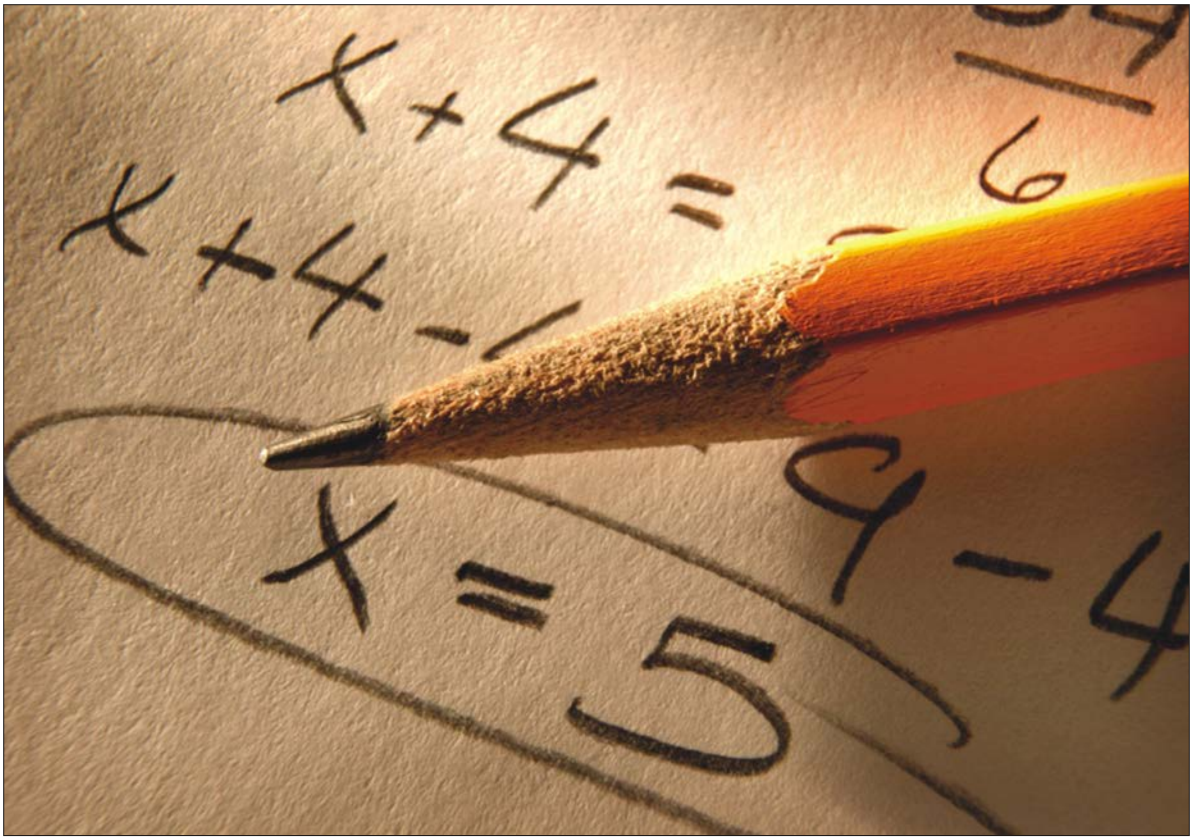

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Prime Antiflexible Derivation Alternator Rings

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Abstract: This study proves that a prime antiflexible derivation alternator R is either associative or the nucleus is equal to the center of R. Also, a prime antiflexible derivation alternator ring R with idempotent e $\neq$ 1 and characteristic $\neq$ 2, 3 is alternative.

Key words: Prime rings, derivation alternator rings, center, characteristic, commutator, associator, nucleus

INTRODUCTION

In the study of non-associative rings one of the important classes of rings is derivation alternator rings. Kleinfeld (1971a) defined two different generalizations of alternative rings, and for each of these generalizations he proved that the simple rings are alternative. Both these generalizations defined by Kleinfeld (1971b) are contained in the varieties of derivation alternator rings. These derivation alternator rings were initially studied by Hentzel *et al.* (1980). Hentzel and Smith (1980) investigated the structure of non-associative, flexible derivation alternator rings and Nimmo (1988) investigated the structure of non-associative, anti-commutative derivation alternator rings. In this study, the structure of non-associative, antiflexible derivation alternator rings is investigated.

In this study, it is proven that a prime antiflexible derivation alternator ring is either associative or N = Z. Also, it is proven that a prime antiflexible derivation alternator ring of characteristic $\neq$ 2, 3 with idempotent e $\neq$ 1 is alternative. At the end of this study an example of an antiflexible derivation alternator ring which is not alternative is provided.

PRELIMINARIES

A non-associative ring with characteristic $\neq$ 2 is called a derivation alternator ring if it satisfies the identities:

$$(x, x, x) = 0 \quad (1)$$

$$(yz, x, x) = y(z, x, x) + (y, x, x)z \quad (2)$$

$$(x, x, yz) = y(x, x, z) + (x, x, y)z \quad (3)$$

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where, we employ the associator $(x, y, z) = (xy)z-x(yz)$. The structure of non-associative, antiflexible rings that satisfy Eq. 1-3. We note that antiflexible derivation alternator rings can be defined simply by Eq. 1 and 2 and the identity:

$$A(x, y, z) = 0 \tag{4}$$

where, $A(x, y, z) = (x, y, z)-(z, y, x)$.

Throughout this study, R will denote antiflexible derivation alternator ring of characteristic $\neq 2$. A ring R is said to be of characteristic $\neq n$ if $nx = 0$ implies $x = 0, \forall x \in R$. The nucleus of N of ring R is defined as $N = \{x \in R: (n, R, R) = (R, n, R) = (R, R, n) = 0\}$. The center Z of R is defined as $Z = \{z \in N: (z, R) = (0)\}$. The middle nucleus M of ring R is defined as $M = \{m \in R: (R, m, R) = 0\}$. A ring R is called purely antiflexible if the nucleus N of R contains no ideal of R.

Consider a ring R with an idempotent e. Since derivation alternator rings are power-associative, it is known that R has pierce decomposition with respect to e (Albert, 1948). Thus, we have $R = R_{11}+R_{10}+R_{01}+R_{00,}$ where $R_{ij} = \{x \in R: ex = ix, xe = jx, i, j = 0, 1\}$.

From Eq. 1 and 4, the following identity is obtained:

$$B(x, y, z) = (x, y, z)+(y, z, x)+(z, x, y) = 0 \tag{5}$$

From Eq. 1-3, the following identity is obtained:

$$(x, yz, x) = y(x, z, x)+(x, y, x)z \tag{6}$$

By linearizing Eq. 6, the following identity is obtained:

$$(x, yz, w) = y(x, z, w)+(x, y, w)z \tag{7}$$

The following identities hold in any ring:

$$F(w, x, y, z) = (wx, y, z)-(w, xy, z)+(w, x, yz)-w(x, y, z) - (w, x, y)z = 0 \tag{8}$$

$$C(x, y, z) = (xy, z)-x(y, z)-(x, z)y-(x, y, z)-(z, x, y)+(x, z, y) = 0 \tag{9}$$

where, $(x, y) = xy-yx$.

The following identity is employed often:

$$(xoy)oz-xo(yoz) = (y, (x, z)) \tag{10}$$

Since, in any ring $(xoy)oz-xo(yoz) = (x, y, z)+(x, z, y)+(y, x, z)-(y, z, x)-(z,x,y)-(z,y,x)+(y,(x, z))$. where, $xoy = xy+yx$. We can verify the following identity in any ring,

$$(xoy, z)+(yoz, x)+(zox, y) = (x, y, z)+(y, z, x)+(z, x, y)+(x, z, y)+ (z, y, x)+(y, x, z)$$

so that from Eq. 5, we get:

$$(xoy, z) + (yoz, x) + (zox, y) = 0 \tag{11}$$

Forming $0 = F(w, x, y, z)-F(x, y, z, w)+F (y, z, w, x)-F(z, w, x, y)$ and using Eq. 5, we obtain,

$$G(w, x, y, z) = (w,(x, y, z)) - (x,(y, z, w)) + (y,(z, w, x)) - (z,(w, x, y)) = 0 \qquad (12)$$

As Anderson and Outcult (1968), expanding $0 = F(w, x, y, z) - F(z, y, x, w) + F(x, y, z, w) - F(w, z, y, x) + F(y, z, w, x) - F(x, w, z, y) + F(z, w, x, y) - F(y, x, w, z) - B((w, x), y, z) - B((x, y), z, w) - B((y, z), w, x) - B((z, w), x, y)$ and using Eq. 4, we get:

$$0 = H(w, x, y, z) = (w, (x, y), z) + (x, (y, z), w) + (y, (z, w), x) + (z, (w, x), y) \qquad (13)$$

From $0 = G(x, x, y, x) + (x, B(x, x, y))$ it follows that $2(x, (x, y, x)) = 0$, so that:

$$(x, (x, y, x)) = 0 \qquad (14)$$

hence from $0 = (x, B(x, y, x))$ and $0 = (x, (A(x, x, y))$, we have,

$$(x, (y, x, x)) = 0 \qquad (15)$$

and

$$(x, (x, x, y)) = 0 \qquad (16)$$

From Eq. 2, the identity Eq. 15 becomes:

$$((y, x), x, x) = 0 \qquad (17)$$

hence from Eq. 4, the identity Eq. 17 becomes:

$$(x, x, (y, x)) = 0 \qquad (18)$$

From Eq. 17, 18 and using Eq. 5, we get:

$$(x, (y, x), x) = 0 \qquad (19)$$

Substituting $x+z$ for x in Eq. 19 and subtracting $0 = A(z,(y, x), x) + A(z,(y, z), x)$, we obtain,

$$2(x, (y, x), z) + 2(x, (y, z), z) + (x, (y, z), x) + (z,(y, x), z) = 0 \qquad (20)$$

Substituting $-z$ for z in Eq. 20 and then adding to Eq. 20 yields.

$$2(x, (y, z), z) + (z, (y, x), z) = 0 \qquad (21)$$

Next, linearize Eq. 21 and add $A(w, (y, x), z) = 0$ to get:

$$K(w, x, y, z) = (x,(y, z),w) + (x,(y, w), z) + (w,(y, x), z) = 0$$

Computing $0 = H(w, x, y, z) + K(w, x, y, z) + K(x, y, w, z) - A(z,(w, x),y)$, we obtain:

$$J(w, x, y, z) = (x,(y, z),w) + (y,(w, x), z) = 0$$

Expanding $0 = J(w, x, y, z) + J(x, w, y, z) - A(w, (y, z), x)$. We get:

$$(w,(x, y), z) = 0 \qquad (22)$$

Lemma 1

Let R be an antiflexible derivation alternator ring. Then,

(i) N(R, R, R) = (NR, R, R)
(ii) (R, R, R)N = (R, R, RN)
(iii) N(R, R, R) = (R, NR, R)
(iv) (R, R, R)N = (R, RN, R)
(v) [N,(R, R, R)] = (0)

Proof

Applying Eq. 8 to $n \in N$. $x, y, z \in R$, we get:

$$(nx, y, z) = n(x, y, z)$$
$$(x, y, zn) = (x, y, z)n$$

which implies (i) and (ii).
(iii) Follows from the following:
(x, ny, z) = n(x, y, z) by Eq. 7
(iv) Follows from the following:
(x, yn, z) = (x, y, z)n by Eq. 7
For (v), subtract (iv) from (iii), [n,(x, y, z)] = 0
Which implies [N, (R, R, R)] = (0).
We can easily verify that $N \subseteq M$ and $MM \subseteq M$ by Eq. 22, $(R, R) \subseteq M$.

Corollary 1

$[R, N] \subseteq N$.

Proof

For any $n \in N$, $x, y, z \in R$
(nx, y, z)= n(x, y, z)
= (x, y, z)n
= (z, y, x)n
= (z, y, xn)
= (xn, y, z)
(or) ((n, x),y, z) = 0, which implies ([R, N], R, R) = (0).
Therefore, $[R, N] \subseteq N$.

Lemma 2

Let R be an antiflexible derivation alternator, if I is the ideal generated by the set (R, N), then,

(i) I = (R, N) + R(R, N)
(ii) $I \subseteq N$

Proof

(i) For any $w, x, y, z \in R$ and $n_1, n_2 \in N$,
$(x, n_1)w = ((x, n_1),w)+w(x, n_1)$,
hence, $(x, n_1)w \in (R, N)+R(R, N)$.

Also, $(y(z, n_2))w = y((z, n_2)w)$

$= y(((z, n_2),w)+w(z, n_2))$

$= y(((z, n_2),w) + (y, w)(z, n_2)$

Hence, $(y(z, n_2))w \in (R, N) + R(R, N)$

Thus, $(R, N)+R(R, N)$ is ideal of R and it contains (R, N).

Therefore, $I \subseteq (R, N) + R(R, N)$.

The converse inclusion is clear. Thus $I = (R, N)+R(R, N)$.

(ii) Since, $I = (R,N) + R(R,N)$ and $R \subseteq (R,N)$, using Eq. 5 and Lemma1, it is immediate that the following are all equivalent:

(a) $I \subseteq N$,
(b) $(I, R, R) = (0)$,
(c) $(R[R, N], R, R) = (0)$
(d) $[R, N](R, R, R) = (0)$

As Celik (1972), let $x, y \in R$ and $n \in N$. Then by Eq. 9 $(xn, y)-x(n, y)-(x, y)n = 0$
(or) $x(y, n) = -(xn, y)+(x, y)n$.
So, $x(y, n) \in (R, R)+(R, R)N$.
But, $(R, R)+(R, R)N \subseteq M$;
hence, $x(y, n) \in M$
which implies that $R(R, N) \subseteq M$.
By definition of middle nucleus M of ring R, $(R, R(R, N), R) = (0)$.
But Lemma 1 and Corollary 1 implies that
$(R, R(R, N), R) = (R, N)(R, R, R)$.
Thus by (d), we have $I \subseteq N$ and the proof of the Lemma is completed.

MAIN RESULTS

Theorem 1

If R is a prime antiflexible derivation alternator ring, then R is either associative or $N = Z$.

Proof

Suppose that R is not associative. Then there exists $x, y, z \in R$, such that $(x, y, z) \neq 0$. Let A be the ideal generated by (x, y, z). A is a non zero ideal of R.
Suppose $p \in I$ and $t \in R$.
Since, I is an ideal and $I \subseteq N$,
$(x, y, z)p = (x, y, zp) = 0$.
Then this identity together with
$(x, y, z)t = -x(y, z, t)+(xy, z, t)-(x, yz, t)+(x, y, zt)$ implies that $AI = (0)$.
Since, $A \neq (0)$ and R is prime, we must have $I = (0)$.
In particular $[R, N] = (0)$ or $N = Z$.
This completes the proof.

Using the definition of purely antiflexible, we state above theorem in a slightly more general form the following corollary:

Corollary 2

A prime antiflexible derivation alternator ring R is either associative (or) purely antiflexible.

Theorem 2

Let R be an antiflexible derivation alternator ring with idempotent e. Then e is the identity element of R if and only if $e \in N$.

Proof

Assume that $e \in N$. So, $e \in Z$.
Consider pierce decomposition $R = R_{11}+R_{10}+R_{01}+R_{00}$ of R with respect to e.
Since, $R_{10} = eR_{10} = R_{10}e = (0)$,
$R_{01} = R_{01}e = eR_{01} = (0)$,
$R = R_{11}+R_{00}$.
Also, $e \in N$ implies that R_{11} and R_{00} are ideal of R. Also we have $R_{11} R_{00} = (0)$.
R is prime, $e \in R_{11}$ implies that $R_{00} = (0)$. Thus $R = R_{11}$ and e is the idempotent element of R.

Corollary 3

If R is simple antiflexible derivation alternator ring with idempotent $e \in N$, then e is the identity element of R.

Corollary 4

If R is a prime antiflexible derivation alternator ring, then the nucleus N of R has no zero division of zero.

Proof

By theorem 1, N = Z = Center of R. By definition of Z, the ideal of R generated by the element $z_i \in Z$ is z_iR+Iz_i, where I = ring of integers. Then, $(z_1 R)(z_2 R) \subseteq (z_1 z_2)R$ implies that Z has no non zero divisions of zero.

Theorem 3

Let R be a prime antiflexible derivation alternator ring with idempotent $\neq 1$ and characteristic $\neq 2,3$. Then R is alternative.

Proof

From Eq. 3, the identity Eq. 16 becomes:

$$(x, x, (x, y)) = 0 \tag{23}$$

In particular, for any idempotent e we have,

$$(e, e, (e, y)) = 0 \tag{24}$$

Thus using Albert decomposition, Eq. 24 and also Eq. 3 and 1 imply $(e, e, x) = (e, e, eox) = 2(e,e,x)e$. Iteration then given $2(e, e, x)e = 4[(e, e, ex)e]e = 4(e, e, x)e$, so that $2(e, e, x)e = 0$. This in turn means $(e, e, x) = 2(e, e, x)e = 0$ for any idempotent e. At this point the argument given in ξ3 of Kleinfeld and Smith (1979) shows that R is alternative, which completes the proof of the theorem.

Corollary 5

Let R be a prime antiflexible derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2, 3$. Then R is associative.

The following example illustrates that an antiflexible derivation alternator ring which is not alternative.

Example

Suppose that the ring R is defined by the following multiplication table together with all finite sums of e, a, b, c, d, h, such that $x+x = 2x \neq 0$.

.	e	a	b	c	d	h
e	e	b	0	0	0	0
a	h	c	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
h	h+b	b	0	0	0	0

We observe that $(a,a,a) = a^2a - aa^2 = ca - ac = 0$.

Therefore, ring R satisfies Eq. 1.

It is enough to check identity Eq. 4, $(e, e, a) = (a, e, e)$

$(e, e, a) = ea - e(ea) \quad = b$

$(a,e,e) = (ae)e - ae \quad = he - h$

$= h + b - h$

$= b.$

Also, to check Eq. 2,

$(ab, e, e) \quad = a(b, e, e) + (a, e, e)b$

$(ab,e,e) - a(b,e,e) - (a,e,e)b \quad = (ab)(e.e) - (ab.e)e - a\{(be)e - b(e.e)\} - \{(ae)e - ae\}b$

$= 0.e - 0.e - a\{0.e - b.e\} - \{he - h\}b$

$= 0 - 0 - a\{0 - 0\} - \{h+b-h\}b$

$= b.b = 0.$

Hence, R is an antiflexible derivation alternator ring, but not alternative.

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