, w_z, i\sqrt{1-\nu^2} \quad (5)$$

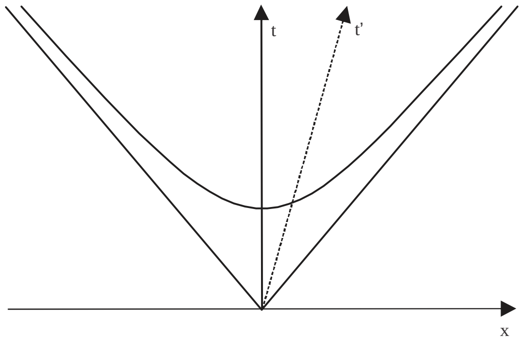

Fig. 1: A reform of Minkowski's Nov. 5th, 1907 demonstration of relativistic velocity space, with a duo of sequential axes, one 3-D axis, a unit hyperbola and its asymptotes[8]

Since, the components of Minkowski's quadruplet do not transform like the coordinates of his vector space x_1, x_2, x_3, x_4 they lack what he knew to be a four-vector property. Minkowski's error in defining four-velocity indicates that we did not yet hold the concept of four-velocity as a four-vector tangent to the worldline of a particle. If we grant ourselves the latter notion, then we can let the square of the differential parameter dτ of a given worldline $d\tau^2 = -\left(dx_1^2+dx_2^2+dx_3^3+dx_4^2\right)$ be such that the 4-velocity w_μ may be defined as the first derivative with respect to $\tau, w_\mu = dx_\mu/d\tau(\mu = 1,2,3,4,....,m)$. Sometime after Minkowski spoke to the Gttingen Mathematical Society, he repaired his de nition of four-velocity and possibly in assembly with this, he came up with the constitutive elements of his concept of spacetime. In precise, he expressed the idea of proper time as the parameter of a hyperlink in spacetime, the light-hypercone structure of spacetime and the spacetime equations of motion of a substantial particle[8]. On 5 April 1908, he expressed his new theory in a sixty-page memoir published in the Gttinger Nachrichten. His memoir, entitled "The basic equations for electromagnetic processes in moving bodies" made for challenging reading. It was packed with new notation, terminology and calculation rules, it made scant reference to the scientific literature and offered no gures or diagrams[9]. Along the same lines, Minkowski rephrased velocity, denoted q in relations to the tangent of an imaginary angle iψ:

$$q = -i\tan i\psi \tag{6}$$

where q<1. From his earlier geometric explanation of hyperbolic velocity space, Minkowski kept the idea that every rotation of a t-diameter agrees to a Lorentz transformation which he now stated in terms of iψ:

$$\begin{aligned} x_1' &= x_1, x_3' = x_3 = x_3\cos i\psi + x_4 \sin i\psi \\ x_2' &= x_2, x_4' = -x_3 \sin i\psi + x_4 \cos i\psi \end{aligned} \tag{4}$$

Minkowski was certainly conscious of the linking between the arrangement of Lorentz transformations and velocity composition, but he did not reference it. Minkowski neither stated Einstein's law of velocity addition nor stated it mathematically. While Minkowski made no demand in "The basic equations" to the hyperbolic geometry of velocity vectors, he reserved the hypersurface in Eq. 4 on which it was created and provided a new explanation of its physical signi cance. This clarification represents an imperative sign to consider how Minkowski revealed the worldline structure of spacetime. The supplement to "The basic equations" rehearses the argument according to which one may choose any point in Eq. 4 such that the line from this point to the origin forms a new time axis and agrees to a Lorentz transformation. He additional defined a "spacetime line" to be the entirety of spacetime points equivalent to any particular point of matter for all-time t[10]. With respect to the new idea of the spacetime line, we noted that its direction is determined at every spacetime point. Here we introduced the concept of "proper time" (Eigenzeit), τ, expressing the increase of coordinate time dl for a point of matter with respect to dτ:

$$d\tau = \sqrt{dt^2-dx^2-dy^2-dz^2} = dt\sqrt{1-lv^2} = \frac{dx_4}{w_4} \tag{8}$$

Where iv^2 is the square of ordinary velocity $dx_4 = idt$ and $w_4 = i/\sqrt{1-iv^2}$ which silently corrects the defective definition of this fourth component of four-velocity in Eq. 5 delivered by Minkowski in his November, 5 lecture. The de nition of four-velocity was properly related by Minkowski to the hyperbolic space of velocity vectors in "The basic equations" and thereby to the light cone structure of spacetime. Sometime before Minkowski came to study the Lorentz transformation in intense, both Einstein and Poincaré understood light waves in space to be the only physical objects resistant to Lorentz contraction. Minkowski observed that when light rays are measured as worldlines, they divide spacetime into three sections, equivalent to the spacetime section inside a future-directed (t>0) hypercone ("Nachkegel"), the region inside a past-directed (t<0) hypercone ("Vorkegel") and the region outside any such hypercone pair. The propagation in space and time of a spherical light wave is defined by a hypercone, or what Minkowski called a light cone ("Lichtkegel"). One instant significance for Minkowski of the light cone structure of spacetime troubled the relativity of simultaneity. Minkowski presented only the Newtonian version of the law of gravitation in "The basic equations", involving the

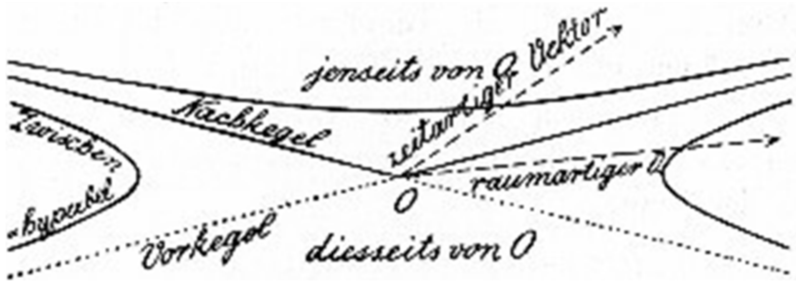

Fig. 2: The light cone structure of spacetime[11]

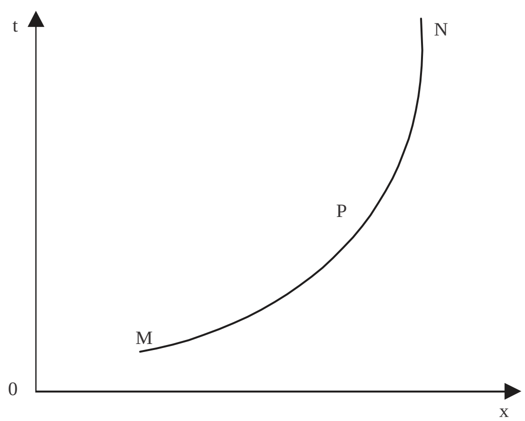

Fig. 3: MN represents the world line of particle, P

states of two massive particles in arbitrary motion and finding an appearance for the spacelike component of the four-force of gravitation. Although, his derivation involved a new spacetime geometry, Minkowski did not establish graphically in his new law, a conclusion that led some physicists to describe his theory as incoherent[12]. According to Minkowski, nevertheless his accomplishment was a formal one in as much as Poincaré had expressed his concept of gravitation by proceeding in what he defined as a “completely difierent way”(Fig. 2).

Galilean transformation as a geometrical representation of motion: The motion of a particle can often be signified graphically. Since, the particle is in motion, the position coordinates in three dimensions (x, y, z) will vary with time. For example, we considered motion along the x axis only. We choose a rectangular coordinate system to represent the displacement of the particle p relative to the S inertial frame, when this displacement is plotted against time t we obtain, for example, the curve MN (Fig. 3). Such a curve is called a word line of particle p and represents the loci of space-time points corresponding to the motion. If the particle P moves in a straight-line uniform velocity, the world line is a straight line for accelerated motion, the world line is curved.

Assume the motion of the particle is observed from another reference frame S’ moving with velocity ν relative to the frame S. Let the origin of the two reference frames concur at t = t’ = 0. According to the Galilean transformation:

$$x' = x - \nu t \tag{9}$$

$$t' = t \tag{10}$$

For the y axis to have the same dimensions as the x axis, multiply t by c where c is the velocity of light, to get ct = w. Therefore, Eq. 9 and 10 can be written as:

$$x' = x - \nu w/c \tag{11}$$

$$ct' = w' = w \tag{12}$$

The x-axis of the S frame is given by w = 0 while the w axis is given by x = 0. Similarly, for the S frame the x’ axis is given by w’ = 0 and using Eq. 12 a, we have that w’= w = 0. This means that the x’ axis concurs with the x-axis. Also, the w’ axis of the S’ frame is given by x’ = 0 Therefore, from Eq. 11 a, the equation for that axis is:

$$0 = x - \nu w/c$$

Or

$$x = \nu w/c \tag{13}$$

This is the equation of a straight line QQ shown in Fig. 4, from (Eq. 13) tan φ = ν/c. Hence, the Galilean transformation involves the transformation from rectangular axes to axes in which the time is slanted. The coordinates of any point on the world line can be determined by the two sets of axes. There is no difficulty with the scales of the x and x’ axes, since, they coincide. However, the time scales are different. Suppose the line ABCD is drawn parallel to the x axis, where OB corresponds to time on the w axis. Then OC corresponds to time w = 1 on the w’ axis. If the time axes are regulated using those distances, it can easily be checked that time

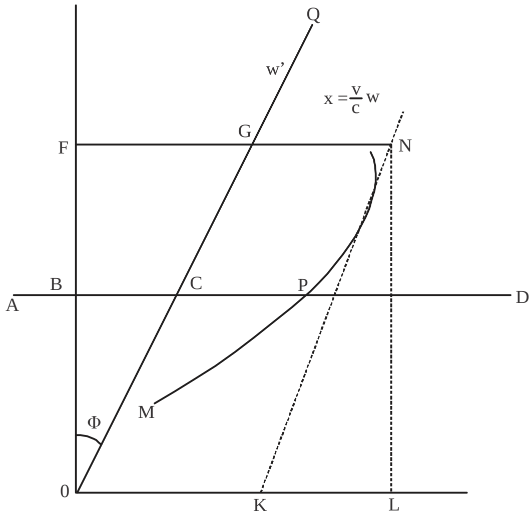

Fig. 4: Geometrical determination of the coordinates of point N from two inertial frames using the Galilean transformation

Coordinates of the world line at point N as represented by OF in the S frame and OG in the frame are equal following the Galilean transformation.

Let consider the space coordinates of the point N in the S frame given by the distance OL where L is the point where the line from N parallel to the axis meets the x axis. For the S frame, we must also draw from N a line parallel to the w' axis. It cuts the x' axis at K, therefore, the space coordinate is given by the distance OK. With this procedure, it is possible to obtain the coordinates of a particle in motion from different inertial frames in uniform relative motion. Once the relative velocity v is known, it will be possible to draw the w' axis using the relation, tanφ. = ν/c. Note that because the classical Galilean transformation is valid at ordinary speeds ν<<c and the angle φ is necessarily small:

Lorentz transformation as a geometrical representation of inertial frames and worldline of a particle: In the Galilean transformation, the time coordinate of one inertial frame does not depend on the space coordinate of one inertial frame. But in special relativity time and space are codependent as seen in Lorentz's transformations where the time coordinate of one inertial frame be contingent on both space and time coordinates of another inertial system. Therefore, instead of creating space and time distinctly as is done in classical physics, it is natural in relativity to treat them together [15]. Minkowski was the first person to show how this could be done; it involved the introduction of 4-dimensional x, y, x, ct space. By taking only the x coordinate and the time Lorentz transformations, we have:

$$x' = \frac{x-\nu t}{\left(1-\nu^2/c^2\right)^{1/2}}$$
$$x = \frac{x'+\nu t'}{\left(1-\nu^2/c^2\right)^{1/2}}$$
$$t' = \frac{t-\nu x/c^2}{\left(1-\nu^2/c^2\right)^{1/2}}$$
$$t = \frac{t+\nu x/c^2}{\left(1-\nu^2/c^2\right)^{1/2}} \tag{14}$$

Again, we want the dimensions of the ordinate and abscissa to be the same, hence, we multiply time by the velocity of light and set the product equal to w viz w = ct and w' = ct'. Also set ν/c = β. With that, the Lorentz transformation Equations in Eq. 12 become:

$$x' = \frac{x-\beta w}{\left(1-\beta^2\right)^{1/2}}; x' = \frac{x+\beta w}{\left(1-\beta^2\right)^{1/2}}$$

$$w' = \frac{w-\beta x}{\left(1-\beta^2\right)^{1/2}}$$

$$w = \frac{w'-\beta x'}{\left(1-\beta^2\right)^{1/2}}$$

$$w = \frac{w'-\beta x'}{\left(1-\beta^2\right)^{1/2}} \tag{15}$$

As before, we start by drawing the axes of the S frame (w, x) orthogonal as shown in Fig. 5. Once again, we can draw a world line at any point given by tanθ = dx/dw = 1/c (dx/dt). The derivative dx/dt is the velocity of the particle = u. Consequently, tanθ = u/c, since, u<c for a material particle, we have that tanθ<1, therefore ,the angle is always <45°. For light wave u = c, therefore, tanθ = 1 making θ = 45°. This means that the world line of a light wave is a straight line inclined at an angle of 45° to the axes (Fig. 5). We have taken the axes of the S frame as orthogonal. The x' axis of S' the frame is given by w' = 0 in Eq. 13. This gives the equation of the x' axis as w = βx (Fig. 6). Similarly, the w' axis is given by x' = 0 changing this in Eq. 13 gives x = βx as the equation for the w' axis (Fig. 7). In both cases, tan∅ = β. What is clear

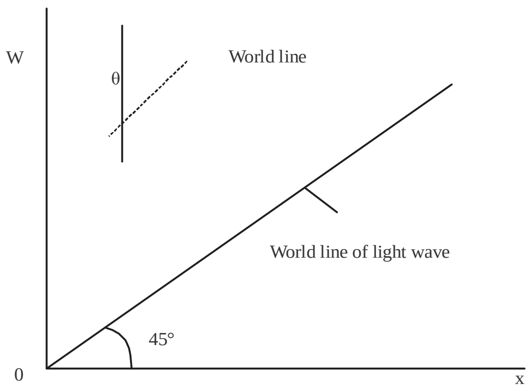

Fig. 5: World line of particle and world line of light wave

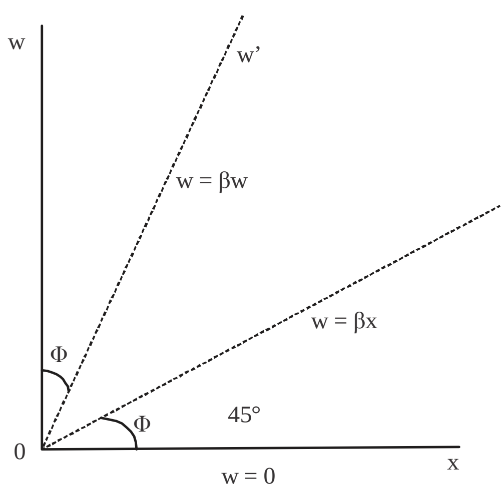

Fig. 6: Illustration of axes of two inertial frames under Lorentz transformation

from the above is that Lorentz transformation comprises transformation from an orthogonal system to a non-orthogonal system. Recall that in the Galilean transformation only one of the axes is oblique on transformation.

To obtain unit length on the axes we have to make use of calibration hyperbolae $w^2-x^2 = 1$ and $x^2-w^2 = 1$ which approach asymptotically the 45° world line of light waves as shown in Fig. 7. The hyperbola, $x^2-w^2 = 1$ cuts the x axis (i.e., w = 0) at x = ±1. Therefore, the point where this hyperbola meets the w axis defines unit length on that axis. Correspondingly, the hyperbola $w^2-x^2 = 1$, meets the w axis (i.e., x = 0) at w = ±1; therefore, where that hyperbola meets the w axis defines unit length on that M. Recall that the equation of that axis is w = βx We attain the point M by substituting this in the equation of the hyperbola.

This gives $x^2-\beta^2x^2 = 1$:

$$x = \pm\frac{1}{\left(1-\beta^2\right)^{1/2}}$$

Substituting the above equation in Eq. 16, we have:

$$x' = \pm\frac{x-\beta^2x}{\left(1-\beta^2\right)^{1/2}} = \pm\frac{\left(1-\beta^2\right)}{\left(1-\beta^2\right)^{1/2}\left(1-\beta^2\right)1/2} \pm 1 \qquad (16)$$

Consequently, the distance OM in Fig. 7 above gives length on the x' axis. Similarly, it can be shown that the point N where the hyperbola $w^2-x^2 = 1$, meets the w' axis gives unit length on that axis.

The development of the special relativity theory using four-vectors: A vector can be signified using a coordinate system fixed in space. If the axes of the coordinate system are rotated in space, the vector itself must not be changed in either magnitude or direction[14]. Therefore, it means that the components of the vector must transform in a particular way when the axes are rotated. We can then subsequently define a vector- if the components of any quantity transform similarly to the mechanisms of the vector when the coordinate axes are rotated in space, then the quantity is termed a vector. A 4-vector in the Y_1, Y_2, Y_3, Y_4 space will be defined as a quantity that transforms under Lorentz transformation, in a similar way as the Y_1, Y_2, Y_3, Y_4 coordinates of a point in four-dimensional space i.e:

$$Y_1' = \gamma\left(Y_1+i\frac{v}{c}X_4\right); Y_2' = Y_2; Y_3' = Y_3; Y_4' = \gamma\left(Y_4-i\frac{v}{c}Y_1\right) \qquad (17)$$

Therefore, B for example is defined to be a 4-vector if under a Lorentz transformation:

$$B_1' = \gamma\left(B_1+i\frac{v}{c}A_4\right); B_2' = B_2; B_3' = B_3; B_4' = \gamma\left(B_4-i\frac{v}{c}B_1\right) \qquad (18)$$

In analogy with Eq. 16 above, we have that:

$$B_1'^2+B_2'^2+B_3'^2+B_4'^2 = \gamma^2\left(B_1+i\frac{v}{c}A_4\right)+B_2^2+B_3^2+\gamma^2\left(B_1-i\frac{v}{c}B_4\right)^2 = B_1^2+B_2^2+B_3^2+B_4^2 \qquad (19)$$

Therefore, the length of a 4-vector is unaffected under rotation of axes (that is by a Lorentz

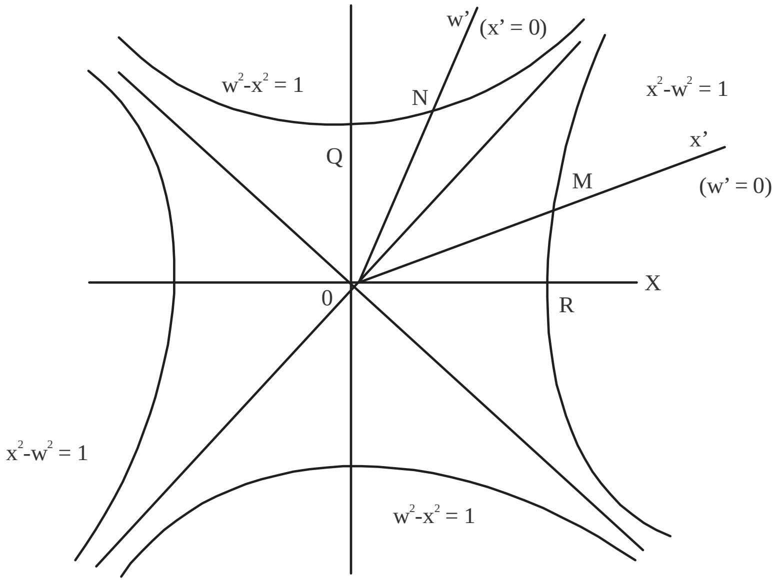

Fig. 7: Using calibrating hyperbolae to determine unit lengths on the axes of two inertial frames

transformation). If the square of the length of a 4-vector is optimistic, it is a space-like vector but if it is adverse, it is a time-like vector. This can be associated with the case of invariant time deliberated above where if the square is positive, we have a time-like interval whereas if it is negative, we have a space-like interval[15]. To advance further, the four components of a position 4-vector can be represented by (Y_1, Y_2, Y_3, Y_4) = (r, ict) The three components Y_1, Y_2 and Y_3 are the components of a vector in ordinary three-dimensional space; the fourth component is equal to a scalar time $\sqrt{-1}$. All 4-vectors are have this property. If the components of another position 4-vector are $Y_1+\delta Y_1$, $Y_2+\delta Y_2+Y_3$, δY_3 $Y_4+\delta Y_4$ we can write the transformation equations as in Eq. 17 (for example $Y_1^{'}+\delta Y_1^{'} = Y\left[Y_1+\delta Y_1+i\frac{v}{c}(Y_4+\delta Y_4)\right]$ and if we proceed to subtract Eq. 17, we obtain this:

$$\delta Y_1^{'} = \gamma\left(\delta Y_1+i\frac{v}{c}\delta Y_4\right); \delta Y_2^{'} = \delta Y_2; \delta Y_3^{'} = \delta Y_3$$
$$\delta Y_4^{'} = \gamma\left(\delta Y_4-i\frac{v}{c}\delta Y_1\right) \quad (20)$$

From Eq. 20, it can be resolved that the increases in a 4-vector from another position 4-vector. The length of a 4-vector is invariant, thus:

$$\delta Y_1^{'2}+\delta Y_2^{'2}+\delta Y_3^{'2}+\delta Y_4^{'2} = \delta Y_1^2+\delta Y_2^2+\delta Y_3^2+\delta Y_4^2$$

This corresponds to the relation of $d\tau^2 = dt^2$-$1/c^2$ ($dx^2+dy^2+dz^2$) which can be re-written in the form:

$$(ic\delta\tau)^2 = \delta x^2+\delta y^2+\delta z^2-c^2\delta t^2$$
$$\frac{\delta x^2}{\delta t^2}+\frac{\delta y^2}{\delta t^2}-c^2 = \left(\frac{ic\delta\tau}{\delta t}\right)^2 \quad (21)$$

If the two events having coordinates (Y_1, Y_2, Y_3, Y_4) and $Y_1+\delta Y_1$, $Y_2+\delta Y_2+Y_3$, δY_3 $Y_4+\delta Y_4$ correspondingly state to the positions of a particle at times t and t+δt in the S frame, then:

$$\frac{\delta x}{\delta t} = u_{x'}\frac{\delta y}{\delta t} = u_{y'}\frac{\delta z}{\delta t} = u_z$$

Hence, Eq. 21 becomes:

$$u^2-c^2 = -c^2\frac{\delta\tau^2}{\delta t^2}\delta\tau = \delta t\left(1-u^2/c^2\right)^{1/2} \quad (22)$$

In Eq. 22, u is the three-dimensional velocity of the particle in the S frame. If all components of a vector are multiplied by a scalar or an α invariant, the result is a new vector of length α times the original vector. Let the 4-vectors Y and Y+δY refer to the positions of a particle at times t and t+δ_t, then δY = (δY_1, δY_2, δY_3, δY_4) by the invariant 1/δt, we attain a quantity that is also a 4-vector. Let this be denoted by U then:

$$U = \left(\frac{\delta Y_1}{\delta\tau}, \frac{\delta Y_2}{\delta\tau}, \frac{\delta Y_3}{\delta\tau}, \frac{\delta Y_4}{\delta\tau}\right) = \begin{pmatrix} \frac{dx}{\sqrt{1-u^2/c^2dt}}, & \frac{dx}{\sqrt{1-u^2/c^2dt}}, \\ \frac{dz}{\sqrt{1-u^2/c^2dt}} & \frac{dz}{\sqrt{1-u^2/c^2dt}} \end{pmatrix}$$

Since, dx/dt = u_x etc., where u is the ordinary 3-dimensional velocity of the particle, we have:

$$U = \begin{pmatrix} \frac{u_x}{(1-u^2/c^2)^{1/2}}, & \frac{u_y}{(1-u^2/c^2)^{1/2}}, \\ \frac{u_z}{(1-u^2/c^2)^{1/2}} & \frac{ic}{(1-u^2/c^2)^{1/2}} \end{pmatrix} = \left(\frac{u}{(1-u^2/c^2)^{1/2}}, \frac{ic}{(1-u^2/c^2)^{1/2}}\right)$$

The 4-vector is called 4-vector. In the frame the 4-vector of the particle is:

$$U' = \left(\frac{u'}{(1-u'^2/c^2)^{1/2}}, \frac{ic}{(1-u'^2/c^2)^{1/2}}\right) \quad (23)$$

where u' is the three-dimensional velocity of the particle measured in S'. If is a 4-vector, it must transform according to Eq. 16, thus:

$$U'_4 = \gamma\left(U_4 - i\frac{v}{c}U_1\right) = \gamma\left[\frac{ic}{\sqrt{1-u^2/c^2}} - i\frac{v}{c}\frac{u_x}{\sqrt{1-u^2/c^2 dt}}\right] = \frac{ic\gamma(1-vu_x/c^2)}{\sqrt{1-u^2/c^2}}$$

But from Eq. 23, we have:

$$U'_4 = \frac{ic}{\sqrt{1-u'^2/c^2}}$$

Hence,

$$\frac{1}{\sqrt{1-u'^2/c^2}} = \frac{(1-vu_x/c^2)}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}} \quad (24)$$

This is the transformation for $1/\sqrt{1-u'^2/c^2}$ obtained:

$$U'_1 = \gamma\left(U_1 + i\frac{v}{c}U4\right) = \gamma\left[\frac{u_x}{\sqrt{1-u^2/c^2}} + i\frac{v}{c}\frac{ic}{\sqrt{1-u^2/c^2}}\right] = \gamma\frac{(u_x = v)}{\sqrt{1-u^2/c^2}}$$

And from Eq. 21, we have $U'_1 = \frac{u'_x}{\sqrt{1-u'^2/c^3}}$

Therefore,

$$U'_x = \frac{\gamma(u_x - v)\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}}$$

Using Eq. 22, we obtain $U'_x = \frac{(u_x - v)}{1-vu_x/c^2}$ This is the transformation Equation of the x component of the three-dimensional velocity of the particle. From Eq. 18 $U'_2 = U_2$ therefore:

$$\frac{u'y}{\sqrt{1-u'^2/c^2}} = \frac{u_y}{\sqrt{1-u^2/c^2}}$$

Or

$$u'_y = u_y\frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}}$$

Using Eq. 24, we have $u'_y = \frac{\sqrt{1-v^2/c^2}}{\sqrt{1-vu_x/c^2}}$

In the same way, we have $u'_z = \frac{u_z\sqrt{1-v^2/c^2}}{\sqrt{1-vu_x/c^2}}$

This completes the velocity transformation derivation using the method of 4-vector. The length of a 4-vector is an invariant. We have:

$$U^2 = U_1^2 + U_2^2 + U_3^2 U_4^2 = \frac{u^2}{(1-u^2/c^2)} + \frac{i^2c^2}{(1-u^2/c^2)} = \frac{u^2-c^2}{(1-u^2/c^2)} = -c^2$$

From the principle of the constancy of the velocity of light, U^2 must be an invariant and since, it is negative, the 4-velocity is a time-like 4-vector. If is the 4-velocity of a particle at a point x+δx, y+δy, z+δz at a time t+δt then the increments in the components of U are the components of a 4-vector that is (δU_1, δU_2, δU_3, δU_4) is a 4-vector. This agrees with what we found for the position of 4-vector. Multiplying the 4-vector by the invariant 1/δt we attain another 4-vector, B which is called the vector acceleration or 4-acceleration. After some algebra we find:

$$B\frac{dU}{dt} = \left[\left\{\frac{c^2}{(c^2-u^2)}a + u\frac{c^2(u.a)}{(c^2-u^2)^2}\right\}, \frac{ic^3(u.a)}{(c^2-u^2)^2}\right] \quad (25)$$

where α is the 3-dimensional acceleration of the particle. Multiplying the 4-vector by the mass of the particle we obtain:

$$p = mU = \left[\frac{mu}{\sqrt{1-u^2/c^2}}, \frac{imc}{\sqrt{1-u^2/c^2}}\right] = (p, iE/c) \quad (26)$$

P is called the 4-vector momentum or 4-momentum while $p = mu/(1-u^2/c^2)^{1/2}$ is the 3-dimensional momentum and $E = mc^2/(1-u^2/c^2)^{1/2}$ is the total energy of the particle. The equivalent 4-momentum in S' is:

$$P' = (p', iE'/c) \tag{27}$$

The length of the 4-vector is an invariant, we have:

$$p^2 + (iE/c)^2 = \text{constant}$$
$$p^2 - E^2/c^2 = \text{constant}$$

where p = 0, $E_0 = mc^2$ (the rest energy of the particle) so that the constant is equal to $-m^2c^2$. Therefore, $E^2 = p^2c^2+m^2c^4$. Since, the length of a 4-vector is invariant in S' we have $p'^2-E'^2/c^2 = -m^2c^4$. Therefore, $E'^2 = p'^2c^2+m^2c^4$.

To obtain the transformation for the momentum of a single particle, we use the fact that (p', iE/c) is a 4-vector. For the first component using (Eq. 18), we have:

$$P_1' = \gamma\left(P_1 + i\frac{v}{c}P_4\right) \text{or } P_x' = \gamma\left(P_x + i\frac{v}{c}iE/c\right) = \gamma\left(p_x - vE/c^2\right)$$

Similarly:

$$P_2' = P_2, \text{ therefore } p_y' = p_y$$
$$P_3' = P_3, \text{ therefore } p_z' = p_z$$

From $p_4' = \gamma\left(p_4 - i\frac{v}{c}P_1\right)$, we have:

$$iE'/c\gamma\left(iE/c - i\frac{v}{c}Px\right)E' = \gamma\left(E - vp_x\right)$$

From Eq. 26 it follows that dP = (dp, i dE/c) is a 4-vector. Multiplying by 1/dτ, we obtain the Minkowski 4-dimensional force which is defined by:

$$F = \frac{dp}{d\tau} = \left(\frac{dp}{d\tau}, \frac{i}{c}\frac{dE}{d\tau}\right) = \left(\frac{dp}{\sqrt{1-u^2/c^2}\,d\tau}, \frac{i}{c}\frac{dE}{\sqrt{1-u^2/c^2}\,d\tau}\right) \tag{28}$$

Space-time four-vectors: In this study, we consider presenting the impression of a vector to describe the separation of two events occurring in spacetime. The crucial idea is to show that the coordinates of an event have transformation properties equivalent to:

$$\begin{pmatrix}\Delta x' \\ \Delta y' \\ \Delta z'\end{pmatrix}_{R'} = \begin{pmatrix}\cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\Delta x \\ \Delta y \\ \Delta z\end{pmatrix}_R$$

For ordinary three-vectors, however with some surprising differences. To begin, we will reflect on two events E_1 and E_2 occurring in space time. For event E_1 with coordinates (x_1, y_1, z_1, t_1) in a frame of reference S and event $E_2\left(x_1', y_1', z_1', t_1'\right)$ in a frame of reference S', these coordinates are related by the Lorentz transformation which we will write as:

$$\begin{aligned} ct_1' &= \gamma ct_1 - \frac{\gamma v_x}{c}x_1 \\ x_1' &= -\frac{\gamma v_x}{c}ct_1 + \gamma x_1 \\ y_1' &= y_1 \\ z_1' &= z_1 \end{aligned} \tag{29}$$

And correspondingly, for event E_2 Then we write:

$$\begin{aligned} c\Delta t' &= c\left(t_2' - t_1'\right) = \gamma c\Delta t - \frac{\gamma v_x}{c}\Delta x \\ \Delta x' &= x_2' - x_1' = -\frac{\gamma v_x}{c}c\Delta t + \gamma\Delta x \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned} \tag{30}$$

Which we can write as:

$$\begin{pmatrix}c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z'\end{pmatrix} s' = \begin{pmatrix}\gamma & -\gamma v_X/c & 0 & 0 \\ -\gamma v_X/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z\end{pmatrix} \tag{31}$$

It is interesting to understand this equation as concerning the components with respect to a coordinate system S' of some sort of 'vector' to the components with respect to some other coordinate system S of the same vector. We would be justified in doing this if this 'vector' has the properties, similar to the length and angle between vectors for ordinary three-vectors which are self-determining of the choice of a reference frame. It turns out that it is 'length' defined as:

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] = (c\Delta t)^2 - (\Delta r)^2 \tag{32}$$

That is invariant for different reference frames i.e.:

$$(\Delta s)^2 = (s\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] = (c\Delta t)^2 - \left[(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2\right] \tag{33}$$

This invariant quantity ΔSs is identified as the interval among the two events E_1 and E_2 Δs is analogous to but essentially different from, the length of a three-vector is that it can be positive zero or negative. We could also talk about the 'angle' between two such 'vectors' and show that:

$$(c\Delta t_1)(c\Delta t_2)-[\Delta x_1\Delta x_2+\Delta y_1\Delta y_2+\Delta z_2] \quad (34)$$

Has the same value in all reference frames. This is equivalent to the scalar product for three-vectors. The quantity is defined by:

$$\Delta\vec{s}=\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (35)$$

Is then understood to link to a property of spacetime representing the separation between two events which has a complete existence independent of the choice of reference frame, and is identified as a four-vector. This four-vector is recognized as the displacement four-vector and signifies the displacement in spacetime between the two events E_1 and E_2. To distinguish a four-vector from an ordinary three-vector, a superscript arrow will be used. As was the case with three-vectors, any quantity which transforms in the same way as Δs is also named as a four-vector. For example, we have shown that:

$$\begin{aligned} E'/c &= \gamma(E/c)-\frac{\gamma v_x}{c}p_x \\ p_x' &= -\frac{\gamma v_x}{c}(E/c)+\gamma p_x \\ p_y' &= p_y \\ p_z' &= p_z \end{aligned} \quad (36)$$

Which we can write as:

$$\begin{pmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{pmatrix}=\begin{pmatrix} \gamma & -\gamma v_x/c & 0 & 0 \\ -\gamma v_x/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \quad (37)$$

Where we see that the same matrix appears on the right-hand side as in the transformation law for Δs'. This appearance relates the components in two different frames of reference S and S' of the four-momentum of a particle. This four-momentum is of course by this transformation property also a four-vector. We can note that the ('length') of this four-vector is given by:

$$\begin{aligned} &(E/c)^2-\left[p_x^2+p_y^2+p_z^2\right]=(E/c)^2-p^2= \\ &\left(E^2-p^2c^2\right)/c^2=m_0^2c^2 \end{aligned} \quad (38)$$

where m_0 is the rest mass of the particle. This quantity is the same (i.e., invariant) in different frames of reference. An additional four-vector is the velocity of four-vector, i.e:

$$\vec{v}=\begin{pmatrix} cdt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} \quad (39)$$

Where:

$$d\tau = ds/c \quad (40)$$

And is recognized as the proper time interval. This is the time interval measured by a clock in its rest frame as it makes its way among the two events an interval ds separately. To see how the velocity four-vector narrates to our normal understanding of velocity, reflect a particle in motion relative to the inertial reference frame S[16-18]. We can recognize two events, E1 wherein the particle is at position (x, y, z) at time t and a second event E_2 wherein the particle is at (x+dx, y+dy, z+dz) at time t+dt. The displacement in space and time between these events will then be represented by the four-vector $d\vec{s}$ defined in Eq. 35. Furthermore, during this time interval dt as measured in S, the particle undergoes a displacement dr = dxi+dyj+dzk and so has a velocity:

$$u=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k=u_x{}^{i}+u_y j+u_z k \quad (41)$$

The time interval among the events E_1 and E_2 as measured by a clock moving with the particle will be just the proper time interval dτ in the rest frame of the particle. We, therefore have by the time dilation equation:

$$dt=\frac{d\tau}{\sqrt{1-(u/c)^2}} \quad (42)$$

where, u is the speed of the particle. So, if we form the four-velocity to be related to the two events E_1 and E_2, we write:

$$\vec{u} = \begin{pmatrix} cdt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} = \frac{1}{\sqrt{1-(u/c)^2}} \begin{pmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \frac{1}{\sqrt{1-(u/c)2}} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} \quad (43)$$

Thus, if u<<c, the three spatial components of the four velocities reduce to the usual components of ordinary three-velocity. Note also that the invariant (‘length’) of the velocity, four-vector is just c_2. Finally, if we take the expression for the four-velocities and multiply by the rest mass of the particle, we get:

$$m_0\vec{u} = \frac{1}{\sqrt{1-(u/c)^2}} \begin{pmatrix} c \\ u_x \\ u_y \\ uz \end{pmatrix} = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \quad (44)$$

Which can be recognized as the four momenta de ned above. We can continue in this way, de ning four-acceleration as:

$$\vec{\alpha} = \frac{d\vec{u}}{d\tau} \quad (45)$$

And the four-force, also known as the Minkowski force $\vec{F}$. A direct generalization of the Newtonian definition would have been but this definition does not apply to zero rest mass particles, hence, the more general alternative in Eq. 46:

$$\vec{F} = \frac{d\vec{p}}{d\tau}$$

CONCLUSION

In this study, the geometric representation of spacetime and the use of four-vectors were vital to the successful findings of three-dimensional, four-dimensional non-Euclidean geometry in Lorentz and Galilean transformations. Thus, the usual opinion that there is a sole set of events presents now in a three-dimensional or four-dimensional spacetime cannot be continual. Minkowski’s path to spacetime, Galilean transformation as a geometrical representation of motion, Lorentz transformation as a geometrical representation of inertial frames and worldline of a particle, the expansion of the special relativity theory using four-vectors and space-time four-vectors were discussed in detail. In the Galilean transformation, the time coordinate of one inertial frame does not depend on the space coordinate of the time inertial frame. But in special relativity, time and space are code pendent as seen in Lorentz’s transformations where the time coordinate of one inertial frame is contingent on both time and space coordinates of another inertial system.

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