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Numerical Evaluation of Real Cauchy Principal Value Integral in Adaptive Environment

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ABSTRACT

In this study, we are concerned with the construction of a three point quadrature rule (analogous to Simpson's 1/3rd rule) for approximate evaluation of the real Cauchy principal value integral. In addition to this, some compound rules have been formed for numerical integration of Cauchy principal value integrals. The rules have been numerically verified with some standard test integrals and also the error bound, asymptotic error has been obtained.

Key words: Analytic function, Cauchy principal value, degree of precision, error bound

INTRODUCTION

Singular integrals of Cauchy type occur abundantly in applied mathematics, particularly in the theory of aerodynamics and in scattering theory. There has been substantial research work on the approximate evaluation of real Cauchy Principal Value (CPV) integrals of the type:

$$I(f) = \int_{-1}^{1} \frac{f(x)}{x-a}\,dx; \quad -1<a<1 \tag{1}$$

where, f (x) is continuous on [-1, 1]. Some of the rules are due to Price, Hunter (Chawla and Jayarajan, 1975), Lokamidis and Theocaris, Elliott and Paget, Theocaris and Kazantazakis, Monegato, etc. Standard integration rules meant for the approximate evaluation of real definite integrals do not accurately integrate when applied to real CPV integrals given in Eq. 1. Among many existing quadrature rules for approximation of real definite integrals, Simpson's 1/3rd rule is widely used. However, this rule does not yield good approximations when applied to real CPV integral of the type:

$$I(f) = \int_{-a}^{a} \frac{f(x)}{x}\,dx \tag{2}$$

Our aim in this paper is to construction of a three point quadrature rule for approximate evaluation of the real Cauchy principal value integral (Conway, 1980) accurately and to develop some compound rules for numerical integration of Cauchy principal value integrals.

FORMULATION OF THE RULE

For the construction of the three point rule the nodes that we have chosen here are $\pm a\alpha$ and 0 and the rule based on these points is denoted by:

$$R\ (f) = w_0 f\ (0)+w_1[f\ (a\alpha)\text{-}f\ (a\alpha)] \tag{3}$$

The weights w_0, w_1 and α are to be determined so that:

$$I((x)^k) = R\ ((x)^k); \qquad \text{for } k = 0, 1, 3 \tag{4}$$

It may be noted that:

$$I\ ((x)^{2k}) = R\ ((x)^{2k}); \qquad \text{for } k = 1, 2, 3,.....$$

since the rule proposed in the Eq. 3 is a fully symmetric quadrature rule.

Thus to determine the weights w_0, w_1 and α in Eq. 3 we make use of the identities given in Eq. 4.

From this, we have the following set of linear equations in weights w_0, w_1, and α as:

$$\left.\begin{array}{ll} w_0 & = 0, \\ \alpha w_1 & = 1 \\ \text{and} & \\ 3\alpha^3 w_1 & = 1 \end{array}\right\} \tag{5}$$

On solving the set of linear equations given in '(5)' on assumption that $\alpha \neq 0$ we have: $w_0 = 0, w_1 = \sqrt{3}$ and:

$$\alpha = \frac{1}{\sqrt{3}} \tag{6}$$

Thus, $\alpha \neq 0$, the rule R (f) given in Eq. 3 with weights w_0, w_1 and the parameter $\alpha \neq 0$ given in Eq. 6 represents a family of one parameter three point rules integrating all polynomials of degree at most three.

Formulation of compound rules: For the construction of the compound rule, we rewrite the integral given in Eq. 1 as:

$$I\ (f) = I_1+I_2\ (\text{say})$$

where:

$$I_1 = \int_{-a}^{a} \frac{f(x)}{x} dx \quad \text{and} \quad I_2 = \int_{a}^{1} \phi(x)\, dx$$

and

$$\phi(x) = \frac{f(x) - f(-x)}{x}$$

which is bounded in the interval [a, 1].

Thus, the integral I_2 is integrable in Riemann sense and can be numerically integrated by any standard quadrature rule preferably of lower precision in compound form. Further, the integral I_1 which is a Cauchy principal value integral can be approximated by the rule given in Eq. 3 with weights and parameter given in Eq. 6 for a suitable value of a.

Here we constructed two compounds rules for approximation of integrals and for this we consider here the Trapezoidal rule and Simpson's (1/3)rd.

Compound trapezoidal rule: The interval of integration i.e., [a, 1] is divided into n-subintervals of equal length by the points:

$$x_0 = a < x_1 < \ldots\ldots\ldots\ldots < x_n = 1$$

and then apply the Trapezoidal rule for approximation of the integral I_2 in each subinterval:

$$[x_{i-1}, x_1]; \qquad \text{for I} = 1, 2, \ldots. n$$

which yields:

$$I_2 \simeq h\sum_{k=0}^{n}{}'' \phi(x_k)$$

where:

$$h = \frac{1-a}{n}$$
$$x_k = a + kh; \qquad k = 0, 1, \ldots\ldots, n$$

and "denotes the first and last terms are halved.

Therefore:

$$I(f) \simeq R(f) + Q_1(f) = R_T(f) \text{ (say)}$$

where:

$$R(f) = \sqrt{3}\left[f\left(\frac{a}{\sqrt{3}}\right) - f\left(-\frac{a}{\sqrt{3}}\right)\right]$$

and

$$Q_1(f) = h\sum_{k=0}^{n}{}^{''}\phi(x_k)$$

is the rule constructed for the approximation of CPV integrals given in Eq. 2.

For any values of a; the sum:

$$h\sum_{k=0}^{n}\phi(x_k)$$

approaches to the integral I_2 as $h\to 0$; since the said sum is a Riemann sum. Thus, the accuracy of the approximations very much depends on the rule:

$$R(f) = \sqrt{3}\left[f\left(\frac{a}{\sqrt{3}}\right) - f\left(-\frac{a}{\sqrt{3}}\right)\right]$$

which in turn depends on a suitable choice of 'a'.

Compound Simpson's 1/3rd rule: To derive the compound Simpson's 1/3rd rule, the interval [a, 1] is divided into 2n-equal subintervals by the points:

$$x_k = a+kh; \qquad k = 0, 1,\ldots.. 2n$$

and

$$h = \frac{1-a}{2n}$$

In this case:

$$I_2 \simeq \frac{h}{3}\left\{2\times\sum_{k=1}^{n-1}{}^{''}\phi(x_k) + 4\times\sum_{k=1}^{n}\phi(x_{2k-1})\right\} = Q_2(f)$$

where "has usual meaning stated as above.

Thus:

$$I(f) \simeq R(f)+Q_2(f) = R_5(f) \text{ (say)}$$

Here $Q_2(f)$ is also a Riemann sum and hence tends to I_2 as $h\to 0$ or equivalently $n\to\infty$. Thus the approximation of the integral I depends upon the rule R (f) which is of order O (a^5). Further, for the particular value of 'a', this rule requires less number of functional evaluations compared to that required in compound Trapezoidal rule. In other words, $Q_2(f)$ converges to $I_2(f)$ rapidly than that of $Q_1(f)$ since the order of convergence of compound Simpson's 1/3rd rule and compound Trapezoidal rule are, respectively:

$$O\left(\frac{1}{n^4}\right)$$

and

$$O\left(\frac{1}{n^2}\right)$$

ERROR ANALYSIS

Let:

$$E\ (f) = I\ (f)\text{-}R\ (f) \tag{7}$$

Then E (f) is the error associated with the rule R (f) meant for the approximation of the real CPV integral I (f) given in Eq. 2. We now establish the following theorem concerning the error.

Theorem 1: If the derivatives of all orders of f (x) exist in $-a \le x \le 0$ then:

$$E\ (f) \in O(a^5) \tag{8}$$

Proof: Taylor's expansion of f (x) in $-a \le x \le 0$ about x = 0 is given by:

$$f(x) \;=\; f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and from this we get:

$$I(f) \;=\; 2af'(0) + \frac{a^3}{9} f^{(3)}(0) + \frac{a^5}{300} f^{(5)}(0) + \frac{a^7}{17640} f^{(7)}(0) + \ldots\ldots \tag{9}$$

On the other hand, by expanding each term of the rule R (f) by Taylor's expansion about x = 0 with determined weights and parameter α, we obtain:

$$R(f) \;=\; 2af'(0) + \frac{2a^3}{3(3!)} f^{(3)}(0) + \frac{2a^5}{9(5!)} f^{(5)}(0) + \frac{2a^7}{27(7!)} f^{(7)}(0) + \ldots\ldots. \tag{10}$$

Thus, Eq. 7, 9 and 10 jointly imply:

$$E(f) \;=\; \frac{8a^5}{45(5!)} f^{(5)}(0) + \frac{40a^7}{189(7!)} f^{(7)}(0) + \ldots\ldots.$$

and from this the desired result follows:

Error bound: To find the error bound of the rule R (f) we are following the technique due to Lether (1971).

To apply the technique due to Lether (1971) we first prove the following Lemma and the theorem.

Lemma 1: If E (f) denote the truncation error in approximation of I (f) by R (f) then:

$$E\,(x^k)\geq 0, \qquad \text{for } k = 0, 1, 2,\ldots.$$

Proof: It is easy to show that:

$$E\,(x^k) = 0 \tag{11}$$

for k = 0, 1, 2, 3 and for k is even.

Thus, it is sufficient to show that:

$$E(x^{2\mu+1})\geq 0 \text{ for } \mu\geq 1$$

Now:

$$\begin{aligned} E\left(x^{2\mu+1}\right) &= \frac{2a^{2\mu+1}}{2\mu+1} - R\left(x^{2\mu+1}\right) \\ &= \frac{2a^{2\mu+1}}{2\mu+1} - \frac{2a^{2\mu+1}}{3^{\mu}} \\ &= 2^{a2\mu+1}\left[\frac{1}{2\mu+1}-\frac{1}{3^{\mu}}\right]\geq 0; \quad \text{for } \mu\geq 1 \text{ and } 0<a<1 \end{aligned}$$

Theorem 2: If f (z) is analytic in a closed disc:

$$\Omega = \{z\in\mathbb{C}: |z|\leq r;\ r>a\}$$

Then, $|E\,(f)|\leq M\,(r)\,e_r$ where $M\,(r) = \underset{|z|=r}{\text{Max}}|f\,(z)|$ and:

$$e_r = r\ln\left(\frac{r+a}{r-a}\right)-\left[\frac{99}{10}\left(\frac{ar}{9r^2-a^2}\right)+\frac{144}{25}\left(\frac{ar}{9r^2-4a^2}\right)+\frac{13}{50}\left(\frac{ar}{r^2-a^2}\right)\right]$$

which $\to 0$ as $r\to\infty$.

Proof: Let, f (z) = f (x); for $z\in[-a, a]$.

Expanding f (z) by Taylor's series expansion about z = 0, we have:

$$f\,(x) = b_0+b_1x+b_2x^2+b_3x^3+\ldots..$$

where:

$$b_k = \frac{f^{(k)}(0)}{(k)!}; \qquad k = 0, 1, 2....... \tag{12}$$

for z∈[-a, a].

Further, since f (z) denotes the truncation error in approximation of integral I (f) by the rule R (f) i.e.:

$$I(f) = R(f) + E(f)$$

and E being a linear operator, we obtain from Eq. 11 and 12:

$$E(f) = \sum_{k=7}^{\infty} b_k E\left(x^k\right) \tag{13}$$

where:

$$E\left(x^k\right) = \int_{-a}^{a} \frac{x^k}{x} dx - R\left(x^k\right); \qquad k \geq 5$$

Further:

$$E(x^k) = 0 \qquad \text{for k being even}$$

Hence Eq. 13 further simplifies to:

$$E(f) = \sum_{\mu=3}^{\infty} b_{2\mu+1} E\left(x^{2\mu+1}\right) \Rightarrow \left|E(f)\right| \leq \sum_{\mu=3}^{\infty} \left|b_{2\mu+1}\right| \left|E\left(x^{2\mu+1}\right)\right|$$

By Cauchy-inequality:

$$\left|b^{2\mu+1}\right| \leq \frac{M(r)}{r^{2\mu+1}}$$

where:

$$M(r) = \operatorname*{Max}_{|z|=r} \left|f(z)\right|$$

Thus:

$$\left|E(f)\right| \leq M(r)\sum_{\mu=3}^{\infty}\frac{1}{r^{2\mu+1}}\left|E\left(x^{2\mu+1}\right)\right| \tag{14}$$

However by Lemma 1:

$$E\,(x^{2\mu+1}) \geq 0; \quad \text{For } \mu \geq 1$$

Therefore by following the technique due to Lether (1971):

$$\sum_{\mu=1}^{\infty}E\left(x^{2\mu+1}\right) = E\left[\left(1-\frac{x}{r}\right)^{-1}\right] \tag{15}$$

Hence from Eq. 14 and 15, we now have:

$$|\,E\,(f) \leq M\,(r)e\,| \tag{16}$$

where:

$$e_r = E\left[\left(1-\frac{x}{r}\right)^{-1}\right]$$

But:

$$E\left[\left(1-\frac{x}{r}\right)^{-1}\right] = r\ln\left(\frac{r+a}{r-a}\right)-\frac{6ar}{3r^2-a^2}$$

From the expressions of e_r it is observed that for fixed a, $e_r \to 0$ as $r \to \infty$ which in term implies that:

$$E\,(f) \to 0 \text{ as } r \to \infty$$

The constant e_r in Eq. 16 is defined to as error constant by Lether (1971).

Numerical verifications: For the numerical verification of the rules $R_T(f)$ and $R_s(f)$; we have taken two standard integrals:

$$J_1 = \int_{-1}^{1}\frac{e^x}{x}dx = 2.11450175 \text{ (Exact value)}$$

and

$$J_2 = \int_{-1}^{1}\frac{\tan^{-1}x}{x}dx = 1.8319308 \text{ (Exact value)}$$

In Table 1 and 3 we have estimated:

$$J_1 = \int_{-1}^{1} \frac{e^x}{x} dx$$

by the rules $R_T(f)$ and $R_S(f)$, respectively and compared the approximated values obtained with the estimated results of Longman. In Table 1, we have got result 2.114525 correct up to five decimal places when n = 51 by the rule $R_T(f)$ where as the value of the same integral is obtained by the rule $R_S(f)$ in Table 3 is exactly same as the Longman's estimated value 2.11450175 when n = 9.

Again in Table 2 and 4, we have estimated another standard integral usually chosen by other researchers as:

Table 1: Evaluation of the integral $J_1 = \int_{-1}^{1} \frac{e^x}{x} dx$ by the rule $R_T(f)$

Rule	No. of intervals n	Approximation of $J_1 = \int_{-1}^{1} \frac{e^x}{x} dx$	Absolute error
$R_T(f)$	3	2.121287	6.78525 e-03
	10	2.115113	6.11250 e-04
	31	2.114565	6.32500 e-05
	40	2.114540	3.82500 e-05
	51	2.114525	2.32500 e-05
	Exact value	2.11450175	

Table 2: Evaluation of the integral $J_2 = \int_{-1}^{1} \frac{\tan^{-1} x}{x} dx$ by the rule $R_T(f)$

Rule	No. of intervals n	Approximation of $J_2 = \int_{-1}^{1} \frac{\tan^{-1} x}{x} dx$	Absolute error
$R_T(f)$	3	1.826660	5.2708 e-03
	10	1.831458	4.7280 e-04
	21	1.831824	1.0680 e-04
	31	1.831882	4.8800 e-05
	40	1.831902	2.8800 e-05
	51	1.831913	1.7800 e-05
	Exact value	1.8319308	

Table 3: Evaluation of the integral $J_1 = \int_{-1}^{1} \frac{e^x}{x} dx$ by the rule $R_S(f)$

Rule	No. of intervals (2n)	Approximation of $J_1 = \int_{-1}^{1} \frac{e^x}{x} dx$	Absolute error
$R_S(f)$	2	2.11459342	9.167 e-05
	6	2.11450286	1.11 e-06
	14	2.11450177	2.0 e-08
	18	2.11450175	0.0
	20	2.11450175	0.0
	Exact value	2.11450175	

Table 4: Evaluation of the integral $J_2 = \int_{-1}^{1} \frac{\tan^{-1} x}{x} dx$ by the rule $R_S(f)$

Rule	No. of intervals(2n)	Approximation of $J_2 = \int_{-1}^{1} \frac{\tan^{-1} x}{x} dx$	Absolute error
$R_S(f)$	2	1.83151609	4.1471 e-04
	4	1.83192307	7.7300 e-06
	8	1.83193049	3.1000 e-07
	16	1.83193082	2.0000 e-08
	20	1.83193083	3.0000 e-08
	40	1.83193084	4.0000 e-08
	Exact value¬	1.83193080	

$$J_2 = \int_{-1}^{1} \frac{\tan^{-1} x}{x} dx$$

by the rules $R_T(f)$ and $R_s(f)$, respectively. The approximated value obtained as 1.831913 by the rule $R_T(f)$ given in Table 2 is correct up to five decimal places when n = 51, where as when the same integral is estimated by the rule $R_s(f)$ as given in Table 4; we got the approximated value 1.83192307 which is correct up to six decimal places when n = 2 and gives more approximate result when n increases gradually.

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