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## Weakly $C^*$ -Normal Subgroups and $p$ -Nilpotency of Finite Groups

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**Abstract:** A subgroup  $H$  is called to be weakly  $c^*$ -normal in  $G$  if there exists a subnormal subgroup  $K$  such that  $G = HK$  and  $H \cap K$  is  $s$ -quasinormal embedded in  $G$ . The following result is established: Let  $G$  be a group such that  $G$  is  $S_4$ -free. Also let  $p$  be the smallest prime dividing the order of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $G$ . If every minimal subgroup of  $P$  of order  $p$  or  $4$  (when  $p = 2$ ) is weakly  $c^*$ -normal in  $N_G(P)$  and when  $p = 2$   $P$  is quaternion-free, then  $G$  is  $p$ -nilpotent. The main result is established and a generalization of some authors'.

**Key words:** Sylow  $p$ -subgroups, weakly  $c^*$ -normal subgroups  $p$ -nilpotent, minimal subgroup

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### INTRODUCTION

A subgroup is quasinormal in  $G$  if for every subgroup  $K$  of  $G$  such that  $HK = KH$ , by Ore (1937) which is a generalization of normality. A subgroup  $H$  of  $G$  is  $s$ -quasinormal if  $H$  permutes with all Sylow subgroups of  $G$ , by Kegel (1962) and extensive studied by Deskins (1963). A subgroup  $H$  is  $c^*$ -normal in  $G$  if there exists a normal subgroup  $K$  of  $G$  such that  $G = HK$  and  $H \cap K$  is  $s$ -quasinormal embedded in  $G$ , by Wei and Wang (2007). Recently, Liu (2009) established some results on the base of weakly  $c^*$ -normal subgroups of finite groups.

### SOME DEFINITION AND PRELIMINARIES

#### Lemma 1

Suppose that  $U$  is  $s$ -quasinormally embedded in a group  $G$ ,  $H \leq G$  and  $K \triangleleft G$ . Then:

- If  $U \leq H$ , then  $U$  is  $s$ -quasinormally embedded in  $H$
- If  $UK$  is  $s$ -quasinormally embedded in  $G$ , then  $UK/K$  is  $s$ -quasinormally embedded in  $G/K$
- $K \leq H$  and  $H/K$  is  $s$ -quasinormally embedded in  $G/K$ , then  $H$  is  $s$ -quasinormally embedded in  $G$

#### Proof

Lemma 1 by Ballester-Boliches and Pearaza-Anguiler (1998).

#### Lemma 2

Let  $G$  be a group. Then we have,

- If  $H$  is weakly  $c^*$ -normal in  $G$  and  $H \leq K$  weakly  $c^*$ -normal in  $K$
- If  $N \triangleleft G$  and  $N \leq H$ . Then  $H$  is weakly  $c^*$ -normal in  $G$  if and only if  $H/N$  is weakly  $c^*$ -normal in  $G/N$
- Let  $\pi$  be a set of primes.  $H$  is a  $\pi$ -subgroup of  $G$  and  $N$  a normal  $\pi$ -subgroup of  $G$ . If  $H$  is weakly  $c^*$ -normal in  $G$ . Then  $HN/N$  is weakly  $c^*$ -normal in  $G/N$

#### Proof

Lemma 2.2 by Liu (2009).

**Lemma 3**

Let  $G$  be a group,  $K$  an  $s$ -quasinormal subgroup of  $G$ ,  $P$  a Sylow  $p$ -subgroup of  $K$ , where,  $p$  is prime number of  $|G|$ . If  $P \leq O_p(G)$  or  $K_G = 1$ , then  $P$  is  $s$ -quasinormal in  $G$ .

**Proof**

Lemma 2.5 by Wei and Wang (2007).

**Lemma 4**

Let  $G$  be a group and  $P$  a  $s$ -quasinormal  $p$ -subgroups of  $G$ , where,  $P$  is a prime number of  $|G|$ , then  $O^p(G) \leq N_G(P)$ .

**Proof**

Lemma 2.2 by Li *et al.* (2003).

**Lemma 5**

Let  $G$  be a group and  $p$  a prime dividing  $|G|$  with  $(|G|, p-1) = 1$ . Then:

- If  $N$  is normal in  $G$  of order  $p$ , then  $N$  is in  $Z(G)$
- If  $G$  has a cyclic Sylow  $p$ -subgroup, then  $G$  is  $p$ -nilpotent
- If  $M \leq G$  and  $|G:M| = p$ , then  $M \triangleleft G$

**Proof**

Lemma 2.8 by Wei and Wang (2007).

**Lemma 6**

Let  $P$  be a  $p$ -subgroup of a group  $G$  and  $N$  a normal  $\pi'$ -subgroup of  $G$  for some prime  $p$ . If  $A$  is a minimal subgroup of  $P$  and  $A$  is weakly  $c^*$ -normal in  $N_G(P)$ , then  $AN/N$  is weakly  $c^*$ -normal in  $N_G(P)N/N$ .

**Proof**

If  $A$  is normal in  $G$ , then  $AN/N$  is weakly  $c^*$ -normal in  $N_G(P)N/N$ . If  $A$  is not normal in  $G$ , then by hypotheses, there exists a subgroup  $K$  of  $N_G(P)$  such that  $N_G(P) = AK$  and  $A \cap K = 1$ . Obviously  $N_G(P)N/N = (AN/N)(KN/N)$ . If  $(AN/N) \cap (KN/N) \neq 1$ , then  $K \leq AN$  and therefore  $N_G(P)N/N = KN/N$ . By comparing the order, there has a contradiction. So,  $AN/N$  is weakly  $c^*$ -normal in  $N_G(P)N/N$ .

**Lemma 7**

Let  $p$  be the smallest prime dividing  $|G|$  and  $P$  a Sylow  $p$ -subgroup of  $G$ . If every minimal subgroup of  $P$  is  $AN/N$  is weakly  $c^*$ -normal in  $G$  and when  $p = 2$  either every cyclic subgroup of  $P$  is weakly  $c^*$ -normal in  $G$  or  $P$  is quaternion-free, then  $G$  is  $p$ -nilpotent.

**Proof**

Suppose that the result is false and Let  $G$  be a minimal counterexample. By lemma 2(1), the hypotheses is inherited by subgroups. Therefore,  $G$  is minimal non- $p$ -nilpotent group. By Ito (Robinson, 2003, Theorem 10.3.3)  $G$  is a minimal non-nilpotent. Then  $G$  is of order  $p^\alpha q^\beta$ , where,  $q$  is a prime,  $q \neq p$ ,  $P$  is normal in  $G$  and any Sylow  $q$ -subgroup  $Q$  of  $G$  is cyclic. Moreover,  $P$  is of exponent  $p$  if  $p$  is odd and exponent at most 4 if  $p = 2$  (Robinson, 2003).

Let  $A$  be a minimal subgroup of  $P$ . Then by hypotheses, there exists a subgroup  $K$  of  $G$  such that  $G = AK$  and  $A \cap K$  is  $s$ -quasinormally embedded in  $G$ . If  $A$  is not normal in  $G$ , then  $K$  is a maximal

subgroup of  $G$  of index  $p$ . Since,  $P$  is the smallest prime divisor of  $G$ . This leads that  $K$  is normal in  $G$  and so the Sylow  $q$ -subgroup of  $K$  are normal in  $G$ , thus  $G$  is nilpotent, a contradiction. So,  $A$  is normal in  $G$ , this leads that  $A$  is in the center of  $G$ . If either  $p$  is odd, or  $p = 2$  and every cyclic subgroup of  $P$  is weakly  $c^*$ -normal in  $G$ , then  $G$  is  $p$ -nilpotent by Ito's lemma. Then let  $B = \langle b \rangle$  be a subgroup of  $P$  of order 4. Then by hypotheses, there exists a subnormal subgroup  $K$  such that  $G = BK$  and  $B \cap K$  is  $s$ -quasinormally embedded in  $G$ . If  $|G:K| = 4$ , then  $K \langle b^2 \rangle$  is a subgroup of index 2 and therefore is normal in  $G$ . This implies that the Sylow  $q$ -subgroup are normal in  $G$ . Then  $G$  is nilpotent. A contradiction. If  $|G:K| = 2$ , then  $K$  is normal in  $G$ , we also get a contradiction. Then  $B$  is normal in  $G$ . If  $B \neq P$ , then, since  $G$  is a minimal non-nilpotent group and the exponent of  $P$  is at most 4, we have  $P \leq C_G(Q)$  and  $G = P \times Q$  is nilpotent, another contradiction. The lemma is proved.

## MAIN RESULTS

### Theorem 1

Let  $G$  be a group such that  $G$  is  $S_4$ -free. Also let  $p$  be the smallest prime dividing the order of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $G$ . If every minimal subgroup of  $P$  of order  $p$  or 4 (when  $p=2$ ) is weakly  $c^*$ -normal in  $N_G(P)$  and when  $p = 2$   $P$  is quaternion-free, then  $G$  is  $p$ -nilpotent.

### Proof

Assume that the theorem is false and let  $G$  be a counterexample of minimal order. Then:

(1)  $O_p(G) = 1$

If  $O_p(G) \neq 1$ , then we can choose a minimal normal subgroup  $N$  of  $G$  such that  $N \leq O_p(G)$ . Now consider the quotient group  $G/N$ . Obviously  $PN/N$  is a Sylow  $p$ -subgroup of  $G/N$ . By lemma 6,  $AN/N$  is weakly  $c^*$ -normal in  $N_G(P)N/N$ . The minimality of  $G$  implies that  $G/N$  is  $p$ -nilpotent and hence  $G$  is  $p$ -nilpotent, a contradiction. Thus  $O_p(G) = 1$ .

(2) For every subgroup  $M$  of  $G$  satisfying  $P \leq M < G$ ,  $M$  must be  $p$ -nilpotent. In particular,  $N_G(P)$  is  $p$ -nilpotent

If  $N_G(P) = G$ , then, by lemma 7,  $G$  is  $p$ -nilpotent. Hence,  $N_G(P) < G$ . Since  $N_G(P) \cap M \leq N_M(P) \leq N_G(P)$ ,  $M$  satisfying the hypotheses of our theorem. The minimal choice of  $G$  implies that  $M$  is  $p$ -nilpotent.

(3)  $G$  is solvable. Furthermore,  $P$  is a maximal subgroup of  $G$  and a Hall  $p'$ -subgroup of  $G$  is an elementary abelian  $q$ -subgroup  $Q$  for some prime  $q$

Since,  $G$  is not  $p$ -nilpotent, by Frobenius' theorem (Robinson, 2003), theorem 10.3.2), there exists a subgroup  $H$  of  $P$  such that  $N_G(H)$  is not  $p$ -nilpotent. So by (2) we think that  $N_G(H)$  is not  $p$ -nilpotent but  $N_G(K)$  is  $p$ -nilpotent for every subgroup  $K$  of  $P$  such that  $K < K \leq P$ . Now we show  $N_G(H) = G$ . Suppose that  $N_G(H) < G$ . Then, we  $H < P^* \leq P$  for some Sylow  $p$ -subgroup  $P^*$  of  $N_G(H)$ . Since every minimal subgroups of  $P^*$  of order  $p$  or 4 is weakly  $c^*$ -normal in  $N_G(P)$ . On the other hand, by the choice of  $H$ ,  $N_G(P^*)$  is  $p$ -nilpotent and so  $N_{N_G(H)}(P^*)$  is  $p$ -nilpotent. This implies that  $N_G(H)$  satisfying the hypotheses of our theorem for its Sylow  $p$ -subgroup  $P^*$  of  $N_G(H)$ . Now, the choice of  $G$  implies that  $N_G(H)$  is  $p$ -nilpotent, a contradiction. Hence,  $O_p(G) \neq 1$  and  $N_G(K)$  is  $p$ -nilpotent for every subgroup  $K$  of  $P$  with  $O_p(G) \leq K \leq P$ . Now, by Frobenius' theorem (Robinson, 2003), theorem 10.3.2),  $G/O_p(G)$  is  $p$ -nilpotent and hence  $G$  is  $p$ -nilpotent. By the odd order theorem,  $G$  is solvable.

Let  $T/O_p(G)$  be a chief factor of  $G$ . Then  $T/O_p(G)$  is an elementary abelian  $q$ -group for some prime  $q \neq p$  and there exists a Sylow  $q$ -subgroup  $Q$  of  $T$  such that  $T = QQ_p(G)$ . It is clear that  $PT = PQ$ . If  $PT < G$ , then, by (2),  $PT$  is  $p$ -nilpotent and so  $Q \leq C_G(O_p(G))$ , which contradicts the fact  $C_G(O_p(G)) \leq O_p(G)$  since,  $G$  is solvable. Hence,  $G = PQ$  and  $Q$  is a Hall  $p'$ -subgroup of  $G$ . The minimality of  $T/O_p(G)$  implies that  $P/O_p(G)$  is a maximal subgroup of  $G/O_p(G)$  and therefore  $P$  is a maximal subgroup of  $G$ .

- (4)  $G = O_p(G)L$ , where,  $L$  is a non-abelian split extension of a normal Sylow  $q$ -subgroup  $Q$  by a cyclic  $p$ -subgroup  $\langle a \rangle$ ,  $a^p \in Z(L)$  and the action of  $a$  on  $Q$  is irreducible

Let  $P_1/O_p(G)$  be a normal  $p$ -complement of  $P/O_p(G)$ . By Schur-Zassenhaus' theorem we have  $D = O_p(G)Q$ .

Let  $P_1/O_p(G)$  be a maximal subgroup of  $P/O_p(G)$ . Then  $N(G)P \neq P$  or  $G$ . If  $N(G)P = P$ , then  $N(H)P_1 = P_1$ , where,  $H = P_1D = PQ$ . Then  $H$  satisfying the hypotheses of the theorem. Since the minimality of  $G$ , we have that  $H$  is  $p$ -nilpotent. Then  $O_p(G)Q = O_p(G) \times Q$  and  $Q$  is normal in  $G$ , a contradiction. Hence,  $P_1 < G$ . So,  $O_p(G) = P_1$  and  $P/O_p(G)$  is a cyclic group. On the other hand, by the Frattini argument,  $G = O_p(G)N_G(Q)$ . Since,  $P$  is not normal in  $G$ , so we assume that  $G = O_p(G)L$ , where,  $L = \langle a \rangle \ltimes Q$  is a non-abelian split extension of a normal Sylow  $q$ -subgroup  $Q$  by a cyclic  $p$ -subgroup  $\langle a \rangle$ . Since,  $|P/O_p(G)| = p$  and  $O_p(G) \cap N_G(Q) \triangleleft N_G(Q)$ ,  $a_p \in Z(L)$ . Also, since  $P$  is a maximal subgroup of  $G$ ,  $O_p(G)Q/O_p(G)$  is a minimal normal subgroup of  $G/O_p(G)$  and therefore the action of  $a$  (by conjugation) on  $Q$  is irreducible.

- (5) If  $\Omega_1(O_p(G)) \cap \langle a \rangle = 1$ , then  $[\Omega_1(O_p(G)), Q] = 1$

Set  $G_1 = \Omega_1(O_p(G))L$ . Obviously,  $\Omega_1(O_p(G))$  is an elementary abelian and characteristic in  $O_p(G)$ . Since, for any  $1 \neq x \in \Omega_1(O_p(G))$ ,  $\langle x \rangle$  is normal in  $G$ ,  $\langle x \rangle \langle a \rangle = \langle a \rangle \langle x \rangle$ . Hence  $x^a \in \Omega_1(O_p(G)) \cap \langle x \rangle \langle a \rangle$ . This implies that  $a$  induces a power automorphism of  $p$ -power order in the elementary abelian  $p$ -group  $\Omega_1(O_p(G))$ . Thus  $[\Omega_1(O_p(G)), a] = 1$ . If there exists an element  $1 \neq x \in \Omega_1(O_p(G))$  and an element  $1 \neq g \in Q$  such that  $x^g = x_1 \neq x$ , then  $x^{a^{-1}ga} = x_1$  and therefore  $x^{a^{-1}ga^{-1}} = x$ . It follows that  $\langle \Omega_1(O_p(G)), \langle a \rangle, a^{-1}ga^{-1} \rangle \in C_{G_1}$ . Since, the action of  $a$  on  $Q$  is irreducible,  $Q\Omega_1(O_p(G))/\Omega_1(O_p(G))$  is a minimal normal subgroup of  $G_1/O_p(G)$  and  $\Omega_1(O_p(G)) \langle a \rangle$  is a maximal subgroup in  $G_1$ . Thus,  $C_{G_1}(x) = \Omega_1(O_p(G)) \langle a \rangle$  or  $G_1$ . But  $1 \neq a^{-1}ga^{-1} \in Q$ . Hence  $C_{G_1}(x) = G_1$ , in contradiction to  $x^g \neq x$ . So  $[\Omega_1(O_p(G)), Q] = 1$ .

- (6) The final contradiction

We consider the following two cases:

**Case 1**

$p > 2$  or  $p = 2$  and  $P$  is quaternion-free. Set  $G_1 = \Omega_1(O_p(G))L$ . If  $\Omega_1(O_p(G)) \cap \langle a \rangle = 1$ , then by (5)  $[\Omega_1(O_p(G)), Q] = 1$ .

Assume that  $\Omega_1(O_p(G)) \cap \langle a \rangle = \langle a^{p^2} \rangle$ . Then  $\langle a^{p^2} \rangle$  is a cyclic group with order  $p$  and  $\langle a^{p^2} \rangle \leq Z(G_1)$  since  $a^p \in Z(L)$ . Consider the quotient  $G_1/\langle a^{p^2} \rangle$ . It is clear that  $(\Omega_1(O_p(G))/\langle a^{p^2} \rangle) \cap (\langle a \rangle / \langle a^{p^2} \rangle) = 1$  and every subgroup  $K$  of  $\Omega_1(O_p(G))$  of order  $p$  is weakly  $c^*$ -normal in  $N_G(P)$  by hypotheses. Then there exists a subnormal subgroup  $H$  such that  $N_G(P) = HK$  and  $H \cap K$  is  $s$ -quasinormally embedded in  $G$ . Let  $W$  denote  $H \cap K$ . Then  $W$  is a Sylow  $p$ -subgroup of some  $s$ -quasinormal subgroup  $M$  of  $G$  and so  $W$  is normal in  $M$  with  $Q \leq M$ . Since  $MP = PM$ ,  $WQ = W \times Q$ . So,  $[O_p(G), Q] = 1$  and  $Q$  is normal in  $G$ , a contradiction.

**Case 2**

$p = 2$  and every cyclic subgroup of with order 2 or 4 is weakly  $c^*$ -normal in  $N_G(P)$ . Let  $G_2 = \Omega_2(O_p(G))L$ . If  $\Omega(O_p(G)) \cap \langle a \rangle = 1$ , then by (5)  $[\Omega(O_p(G)), Q] = 1$ . Now assume that  $\Omega(O_p(G)) \cap \langle a \rangle = A = \langle c \rangle$  is a cyclic group of order 2. It is clear that  $A \leq Z(\Omega_1(O_p(G)))$ . Let  $x \in \Omega_2(O_p(G))$  with order 4. By hypothesis,  $\langle x \rangle$  is weakly  $c^*$ -normal in  $N_G(P)$ . There exists a subnormal  $K$  of  $N_G(P)$  such that  $N_G(P) = \langle x \rangle K$  and  $\langle x \rangle \cap K$  is  $s$ -quasinormally embedded in  $G$ . Set  $W = \langle x \rangle \cap K$ , then there exists a  $s$ -quasinormal subgroup  $H$  of  $G$  such that  $W$  is a Sylow  $p$ -subgroup of  $H$ . If  $W = H$ , then  $W$  is  $s$ -quasinormal in  $G$ , so  $WQ = QW$  is  $p$ -nilpotent by lemma 5 and by lemma 4  $Q$  is normal in  $G$ , a contradiction. If  $H \neq G$ , there has nothing to prove. So, we have  $W < H < G$ . If  $G = PH$ , then  $G = PH = PQ$ . Therefore,  $Q < H$  and  $PQ < G$ , a contradiction. Then  $PH < G$  and by (2)  $PH$  is  $p$ -nilpotent. Let  $Q^*$  be the normal  $p$ -complement of  $PH$ ,  $PH = PQ^* = Q^*P$  and so  $WQ^* = Q^*W = W \times Q^*$ . So,  $W$  is normal in  $G$ , by (1)  $G/W$  is  $p$ -nilpotent, then  $G$  is  $p$ -nilpotent, contradiction.

**Remark 1**

The hypothesis that  $G$  is  $S_4$ -free can't be removed. Let  $G = S_4$ ,  $P$  the Sylow  $p$ -subgroup. Then  $P = N_G(P)$  and every minimal subgroup of  $P$  is weakly  $c^*$ -normal in  $N_G(P)$ , But  $G$  is not 2-nilpotent.

**Remark 2**

The hypothesis that  $P$  is quaternion-free can not be removed. Let  $A = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$  be a quaternion group, then  $A$  has an automorphism of order 3. Let  $G = \langle \alpha \rangle \ltimes A$ , clear then every element of  $G$  with order 2 lies in that center of  $G$  and is weakly  $c^*$ -normal in  $N_G(P)$ , Bu  $G$  is not 2-nilpotent.

**Corollary 1**

Let  $G$  be a finite group,  $p$  a prime dividing the order of  $G$  such that  $(|G|, p-1) = 1$ . If there exists a normal subgroup  $N$  of  $G$  such that  $G/N$  is  $p$ -nilpotent and every subgroup of prime and order 4 of  $G$  is  $s$ -quasinormally embedded in  $G$ , then  $G$  is  $p$ -nilpotent.

**Proof**

Theorem 4.1 by Li *et al.* (2005).

**CONCLUSION**

Let  $G$  be a group such that  $G$  is  $S_4$ -free. Also let  $p$  be the smallest prime dividing the order of  $G$  and  $P$  a Sylow  $p$ -subgroup of  $G$ . If every minimal subgroup of  $P$  of order  $p$  or 4 (when  $p = 2$ ) is weakly  $c^*$ -normal in  $N_G(P)$  and when  $p = 2$   $P$  is quaternion-free, then  $G$  is  $p$ -nilpotent.

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