Asian Journal of
Earth Sciences

ISSN 1819-1886

Academic
Journals Inc.

www.academicjournals.com

Asian Journal of Earth Sciences 2 (2): 39-48, 2009
ISSN 1819-1886

Symmetry of Hamiltonian Systems

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Abstract: In the present study we use the formalism of Hamiltonian system on symplectic manifold due to Reeb, given in Abraham and Marsden and Arnold to derive the equation of motion for a particle on a line in a plane with a spring force and for a free particle in n-space. The time flows for both the problems mentioned above are also determined and proved that the determined flow is a Hamiltonian flow i.e., the symmetry of a Hamiltonian system. A non-Hamiltonian flow is also considered and it is shown that by changing the symplectic form and the phase space of the system we can convert it into a Hamiltonian flow. The translation and rotational symmetry related to linear and angular momentum respectively for the motion of a free particle in n-space is also considered, which is useful in reducing the phase space of a mechanical system.

Key words: Hamiltonian system, lie-group action, hamiltonian flow, linear and angular momentum

INTRODUCTION

The use of differential form in mechanics and its eventual formulation in terms of symplectic manifolds has been slowly evolving since Cartan (1922). The first modern exposition of Hamiltonian systems on symplectic manifolds seems to be due to Reeb (1952).

In this study the Hamiltonian systems formalism given in Abraham and Marsden (1978) and Arnold (1989) is used to derive the equations of motion for a particle on a line with a spring force and for a free particle in n-space from the energy function and the kinetics of the phase space.

The study of symmetry provides one of the most appealing applications of group theory. Groups were first invented to analyze symmetries of certain algebraic structures called field extensions and because symmetry is a common phenomenon in all sciences, it is still one of the two main ways in which group theory is applied the other way is through group representations. One can study the symmetry of plane figures in terms of groups of rigid motions of the plane. Plane figures provide a rich source of examples and a background for the general concept of group operations. Plane figures have generally bilateral symmetry, rotational symmetry, translational symmetry, glide symmetry and their combination.

HAMILTONIAN SYSTEM

A general Hamiltonian system consists of a manifold X, possibly infinite dimensional together with a (weakly) non-degenerate closed two-form ω on X (i.e., ω is an alternating bilinear form on each tangent space T_xX of X, $d\omega = 0$ and for $x \in X$, $\omega_x(u, v) = 0$ for all $u \in T_xX$ implies $v = 0$) and a Hamiltonian function $H: X \to \mathbb{R}$. Then X, H, ω determine in nice cases, a vector field X_H called the Hamiltonian vector field determined by the condition:

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$$C_{X_H}\omega = dH \tag{1}$$

Flows

Let X be a smooth manifold. A C^∞-function $F:\mathbb{R}\times X\rightarrow X$ is called a flow for the vector field *v* if F_x: $\mathbb{R}\rightarrow X$ is an integral solution for *v* i.e.,

$$\frac{d}{dt}F_x(t) = v \circ F_x(t)$$

or

$$\frac{d}{dt}F(t,x) = v \circ F(t,x)$$

and

$$F_x(0) = F(0,x) = x \quad \forall\ t \in \mathbb{R}, x \in X$$

Hamiltonian Flow

Let (X, H, ω) be a Hamiltonian system. A flow F is called a Hamiltonian flow if it preserves the symplectic form and the Hamiltonian function (i.e., $F_t^*\ \omega = \omega$ and $F_t^*H = H$ for $t\in\mathbb{R}$) (Abraham and Marsden, 1978).

Group Actions

Let G be a group and let X be a set. An action of G on X is an assignment of a function S_g: X→X to each element g∈G such that:

- If I is the identity element of the group G, then S_I is the identity map, i.e., for any x∈X we have $S_I\,(x) = x$
- For any g, h∈G we have $S_g \circ S_h = S_{gh}$, i.e., for every x∈Xwe have $S_g(S_h\,(x)) = S_{gh}\,(x)$

A Lie-group action should satisfy certain differentiability properties in addition to the algebraic properties given above. The action is called effective if S_g = Identity map for only t = 0.

SYMMETRY OF HAMILTONIAN SYSTEMS

The symmetry of Hamiltonian system (X, ω, H) is a function S: X→X that preserves both the symplectic form ω and the Hamiltonian function H.

Motion of a Particle on a Line in the Plane with the Spring Force

The phase space of such a physical system Simmons (1991) is the simplest non-trivial symplectic manifold, the two-dimensional plane X = R^2 = {(q, p): q∈R, p∈R} with the area two-form ω = dq∧dp.

The Hamiltonian function for such a particle is:

$$H = \frac{1}{2m}p^2 + \frac{k}{2}q^2 \tag{2}$$

where, second term in the Hamiltonian is the potential energy of the spring.

Using the Eq. 1, we have for $q \in R$ and $v \in T_q R$

$$\omega(X_H(q), v) = dH(q).v$$

Taking

$$X_H = x_q \frac{\partial}{\partial q} + x_p \frac{\partial}{\partial p}$$

and

$$v = v_q \frac{\partial}{\partial q} + v_p \frac{\partial}{\partial p}$$

as an arbitrary vector field. We find:

$$(dq \wedge dp)(X_H(q), v) = dH(q)(v_q \frac{\partial}{\partial q} + v_p \frac{\partial}{\partial p})$$

or

$$(dq \wedge dp)(x_q \frac{\partial}{\partial q} + x_p \frac{\partial}{\partial p}, v_q \frac{\partial}{\partial q} + v_p \frac{\partial}{\partial p}) = (\frac{p}{m} dp + k\, q\, dq)(v_q \frac{\partial}{\partial q} + v_p \frac{\partial}{\partial p})$$

or

$$x_q v_p - x_p v_q = \frac{p}{m} v_p + k\, q\, v_q \Rightarrow x_q = \frac{p}{m}$$

and

$$x_p = -k\, q$$

Thus we have:

$$X_H = \frac{p}{m} \frac{\partial}{\partial q} - k\, q \frac{\partial}{\partial p}$$

since, ∂/∂q and ∂/∂p are functions of time t (along a particular trajectory) we can write the vector field:

$$X_H = \frac{dq}{dt} . \frac{\partial}{\partial q} + \frac{dp}{dt} . \frac{\partial}{\partial p}$$

as time derivative along trajectories on the plane, since ∂/∂q and ∂/∂p are linearly independent we have:

$$\frac{p}{m} = \frac{dq}{dt} \text{ and } \frac{dp}{dt} = -kq \tag{3}$$

which shows that equation of motion for a particle in a line with spring force is a linear differential equation:

$$\frac{d^2q}{dt^2} = -\frac{k}{m}q \tag{4}$$

We can draw the useful picture by using the conservation of the Hamiltonian by the Hamiltonian flow because it implies that the orbits of the system must lie inside level sets of H (An orbit is set of all points in phase space that the system must passes through during one particular motion. In other words it is the set of all points on one particular trajectory). The beautiful features of Hamiltonian systems is that we can get information about orbits of the differential equations of motion by solving the algebraic equation H = constant, which is easy to solve. So, here first we determine the Hamiltonian flow of the spring problem.

HAMILTONIAN FLOW OF THE SPRING PROBLEM

For finding the bona fide solutions to our differential equations, i.e., not only the orbit of a trajectory but the trajectory itself (i.e., the position as a function of time). We use the algebraic equation H_0 = constant to reduce our original system of differential equations:

$$\frac{dq}{dt} = \frac{p}{m} \text{ and } \frac{dp}{dt} = -k\,q$$

into one scalar differential equation:

$$\frac{dq}{dt} = \pm\frac{\sqrt{4\,m\,H_0 - k\,q^2 m}}{m} = \pm\sqrt{\frac{4\,H_0}{m} - \frac{k\,q^2}{m}} \tag{5}$$

which on integration gives:

$$q(t) = \pm 2\sqrt{\frac{H_0}{k}}\,\sin\left(\sqrt{\frac{k}{m}}\,t + q(0)\right) \tag{6}$$

The Hamiltonian flow for the linear differential Eq. 4 is given by the function:

$$f_t : R^2 \ni \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \cos\sqrt{\frac{k}{m}}\,t & \frac{1}{\sqrt{km}}\sin\sqrt{\frac{k}{m}}\,t \\ -\sqrt{km}\,\sin\sqrt{\frac{k}{m}}\,t & \cos\sqrt{\frac{k}{m}}\,t \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \in R^2 \tag{7}$$

or

$$f_t : R^2 \ni \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{k\,m}} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\sqrt{\frac{k}{m}}\,t & \sin\sqrt{\frac{k}{m}}\,t \\ -\sin\sqrt{\frac{k}{m}}\,t & \cos\sqrt{\frac{k}{m}}\,t \end{pmatrix}\begin{pmatrix} \sqrt{k\,m} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \in R^2 \tag{8}$$

Geometrically, the flow at time t in phase space is effected by first scaling the q-axis by a factor of $\sqrt{k\,m}$, which takes the orbits to circles, second, rotating these circles clockwise through an angle $\sqrt{\frac{k}{m}}\,t$ and finally rescaling the q-axis back to its original scale. Since, for each t the function f_t is a linear function from $R^2$to R^2 and because the determinant of the matrix representing f_t is 1, f_t is area preserving. So, the flow preserves the symplectic form.

Also the Hamiltonian flow preserves the Hamiltonian for $f_t^* H = H$, i.e., $H \circ f_t$, we have, for $g \in SO(2)$, (special orthogonal group):

$$H \circ f_t(q,p) = H(gq, pg^T) = \frac{1}{2m}\left|pg^T\right|^2 + \frac{k}{2}\left|gq\right|^2$$

$$= \frac{1}{2m}\, p\, g^T g\, p^T + \frac{k}{2}(gq)\left(q^T g^T\right) = \frac{p^2}{2m} + \frac{k}{2}q^2 = H,\ \left(g^T g = g\, g^T = 1\right)$$

the level sets of H in phase space are ellipses (Fig. 1),

Here if k is large the ellipses are tall and skinny, while if k is close to 0 then the ellipses are short and wide. If k = 1/m the ellipses degenerate to circles. As the flow preserves the Hamiltonian, each solution of the system must lie entirely with in one ellipse in phase space. The conservation of the Hamiltonian by the Hamiltonian flow tells us that orbits must lie inside sets of the form $H = \frac{p^2}{2m} + \frac{k}{2}q^2 = \text{constant}$. Since the motion is continuous, it follows that each orbit is contained in the curve $\frac{p^2}{2m} + \frac{k}{2}q^2 = \text{constant}$ (Fig. 1):

The spring Hamiltonian given in Eq. 2 is an action of the group (R, +) on R^2, for:

- $$f_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Identity matrix, for } 0 \in R$$

$$f_t \circ f_s = \begin{pmatrix} \cos\left(\sqrt{k/m}\right)t & \frac{1}{\sqrt{km}}\sin\left(\left(\sqrt{k/m}\right)t\right) \\ -\sqrt{km}\sin\left(\left(\sqrt{k/m}\right)t\right) & \cos\left(\sqrt{k/m}\right)t \end{pmatrix} \times \begin{pmatrix} \cos\left(\sqrt{k/m}\right)s & \frac{1}{\sqrt{km}}\sin\left(\left(\sqrt{k/m}\right)s\right) \\ -\sqrt{km}\sin\left(\left(\sqrt{k/m}\right)s\right) & \cos\left(\sqrt{k/m}\right)s \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\sqrt{k/m}\right)(t+s) & \frac{1}{\sqrt{km}}\sin\left(\left(\sqrt{k/m}\right)(t+s)\right) \\ -\sqrt{km}\sin\left(\left(\sqrt{k/m}\right)(t+s)\right) & \cos\left(\sqrt{k/m}\right)(t+s) \end{pmatrix} = f_{t+s}, \text{ for any s,t in R}$$

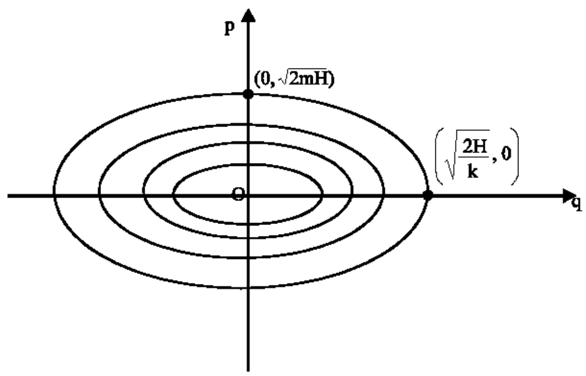

Fig. 1: Phase space of the particle on the line with level sets of the spring Hamiltonian

This action is not effective because if *t* is an integer multiple of $2\pi\left(\sqrt{m/k}\right)$ then:

$$\begin{pmatrix} \cos\left(\sqrt{k/m}\right)t & \frac{1}{\sqrt{km}}\sin\left(\left(\sqrt{k/m}\right)t\right) \\ -\sqrt{km}\sin\left(\left(\sqrt{k/m}\right)t\right) & \cos\left(\sqrt{k/m}\right)t \end{pmatrix} = I$$

Which also shows that the flow is periodic with period $2\pi\left(\sqrt{m/k}\right)$.

MOTION OF A FREE PARTICLE IN n-SPACE

Consider the motion of a free particle in n space. Let $q = (q_1, ..., q_n)$ be the position vector of the particle and $p = (p_1, ..., p_n)$ be the corresponding momentum vector of the particle. Then the phase space of the particle is the manifold $M = \{(q_1,...,q_n,p_1,...,p_n): q_1,...,q_n,p_1,...,p_n \in \mathbb{R}\}$ with the symplectic form:

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

and the Hamiltonian function

$$H = \frac{1}{2m}\sum_{i=1}^{n} p_i^2$$

Then ω, H determine the vector field X_H by the condition (1)

Let $X_H = a_1\frac{\partial}{\partial q_1} + ... + a_n\frac{\partial}{\partial q_n} + b_1\frac{\partial}{\partial p_1} + ... + b_n\frac{\partial}{\partial p_n}$ and $v = \sum_{i=1}^{n} a_i'\frac{\partial}{\partial q_i} + b_i'\frac{\partial}{\partial p_i}$ be arbitrary vector fields, then using (1), we have:

$$\left(\sum_{i=1}^{n} dq_i \wedge dp_i\right)\left(\sum_{i=1}^{n} a_i\frac{\partial}{\partial q_i} + b_i\frac{\partial}{\partial p_i}, \sum_{i=1}^{n} a_i'\frac{\partial}{\partial q_i} + b_i'\frac{\partial}{\partial p_i}\right) = \left[\frac{1}{m}\left(\sum_{i=1}^{n} p_i\, dp_i)\right)\right]\left(\sum_{i=1}^{n} a_i'\frac{\partial}{\partial q_i} + b_i'\frac{\partial}{\partial p_i}\right)$$

or

$$\sum_{i=1}^{n}\left(a_i b_i' - a_i' b_i\right) = \frac{1}{m}\left(\sum_{i=1}^{n} p_i b_i'\right)$$

This gives;
$a_i = p_i/m$ and $b_i = 0$, $(i=1,\dots,n)$.

Thus the vector field is given by:

$$X_H = \frac{p_1}{m}\frac{\partial}{\partial q_1} + ... + \frac{p_n}{m}\frac{\partial}{\partial q_n} \quad (9)$$

Taking the vector field:

$$X_H = \frac{dq_1}{dt}\frac{\partial}{\partial q_1} + ... + \frac{dq_n}{dt}\frac{\partial}{\partial q_n} + \frac{dp_1}{dt}\frac{\partial}{\partial p_1} + ... + \frac{dp_n}{dt}\frac{\partial}{\partial p_n} \quad (10)$$

as time derivative along trajectories, we have:

$$\frac{p_i}{m} = \frac{dq_i}{dt} \text{ and } \frac{dp_i}{dt} = 0, \ (i = 1, \ldots, n) \tag{11}$$

This gives:

$$m\frac{d^2}{dt^2}(q) = 0 \tag{12}$$

the required equation of motion of the free particle in n- space

HAMILTONIAN FLOW OF THE PARTICLE IN n-SPACE

The Hamiltonian flow of the Particle in n-space is determined by taking the algebraic equation:

$$H_0 = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 = \text{constant} \tag{13}$$

with the system of differential equations (1.9.3) and initial condition p(t) = p(0), t = 0, we have:

$$q(t) = \frac{t}{m}p(0) + q(0) \tag{14}$$

Thus for any fixed time t, the map:

$$f_t : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}^n \times (\mathbb{R}^n)^*$$

defined by:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + \frac{p}{m}t \\ p \end{pmatrix} \tag{15}$$

is a Hamiltonian flow, for:

$$f_t^*\omega = f_t^*\left(\sum_{i=1}^{n} dq_i \wedge dp_i\right) = \sum_{i=1}^{n}\left(f_t^* dq_i \wedge f_t^* dp_i\right) = \sum_{i=1}^{n}\left(dq_i + \frac{t}{m}dp_i\right) \wedge dp_i = \sum_{i=1}^{n}(dq_i \wedge dp_i) = \omega$$

and f_t^* H = H.

To show that every flow is not a Hamiltonian flow. If we take the flow of the problem particle in n-space as:

$$g_t : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}^n \times (\mathbb{R}^n)^*$$

defined by:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} qe^t \\ pe^t \end{pmatrix} \tag{16}$$

for any $t \in \mathbb{R}$, then $g_t^{*}\omega = e^{2t}\omega$. Thus g_t is not a Hamiltonian flow of a Hamiltonian system with the canonical symplectic form on $\mathbb{R}^{2n}$.

Taking $\omega = \sum_{i=1}^{n} \frac{1}{q_i p_i}(dq_i \wedge dp_i)$ as the symplectic form on $\mathbb{R}^{2n} \sim \{0\}$ then the flow g_t defined Eq. 16 preserves ω, for:

$$\begin{aligned} g_t^{*}\omega &= g_t^{*}\left(\sum_{i=1}^{n} \frac{1}{q_i p_i}(dq_i \wedge dp_i)\right) = \sum_{i=1}^{n} g_t^{*}\left(\frac{1}{q_i p_i}\right)(dq_i \wedge dp_i) = \sum_{i=1}^{n} \frac{1}{(q_i e^t p_i e^t)}\left(d(q_i e^t) \wedge d(p_i e^t)\right) \\ &= \sum_{i=1}^{n} \frac{1}{(q_i p_i e^{2t})} e^{2t}(dq_i \wedge dp_i) = \sum_{i=1}^{n} \frac{1}{(q_i p_i)}(dq_i \wedge dp_i) = \omega \end{aligned}$$

The Hamiltonian function for this system can be determined by taking

$$X_H = q_1 \frac{\partial}{\partial q_1} + \ldots + q_n \frac{\partial}{\partial q_n} + p_1 \frac{\partial}{\partial p_1} + \ldots + p_n \frac{\partial}{\partial p_n}$$

and

$$v = \sum_{i=1}^{n} a_i' \frac{\partial}{\partial q_i} + b_i' \frac{\partial}{\partial p_i}$$

as arbitrary vector fields, then Eq. 1, we have:

$$\sum_{i=1}^{n}\left(\frac{1}{q_i p_i}(dq_i \wedge dp_i)\right)\left(\sum_{i=1}^{n}\left(q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}\right), \sum_{i=1}^{n}\left(a_i' \frac{\partial}{\partial q_i} + b_i' \frac{\partial}{\partial p_i}\right)\right) = dH(v)$$

or

$$\sum_{i=1}^{n}\left(\frac{1}{q_i p_i}\left(q_i b_i' - a_i' p_i\right)\right) = \left(\sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right)\right)\left(\sum_{i=1}^{n}\left(a_i' \frac{\partial}{\partial q_i} + b_i' \frac{\partial}{\partial p_i}\right)\right)$$

which gives:

$$\frac{\partial H}{\partial q_i} = -\frac{1}{q_i}, \quad \frac{\partial H}{\partial p_i} = \frac{1}{p_i}, \quad (i=1,\ldots,n)$$

which on integration yields:

$$H = \log\left(\frac{p_1 \ldots p_n}{q_1 \ldots q_n}\right) + c \tag{17}$$

Now:

$$g_t^{*}H = g_t^{*}\left(\log\left(\frac{p_1 \ldots p_n}{q_1 \ldots q_n}\right) + c\right) = \log\left(\frac{(p_1 e^t)\ldots(p_n e^t)}{(q_1 e^t)\ldots(q_n e^t)}\right) + c = H$$

Hence, g_t preserves H. Thus g_t defined Eq. 16 is a Hamiltonian flow for the Hamiltonian system (M, ω, H), where,

$M = \mathbb{R}^{2n}$-{0}

$$\omega = \sum_{i=1}^{n} \frac{1}{q_i p_i}(dq_i \wedge dp_i)$$

and H is given Eq. 17.

The Hamiltonian flow of the Particle in n-space given Eq. 15 can be written as:

$$f_t : \begin{pmatrix} q \\ p \end{pmatrix} \to \begin{pmatrix} q + (t/m)p \\ p \end{pmatrix} = \begin{pmatrix} 1 & (t/m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

and satisfying the condition of group action, for:

- $0 \in R$, f_0 is indeed an identity matrix
- $f_s \circ f_t = \begin{pmatrix} 1 & (s/m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (t/m) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (s+t/m) \\ 0 & 1 \end{pmatrix} = f_{s+t} \quad s, t \in R$

The action is also effective. Thus the time flows of the Spring problem and Particle in n-space problem are symmetry of Hamiltonian system. But a Hamiltonian system may have other type of symmetries in addition to the time flow. For the problem the phase space of such a particle motion is:

$$X = T^* R^n = R^n \times (R^n)^* = \{(q_1, \ldots, q_n, p_1, \ldots, p_n) | q_1, \ldots, q_n, p_1, \ldots, p_n \in R\}$$

with symplectic form $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ and the Hamiltonian function $H = \frac{1}{2m} \sum_{i=1}^{n} p_i^2$. Consider the translation action of the group $(R^n, +)$ on X, for each $g = (g_1, \ldots, g_n)$ in R^n, define the function:

$$S_g : X = R^{2n} \to X = R^{2n}$$

by

$$S_g(q, p) = (q + g, p) \qquad (18)$$

Then S_g is the symmetry of the Hamiltonian system for any $g \in R^n$, for:

$$S_g^* \cdot \; = d(q+g) \quad dp = dq \quad dp = \quad \text{for} \;\; dg = 0$$

and S_g^* H =H.

Since, S_g gives a one-to-one correspondence from X to X, shows that S_g preserves the symplectic manifold.

Next, consider the rotational symmetry of a free particle in n-space.

The Hamiltonian system for such a particle is:

$$\left(\left(R^n \times (R^n)^* \right), \;\; (dq \wedge dp), \;\; \left(\frac{1}{2m} |p|^2 \right) \right)$$

The action of SO(n) on X is defined by $S_g(p, q) = (gq, pg^T)$ and is called the rotation action. Here, S_g preserves manifold X and symplectic form, since, g is constant and an orthogonal matrix $g\, g^T$, also we have:

$$S_g^* \omega = d(g\,q) \wedge d(p\,g^T) = (g\,g^T) dq \wedge dp = dq \wedge dp = \omega$$

and $S_g^* H = H$, i.e., $H \circ S_g = H$, for:

$$H \circ S_g(q,p) = H(g\,q, p\,g^T) = \frac{1}{2m}|p\,g^T|^2 = \frac{1}{2m}(p\,g^T)(p\,g^T)^T = \frac{1}{2m}(p\,g^T)(g\,p^T) = \frac{1}{2m}p\,p^T = \frac{1}{2m}|p|^2 = H(q,p)$$

Hence, the action of the Lie group $(R^n, +)$ on :

$$\left(R^{2n}, (dq \wedge dp), \frac{1}{2m}|p|^2\right)$$

preserves both the symplectic form and the Hamiltonian function called the translation and rotation symmetry of the mechanical system. These symmetries are linear symmetries so they can express in matrix form.

CONCLUSION

In this study we have shown that every flow is not Hamiltonian but by changing the symplectic form and with some restriction on the phase space one can successfully change the non Hamiltonian flow into the Hamiltonian flow. Also, we have discussed the symmetry group properties of the mechanical system. For two body problem only these symmetries are sufficient for consideration but for other system nonlinear symmetries also arise (for further discussion about symmetry of differential equations and Hamiltonian systems see Artin (1991) and Marsden and Ratiu (1999)). The above symmetries of a mechanical system are useful in reducing the phase space of the system using the Marsden-Weinstein theorem.

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