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Prime and Maximal Ideals of Pre A*-Algebra

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ABSTRACT

In this study we formulate the definitions of an ideal, prime ideal, maximal ideal of Pre A*-algebra A and discuss certain examples. We prove important fundamental properties of Ideals. In particular we extend to prove that every ideal I of a Pre A*-algebra A is the intersection of all prime ideals of A containing I. We also show that every maximal ideal is necessarily prime, while the converse is true for special cases only.

Key words: Pre A*-algebra, ideal, principal ideal, prime ideal, maximal ideal, minimal prime ideal

INTRODUCTION

The study of lattice theory had been made by Birkhoff (1948). In a draft paper, the equational theory of disjoint alternatives by Manes (1989) introduced the concept of Ada (Algebra of disjoint alternatives) $(A, \wedge, \vee, (-)', (-)_\pi, 0,1,2)$ which is however differ from the definition of the Ada of E.G. Manes (1993) later study Adas and the equational theory of if-then-else. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean Algebra and the later concept is based on C-algebra $(A, \wedge, \vee, ')$ introduced by Guzman and Squier (1990).

Koteswara (1994) first introduced the concept A*-Algebra $(A, \wedge, \vee, *, (-)^{\sim}, (-)_\pi, 0,1,2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with 3- Ring, Stone type representation but also introduced the concept of A*-clone. Venkateswara Rao and Koteswara Rao (2003) studied about Boolean algebras and A*-algebras and the methods of generating A*-algebras from Boolean algebras and vice-versa. Venkateswara Rao and Koteswara Rao (2008) studied about If-Then-Else structure over A*-algebra and also Venkateswara Rao and Koteswara Rao (2004) studied about Prime ideals of A*-algebra. Venkateswara Rao and Koteswara Rao (2005) obtained Cayley theorem for A*-Algebras. Venkateswara Rao (2000) introduced the concept of Pre A*-algebra $(A, \wedge, \vee, *, (-)^{\sim})$ as the variety generated by the 3-element algebra A={0,1,2} which is an algebraic form of three valued conditional logic. It was proved that the only sub directly irreducible Pre A*-algebra are either A or two element Boolean algebra B={0,1}. Venkateswara Rao *et al.* (2009) generated Pre A*-algebras from Boolean algebras and defined congruence relation and Ternary operation on Venkateswara Rao and Srinivasa Rao (2009) defined a partial ordering on a Pre A*-algebra A and the properties of A as a poset are studied.

In this study we formulate the definitions of an ideal, prime ideal, maximal ideal of Pre A*-algebra A and discuss certain examples. We also show that for any ideals I and J of a Pre A*-algebra A, the ideal (I: J) is precisely equal to the intersection of all prime ideals containing J and not containing I. We also show that every maximal ideal is necessarily prime, while the converse is true for special cases only.

PRELIMINARIES

Definition 1: Jocobson (1994) Boolean algebra is an algebra $(B, \vee, \wedge, (-)', 0,1)$ with two binary operations, one unary operation (called complementation) and two nullary operations which satisfies:

- $(B, \vee, \wedge)$ is a distributive lattice
- $x \wedge 0 = 0$, $x \vee 1 = 1$
- $x \wedge x' = 0$, $x \vee x' = 1$

We can prove that $x'' = x, (x \vee y)' = x' \wedge y', (x \wedge y)' = x' \vee y'$, for all $x, y \in B$.

Here, we concentrate on the algebraic structure of Pre A*-algebra and state some results which will be used in the later text.

Definition 2: Venkateswara Rao (2000): An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying:

- $x^{\sim\sim} = x, \forall x \in A$
- $x \wedge x = x, \forall x \in A$
- $x \wedge y = y \wedge x, \forall x, y \in A$
- $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$
- $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y, z \in A$ is called a Pre A*-algebra

Example (Venkateswara Rao, 2000): $3 = \{0, 1, 2\}$ with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	0	1

x	$x^{\sim}$
0	1
1	0

Note: The elements 0, 1, 2 in the above example satisfy the following laws:

- $2^{\sim} = 2$
- (b) $1 \wedge x = x$ for all $x \in 3$
- $0 \vee x = x$ for all $x \in 3$
- $2 \wedge x = 2 \vee x = 2$ for all $x \in 3$

Example: $2 = \{0, 1\}$ with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	x~
0	1
1	0
2	2

Note

- $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So, every Boolean algebra is a Pre A* algebra
- The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1(b) to 1.1(g)

Note: Let, A be a Pre A*-algebra then A is Boolean algebra iff $x\vee(x\wedge y) = x$, $x\wedge (x\vee y) = x$ (absorption laws holds).

Lemma 1: Venkateswara Rao and Srinivasa Rao (2009). Every Pre A*-algebra satisfies the following laws.

- $x\vee (x^{\sim}\vee x) = x$
- $(x\vee x^{\sim})\wedge y = (x\wedge y) \vee (x^{\sim}\wedge y)$
- $(x\vee x^{\sim}) \wedge x = x$
- $(x\vee y)\wedge z = (x\wedge z)\vee(x^{\sim}\wedge y\wedge z)$

Definition 3: (Venkateswara Rao and Srinivasa Rao, 2009): Let, A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x\vee x^{\sim} = 1$ and the set $\{x \in A / x\vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A). Note that if A is a Pre A*-algebra with 1 then $1, 0\in B(A)$. If the centre of Pre A*-algebra coincides with $\{0, 1\}$ then we say that A has trivial centre.

Theorem 1: Venkateswara Rao and Srinivasa Rao (2009). Let, A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$.

Lemma 2: Venkateswara Rao and Srinivasa Rao (2009). Let, A be a Pre A*-algebra with 1:

- If $y \in B(A)$ then $x\wedge x^{\sim}\wedge y = x\wedge x^{\sim}$, $\forall\ x\in A$
- $x\wedge(x\vee y) = x\vee(x\wedge y) = x$ if and only if x, $yB\in(A)$

IDEALS OF PRE A*-ALGEBRA

Definition 4: A nonempty subset U of a Pre A*-algebra A is said to be an ideal of A if the following hold:

- $a, b \in U \Rightarrow a\vee b\in U$
- $a \in U \Rightarrow x\wedge a\in U$ for each $x\in A$

Example: All the ideals of Pre A*-algebra A=\{0, 1, 2\} are $I_1 =\{2\}$ $I_2=\{0, 2\}$ and A itself.

Now we give some examples of Pre A*-algebras and collect all the ideals of these.

Example: Let, $G=\{a_1, a_2, a_3, a_4, a_5\}$ where $a_1 =(1, 2)$, $a_2 = (0, 2)$, $a_3 = (2, 1)$, $a_4 = (2, 0)$, $a_5 = (2, 2)$. Then G is a Pre A*-algebra (a sub algebra of A×A) under the point wise operation. This algebra $(G, \wedge, \vee, (-)^{\sim})$ is a Pre A*-algebra without 1.All the ideals of G are"

$I_1=\{a_5\}$

I_2={a_2, a_5}
I_3={a_4, a_5}
I_4={a_1, a_2, a_5}
I_5={a_2, a_4, a_5}
I_6={a_3, a_4, a_5}
I_7={a_1, a_2, a_4, a_5}
I_8={a_2, a_3, a_4, a_5}
I_9= G

Example: Let, H = {b_1, b_2, b_3, b_4, b_5, b_6, b_7} where b_1 =(1, 2), b_2 = (0, 2), b_3= (2, 1), b_4 = (2, 0), b_5 =(2, 2), b_6 =(1, 1), b_7=(0, 0). Then H is a Pre A*-algebra (a sub algebra of A×A) under the point wise operation. This algebra (H, ∧, ∨, (-)˜) is a Pre A*-algebra with 1 and 1= b_6. All the ideals of H are:

I_1={b_5}
I_2 ={b_2, b_5}
I_3={b_4, b_5}
I_4={b_1, b_2, b_5}
I_5={b_3, b_4, b_5}
I_6={b_1, b_2, b_4, b_5}
I_7={b_2, b_3, b_4, b_5}
I_8={b_2,b_4,b_5, b_7}
I_9={b_1,b_2, b_3, b_4,b_5}
I_{10}={b_1,b_2, b_4,b_5, b_7}
I_{11}={b_1, b_2,b_3, b_4, b_5, b_7}
I_{12}={b_2,b_3, b_4, b_5, b_7}
I_{13}= H

Example: Let, F= A×A = {c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9} where c_1 =(1, 2), c_2 =(0, 2), c_3 = (2, 1), c_4 = (2, 0), c_5 = (2, 2), c_6=(1, 1), c_7=(0, 0), c_8=(1, 0), c_9=(0, 1). Then F is a Pre A*-algebra under the point wise operation.

This algebra (F, ∧, ∨, (-)˜) is a Pre A*-algebra with 1 and 1= c_6. The ideals of A×A need not be of the form I_1×I_2 where, I_1, I_2 are ideals of A for such ideals of A×A are 9 in number (since, C has only 3 ideals namely {2}, {0, 2} and A) where, we are exhibiting 14 ideals in A×A.

All the ideals of F are:

I_1={c_5}
I_2 ={c_2, c_5}
I_3={c_4, c_5}
I_4={c_1, c_2, c_5}
I_5={c_3, c_4, c_5}
I_6={c_1, c_2, c_4, c_5}
I_7={c_2, c_3, c_4, c_5}
I_8={c_2, c_4, c_5, c_7}
I_9={c_1, c_2, c_3, c_4, c_5}

I_{10}={c_1, c_2, c_4, c_5, c_7}
I_{11}={c_1, c_2, c_3, c_4, c_5, c_7}
I_{12}={c_1, c_2, c_4, c_5, c_7, c_8}
I_{13}= {c_2, c_3, c_4, c_5, c_7, c_9}
I_{14}=F

Lemma 3: Let, A be a Pre A*-algebra and p∈A. Then {x∧p /x∈A} is the smallest ideal containing p.

Proof: Let, U={x∧p /x∈A}. If s, t∈U then s= x∧p and t= y∧p for some x, y∈A s∨t=(x∧p) ∨ (y∧p)= (x∧p∧p) ∨ (y∧p∧p) = [(x∧p)∨ (y∧p)]∧p =(s∨t) ∧ p∈U. Therefore, s∨t∈U. Let, x∧p∈U and z∈A. Then, z∧(x∧p) = (z∧x)∧p∈U (since, z, x∧A∈z⇒x∈A). Therefore, U is an ideal of A. Also, U is an ideal containing p (since, p=p∧p∈U). Let, I be the ideal containing p. Then, xp∧I for any x∈A and hence U⊆I. Thus, U is the smallest ideal containing p.

Definition 5: Let, A be a Pre A*-algebra and p ∈A then {x∧p /x∈A} is called the Principal ideal generated by p and is denoted by < p >.

Now we give some examples of principal ideals of certain Pre A*-algebras.

Example: Principal ideals of the Pre A*-algebra A={0, 1, 2} are:

< 0 > = {0, 2}
< 1 > = {0, 1, 2}
< 2 > = {2}

Example: Principal ideals of the Pre A*-algebra G={a_1, a_2, a_3, a_4, a_5} earlier are:

< a_1 > = {a_1, a_2, a_5}
< a_2 > ={a_2, a_5}
< a_3 > = {a_3, a_4, a_5}
< a_4 > = {a_4, a_5}
< a_5 > = {a_5}

Example: Principal ideals of the Pre A*-algebra H ={b_1, b_2, b_3, b_4, b_5, b_6, b_7} earlier are:

< b_1 > = {b_1, b_2, b_5}
< b_2 > = {b_2, b_5}
< b_3 > = {b_3, b_4, b_5}
< b_4 > = {b_4, b_5}
< b_5 > = {b_5}
< b_6 > = H
< b_7 > = {b_2, b_4, b_5, b_7}

Example: Principal ideals of the Pre A*-algebra F= {c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9} earlier are:

$< c_1 > = \{c_1, c_2, c_5\}$
$< c_2 > = \{c_2, c_5\}$
$< c_3 > = \{c_3, c_4, c_5\}$
$< c_4 > = \{c_4, c_5\}$
$< c_5 > = \{c_5\}$
$< c_6 > = F$
$< c_7 > = \{c_2, c_4, c_5, c_7\}$
$< c_8 > = \{c_1, c_2, c_4, c_5, c_7, c_8\}$
$< c_9 > = \{c_2, c_3, c_4, c_5, c_7, c_9\}$

Lemma 4: Let, A be a Pre A*-algebra and p, $y \in A$ then $y \in < p > \Leftrightarrow y = y \wedge p$.

Proof: Suppose, $y \in < p > \Rightarrow y = x \wedge p$ for some $x \in A$. Now $y \wedge p = (x \wedge p) \wedge p = x \wedge p = y$.
Conversely suppose that $y = y \wedge p$. Then, $y \wedge p \in < p >$. Therefore $y \in < p >$.

Lemma 5: Let, A be a Pre A*-algebra and p, $q \in A$ then $< p > = < q >$ if and only if $p = q$.

Proof: Suppose $< p > = < q >$. Clearly $p \in < p > = < q >$ and $q \in < q > = < p >$:

$\Rightarrow$ $p = p \wedge q$ and $q = q \wedge p$ (by lemma 5)
$\Rightarrow$ $p = q$

On the other hand, $p = q \Rightarrow p \wedge q = p$ and $q \wedge p = q$:

$\Rightarrow$ $p \in < q >$ and $q \in < p >$
$\Rightarrow$ $< p > \subseteq < q >$ and $< q > \subseteq < p >$
$\Rightarrow$ $< p > = < q >$

Lemma 6: Let A be a Pre A*-algebra and p, $q \in A$. Then $< p > \cap < q > = < p \wedge q >$.

Proof: Let, p, $q \in A$. Let, $x \in < p \wedge q > \Rightarrow x = x \wedge p \wedge q$:

$\Rightarrow$ $x \wedge p = x \wedge p \wedge q \wedge p = x \wedge p \wedge q = x$
$\Rightarrow$ $x \in < p >$

Also, $x \wedge q = x \wedge p \wedge q \wedge q = x \wedge p \wedge q = x$:

$\Rightarrow$ $x \in < q >$
$\Rightarrow$ $x \in < p > \cap < q >$

Therefore, $< p \wedge q > \subseteq < p > \cap < q >$. Let, $x \in < p > \cap < q >$. Then, $x \in < p >$ and $x \in < q >$:

$\Rightarrow$ $x = x \wedge p$ and $x = x \wedge q$
$\Rightarrow$ $x = x \wedge q = (x \wedge p) \wedge q = x \wedge (p \wedge q)$
$\Rightarrow$ $x \in < p \wedge q >$

Therefore, $< p > \cap < q > \subseteq < p \wedge q >$. Thus, $< p > \cap < q > = < p \wedge q >$.

Note: We do not have $< p > \cup < q > = < p \wedge q >$, in general for example in the Pre A*-algebra $G=\{a_1, a_2, a_3, a_4, a_5\}$ $< a_1 > \cup < a_3 >=\{a_1, a_2, a_5\}$ $\{a_3, a_4, a_5\}$= G. But $a_1 \vee a_3 = a_5$ and $< a_5 > \neq G$.

Definition 6: For any subset S of a Pre A*-algebra A and an ideal U, we define $(S : U)=\{a\in A\setminus s\wedge a\in U$ for each $s\in S\}$. If S={c} then we write it as (c : U)

Theorem 2: (S : U) is an ideal of A.

Proof: Let, a, b $\in$ (S : U). Then $s\wedge a$, $s\wedge b \in U$ for all $s\in S$. $s\wedge(a\vee b) = (s\wedge a) \vee (s\wedge b) \in U$ (since, $s\wedge a$, $s\wedge b\in U$ and U is ideal). Hence, $a\vee b\in (S : U)$. Let, $a \in (S : U)$, $x\in A \Rightarrow a\wedge s\in U$ for some $s\in S$. Then, $(x\wedge a) \wedge s=x\wedge(a \wedge s) \in U$ (since, $a\wedge s\in U$, $x\in A$ and U is ideal). Therefore, $x\wedge a\in (S : U)$. Thus, (S : U) is an Ideal of A.

Theorem 3: Let, S be a subset and U and W are the two ideals of Pre A*-algebra A then, $(S : U\cap W)= (S : U) \cap (S : W)$.

Proof: Let, $x\in A$. Then, $x\in (S : U\cap W) \Leftrightarrow x \wedge s \in U\cap W$, for all $s\in S$.

$\Leftrightarrow$ $x \wedge s \cap U$ and $x \Leftrightarrow s \wedge W$ for all $s\in S$
$\Leftrightarrow$ $x\in(S : U)$ and $x\in(S : W)$
$\Leftrightarrow$ $x\in(S : U) \cap (S : W)$
Therefore, $(S : U\cap W) = (S : U) \cap (S : W)$.

Theorem 4: Let, S and T are be any two subsets of Pre A*-algebra A and $S \subseteq T$ then, $(T : U) \subseteq (S :U)$.

Proof: Let, $x\in(T : U)$ then $x \wedge t \in U$ for all $t \in T$ and hence, $x \wedge s \in U$ for all $s\in S$ (since $S \subseteq T$). So, that $x \in (S : U)$. Therefore, $(T : U) \subseteq (S : U)$.

Theorem 5: Let, S and T are be any two subsets of Pre A*-algebra A then, $(S \cup T: U) = (S : U)\cap (T: U)$.

Proof: We know that S and $T \subseteq S \cap T$ and hence by lemma 6. $(S \cup T: U) \subseteq (S : U) \cap (T : U)$. Now let, $x\wedge (S : U) \cap (T : U)$. Then, $x \wedge s \in U$ and $x \wedge t \in U$ for all $s \in S$ and $t \in T$. Therefore, $x \wedge a\in U$ for all $a \in S \cup T$.

$\Rightarrow$ $x\in(S \cup T: U)$

Therefore, $(S : U) \cap (T : U) \subseteq (S \cup T: U)$. Thus, $(S \cup T: U) = (S : U) \cap (T : U)$.

Theorem 6: Let A be a Pre A*-algebra $S\subseteq A$ for any ideals U and W of A $< S >\cap W\subseteq U \Leftrightarrow W\subseteq (S : U)$.

Proof: Suppose that $W\subseteq (S : U)$. Let, $x\in< S >\cap(S : U)$. Then:

$$x = \bigvee_{i=1}^{n} (y_i \wedge s_i)$$

and x∈(S : U) where, s_i∈S, y_i∈ A:

$$= \bigvee_{i=1}^{n} (y_i \wedge s_i)$$

and x∧s_i∩ U for each i. Now x = x ∧ x:

$$= \bigvee_{i=1}^{n} x \wedge (y_i \wedge s_i)$$

$$= \bigvee_{i=1}^{n} (x \wedge y_i \wedge s_i)$$

$$= \bigvee_{i=1}^{n} (x \wedge y_i) \wedge (x \wedge s_i)$$

Therefore, x ∈ U (since, x∧s_i∈ U). Therefore, < S >∩ (S : U) ∩ U and hence < S >∩W⊆ U (since, W⊆ (S : U)). On the other hand W is a ideal of A such that < S >∩W⊆U. Let, x∈W. Then, for any s ∈ S, x∧s∈< S >∩W (since W and < S > are ideals).

⇒ x∧s∈U (by supposition)

Therefore, x ∈ (S : U). Thus, W⊆ (S : U).

Lemma 7: Let, S be any subset of a Pre A*-algebra A. Then (< S > : U) = (S : U).

Proof: We know that S ⊆ < S > and therefore by theorem 4 (< S > : U) ⊆ (S : U). Let, x ∈(S:U).Then, x = x ∧s ∈ U for all s∈S. Let, t∈< S >. Then,

$$t = \bigvee_{i=1}^{n} (y_i \wedge s_i)$$

for some s_i ∈ S, y_i ∈ A. Now,

$$x \wedge t = x \wedge \left(\bigvee_{i=1}^{n} (y_i \wedge s_i) \right)$$

$$= \bigvee_{i=1}^{n} x \wedge (y_i \wedge s_i)$$

$$= \bigvee_{i=1}^{n} (x \wedge y_i \wedge x \wedge s_i)$$

$$= \bigvee_{i=1}^{n} (x \wedge y_i) \wedge (x \wedge s_i)$$

Then, $x \wedge t \in U$. Thus, $x \in (< S > : U)$. Therefore, $(S : U) \subseteq (< S > : U)$ and hence $(< S > : U) = (S : U)$

PRIME IDEALS OF PRE A*-ALGEBRA

Definition 7: Let, A be a Pre A*-algebra. A proper ideal of P of A is called a prime ideal if for any $a, b \in A$, $a \wedge b \in P \Rightarrow$ either $a \in P$ or $b \in P$.

Example: Consider the Pre A*-algebra A={0, 1, 2}. In this {0, 2} is the only prime ideal.

Example: The prime ideals of the Pre A*-algebra $G=\{a_1, a_2, a_3, a_4, a_5\}$ (see 2.3 example) are $I_4=\{a_1, a_2, a_5\}$, $I_6=\{a_3, a_4, a_5\}$, $I_7=\{a_1, a_2, a_4, a_5\}$, $I_8=\{a_2, a_3, a_4, a_5\}$ where as $I_1=\{a_5\}$, $I_2=\{a_2, a_5\}$, $I_3=\{a_4, a_5\}$, $I_5=\{a_2, a_4, a_5\}$ are not prime ideals.

Example: Some of the prime ideals of the Pre A*-algebra $H= \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ (see 2.4 example) are, $I_{10}=\{b_1, b_2, b_4, b_5, b_7\}$, $I_{12}=\{b_2, b_3, b_4, b_5, b_7\}$ where as $I_8=\{b_2, b_4, b_5, b_7\}$is an ideal but not a prime ideal.

Example: Some of the prime ideals of the Pre A*-algebra $F= \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}$ (see 2.5 example) are $I_{12}=\{c_1, c_2, c_4, c_5, c_7, c_8\}$, $I_{13}= \{c_2, c_3, c_4, c_5, c_7, c_9\}$ where as $I_8=\{c_2, c_4, c_5, c_7\}$ is an ideal but not a prime ideal.

Theorem 7: The following are equivalent for any proper ideal P of Pre A*-algebra A

- P is prime ideal
- For any ideal I and J of A, $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$
- For any ideal I and J of A, $P= I \cap J \Rightarrow P=I$ or $P=J$

Proof: (1) ⇒(2): Let, P be prime ideal and I and J are ideals of A such that $I \cap J \subseteq P$. Assume that $I \not\subseteq P$. Then there exist $a \in I$ such that $a \notin P$. Let, $b \in J$. Then, $a \wedge b \in I \cap J \subseteq P$ (since, $I \in J \subseteq P$). Since, P is prime, $a \in P$ or $b \in P$. But $a \notin P$ we have $b \in P$. Thus, $J \subseteq P$.

- (2) ⇒ (3) is trivial
- (3) ⇒ (1)

Suppose that $I \cap J=P \Rightarrow P=I$ or $P=J$ for any ideals I and J of A. Let, a and $b \in A$ such that $a \wedge b \in P$. Then, by lemma 7 $< a > \cap < b > = < a \wedge b > \subseteq P$ and hence $(< a > \vee P) \cap (< b > \vee P) = (< a > \cap < b >) \vee P = P$ so that $< a > \vee P = P$ or $< b > \vee P = P$ or equivalently $< a > \subseteq P$ or $< b > \subseteq P$ and therefore $a \in P$ or $b \in P$. Thus, P is prime ideal.

Theorem 8: Let, I be an ideal of a Pre A*-algebra A and $a \in A \setminus I$. Then there exist a prime ideal P containing I and not containing a.

Proof: Let, $a \notin I$. Let $\mathcal{T} = \{J / J$ is an ideal of A, $I \subseteq J$ and $a \notin J\}$. Clearly $I \notin \mathcal{T}$. Therefore, $\mathcal{T}$ is a nonempty and partially ordered set under set inclusion. By zorn's lemma there exist a maximal member M in $\mathcal{T}$. Then clearly M is a proper ideal of A and $I \subseteq M$.

We shall prove that M is a prime ideal of A. Let $x, y \in A$ such that $x \notin M$ and $y \notin M$. Let, M is properly contained in $M \vee < x >$ and $M \vee < y >$ and hence by maximality of M, we have $a \in M \vee < x > \cap M \vee < y > = M \vee (< x > \wedge < y >) = M \vee (< x \wedge y >)$.

Therefore, $x \wedge y \notin M$ (for if $x \wedge y \in M$ then $M \vee (< x > \wedge < y >) = M$ which is a contradiction, since $a \notin M$). Therefore, M is prime ideal containing I and not containing a.

Corollary: For any ideal I of a Pre A*-algebra A, $I = \cap\{P / P$ is a prime ideal of A and $I \subseteq P\}$.

Proof: If $a \notin P$ then by theorem 8, there exist a prime ideal P containing I and not containing a and therefore, a:

$$\notin \bigcap_{I \subseteq P, P prime} P$$

Thus, $I = \cap \{P / P$ is a prime ideal of A and $I \subseteq P\}$

Theorem 9: Let, I be an ideal of a Pre A*-algebra A and S a nonempty subset of A which is closed under the operation $\wedge$ and is disjoint with I. Then there exist a prime ideal P of A containing I and disjoint with S.

Proof: Let, I be an ideal of a Pre A*-algebra A, $S \subseteq A$ and $S \cap I = \phi$. Let, $\mathcal{T} = \{J / J$ is an ideal of A, $I \subseteq J$ and $J \cap S = \phi\}$.

Clearly $I \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a nonempty and partially ordered set under set inclusion. $\mathcal{T}$ is clearly closed under unions of chains. Therefore, by zorn's lemma there exist a maximal member M in $\mathcal{T}$. Then $I \subseteq M$ and $M \cap S = \phi$.

We shall prove that M is prime ideal of A. Since, S is nonempty and $M \cap S = \phi$, M is proper ideal of A. Let, $x, y \in A$ such that $x \notin M$ and $y \notin M$.

Then $M \vee < x >$ and $M \vee < y >$ are ideals containing M properly. By the maximality of M, we get $(M \vee < x >) \cap S \neq \phi$, $(M \vee < y >) \cap S \neq \phi$.

Choose $s \in (M \vee < x >) \cap S$ and $t \in (M \vee < y >) \cap S$. Then, $s \wedge t \in S$ (since, S is closed under $\wedge$ and $s, t \in S$) and $s \in M \vee < x >$ and $t \in M \vee < y >$.

Then, $s \wedge t \in (M \vee < x >) \cap (M \vee < y >) = M \vee (< x > \cap < y >) = M \vee (< x \wedge y >)$ and hence, $s \wedge t \in M \vee (< x \wedge y >) \cap S$.

Then it follows that $x \wedge y \notin M$ (for, if $x \wedge y \in M$ then $M \vee (< x \wedge y >) = M$ and hence $s \wedge t \in M \cap S$, which is contradiction to the fact that $M \cap S = \phi$)

Thus, M is the prime ideal of A, $I \subseteq M$ and $M \cap S = \phi$.

Theorem 10: For any ideals I and J of Pre A*-algebra A, $(I: J) = \cap \{P / P$ is a prime ideal of A, $J \subseteq P$ and $I \not\subseteq P\}$.

Proof: We have $I \cap (I: J) \subseteq J$. If P is prime ideal containing J then $I \cap (I: J) \subseteq J \subseteq P$ and if in addition $I \not\subseteq P$ then $(I: J) \subseteq P$. Therefore, $(I: J) \subseteq \cap \{P/P$ is a prime ideal of A, $J \subseteq P$ and $I \not\subseteq P\}$. On the other hand, suppose $x \notin (I: J)$.

Then there exist $a \in I$ such that $x \wedge a \notin J$. Therefore, there exists prime ideal P such that $J \subseteq P$ and $x \wedge a \notin P$. Then, $x \notin P$ and $a \notin P$ (since, P is prime) and hence, $I \not\subset P$ (since, $a \in I$ and $a \notin P$). Therefore, $x \notin \cap \{P / P$ is a prime ideal of A, $J \subseteq P$ and $I \not\subset P\}$ and hence, $\cap\{P / P$ is a prime ideal of A, $J \subseteq P$ and $I \not\subset P \subseteq (I: J)$. Thus, $(I: J) = \cap \{P / P$ is a prime ideal of A, $J \subseteq P$ and $I \not\subset P\}$.

MAXIMAL IDEALS OF PRE A*-ALGEBRA

Definition 8: Let A be a Pre A*-algebra. A proper ideal of M of A is called a Maximal ideal of A if M is maximal among the proper ideals of A.

Lemma 8: Let A be a Pre A*-algebra. Every maximal ideal in A is a prime ideal.

Proof: Let M is a maximal ideal of A. Then clearly M is a proper ideal of A. Suppose $a, b \in A$ such that $a \notin M$ and $b \notin M$.

Then, $M \subsetneq < M \cup \{a\} > = A$; $M \subsetneq < M \cup \{b\} > = A$ (since M is maximal). Then:

$$b \in < M \cup \{a\} > = \left\{ \bigvee_{i=1}^{n} \left(y_i \wedge x_i / y_i \in A, x_i \in M \cup \{a\} \right) \right\}$$

since, $b \in A$ we have $b \in < M \cup \{a\}$.

That is:

$$b = \bigvee_{i=1}^{n} (y_i \wedge x_i)$$

for some $y_i \in A$, $x_i \in M \cup \{a\}$. Now $b = b \wedge b$:

$$= b \wedge \left(\bigvee_{i=1}^{n} (y_i \wedge x_i) \right)$$

$$= \bigvee_{i=1}^{n} (b \wedge y_i \wedge x_i)$$

If $x_i \in M$ then clearly $b \wedge y_i \wedge x_i \in M$. Since, $b \notin M$ and

$$b == \bigvee_{i=1}^{n} (b \wedge y_i \wedge x_i)$$

it follows that $b \wedge y_i \wedge a \notin M$ for some y_i (since, $x_i \in M \cup \{a\} \Rightarrow x_i = M$ or $x_i = a$). Therefore, $b \wedge a \notin M \Rightarrow a \wedge b \notin M$ (since M is ideal).

Thus, M is prime ideal of A. The validity of the converse of the theorem 10 is not known. That is, are there prime ideals which are not maximal. In certain special cases, we know that every prime ideal is maximal, for example Boolean algebras. In the following we discuss another class of Pre A*-algebra when every prime ideal is maximal.

Theorem 11: Let, A be a Pre A*-algebra with 1.Suppose that for any $x \in A$, there exists a smallest $x_0 \in B(A)$ such that $x \wedge x_0 = x$ and $x_0^{\sim} \wedge x = 1$. Then, every prime ideal of A is a maximal ideal.

Proof: Let, P is a prime ideal of A and Q be any ideal such that $P \subsetneq Q$. Since, $P \subsetneq Q$ there exits $x \in Q$ such that $x \notin P$.

Then there exists a smallest $x_0 \in B(A)$ such that $x \wedge x_0 = x$ and $x_0^{\sim} \vee x = 1$. Therefore, $x_0 \notin P$ (for if $x_0 \in P$ then $x \in P$, a contradiction).

Since, P is prime $x_0^{\sim} \in P \subseteq Q$. Therefore both x and $x_0^{\sim}$ belongs to Q and hence, $1 = x_0^{\sim} \vee x \in Q$ (since, Q is ideal).

Then $a = 1 \wedge a \in Q$ for all $a \in A$ (since, $1 \in Q$, $a \in A$ and Q is ideal, $1 \wedge a \in Q$). Therefore, Q=A. Thus P is maximal ideal of A.

Definition 9: A prime ideal P of a Pre A*-algebra A is said to be a minimal prime ideal of A if there is no prime ideal Q of A such that $Q \subset P$.

Lemma 9: Let, A be a Pre A*-algebra. Every prime ideal contains a minimal prime ideal.

Proof: Let, P be a minimal prime ideal of A. Let, $\mathcal{T} = \{Q \,/\, Q \text{ is a prime ideal of A}, Q \subseteq P\}$. Then, $\mathcal{T}$ is a nonempty ($P \in \mathcal{T}$) and T is a partially ordered set with respect to inclusion. If $\{Q_\alpha\}_{\alpha \in \Delta}$ is a chain in $\mathcal{T}$ then:

$$\bigcap_{\alpha \in \Delta} Q_\alpha$$

in $\mathcal{T}$.

Therefore every chain in T has a lower bound in $\mathcal{T}$. Thus, by zorn's lemma $\mathcal{T}$ has a minimal member say M. Then M is a prime ideal of A and $M \subseteq P$. Suppose there is a prime ideal R of A such that $R \subseteq M$. Then, $R \subseteq M \subseteq P$. Therefore, $R \subseteq \mathcal{T}$. By the minimality of M in $\mathcal{T}$ we get that R=M. Hence, M is the minimal prime ideal contained in P.

REFERENCES

Birkhoff, G., 1948. Lattice Theory. American Mathematical Society, Colloquium Publications, New York.

Guzman, F. and C.C. Squier, 1990. The algebra of conditional logic. Algebra Universalis, 27: 88-110.

Jocobson, N., 1994. Basic Algebra. Vol. 1, Hindustan Publishing Corporation, India.

Koteswara, R.P., 1994. A*-Algebras and if-then-else structures. Ph.D. Thesis, Acharya Nagarjuna University andhra Pradesh, India.

Manes, E.G., 1989. The equational theory of disjoint alternatives. Personal Communication to Prof. N.V. Subrahmanyam.

Manes, E.G., 1993. Adas and the equational theory of if-then-else. Algebra Universalis, 30: 373-394.

Venkateswara Rao, J., 2000. On A*-algebras. Doctoral Thesis, Nagarjuna University, A.P., India.

Venkateswara Rao, J. and P. Koteswara Rao, 2003. Boolean algebras and A-algebras. J. Pure Math., 20: 33-38.

Venkateswara Rao, J. and P. Koteswara Rao, 2004. Prime ideals and congruences in A-algebras. Southeast Asian Bull. Math., 28: 1099-1119.

Venkateswara Rao, J. and P. Koteswara Rao, 2005. A Cayley theorem for A-Algebras. Sectunia Matematica, S1, F1, pp: 1-6.

Venkateswara Rao, J. and P. Koteswara Rao, 2008. A-modules and If-then-else algebras over A-algebras. Int. J. Comput. Math. Appl., 2: 103-108.

Venkateswara Rao, J. and K. Srinivasa Rao, 2009. Pre A*-algebra as a poset. Afr. J. Math. Comput. Sci. Res., 2: 73-80.

Venkateswara Rao, J. K. Srinivasa Rao, T. Nageswara Rao and R.V.N. Srinivasa Rao, 2009. Exploring pre A-algebra as a new paradigm. Int. J. Syst. Cybernet. Inform., 1: 14-19.