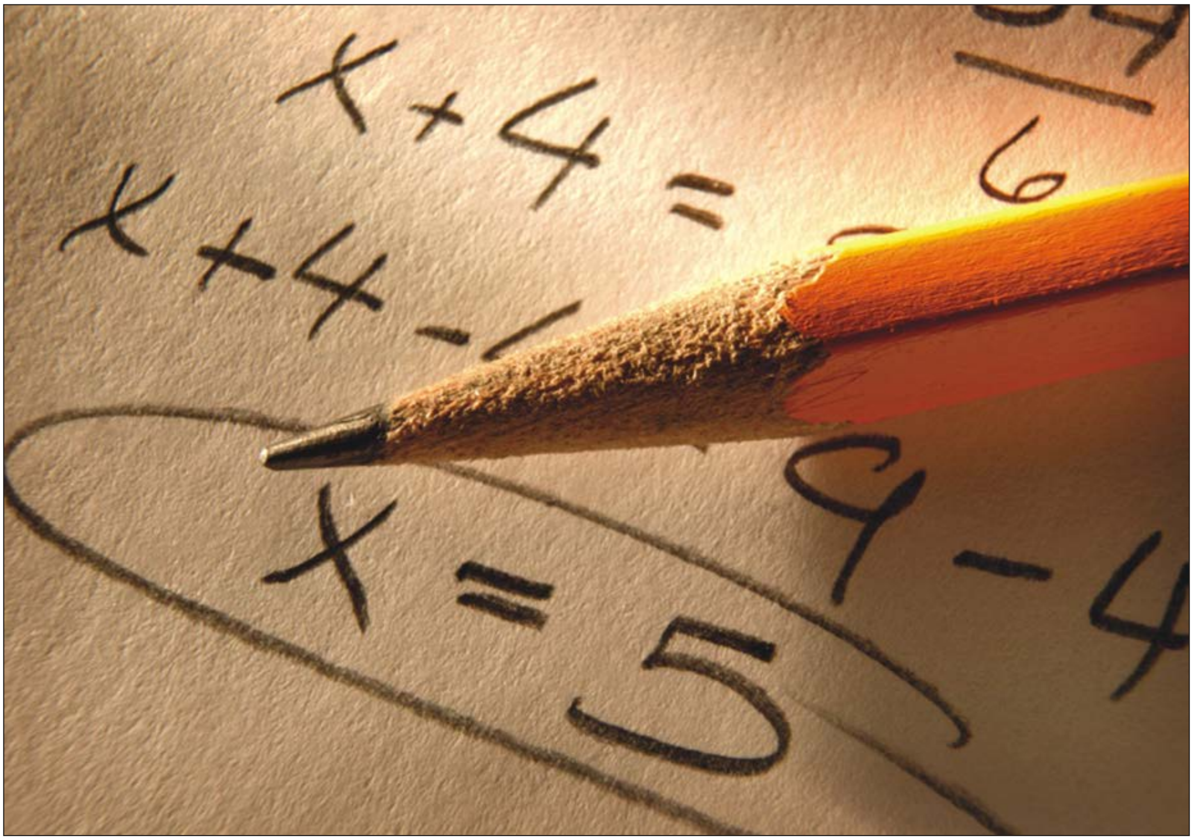

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Intuitionistic Fuzzy Groups*

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Abstract: The aim of this study is to introduce the notion of intuitionistic fuzzy groups based on the notion of intuitionistic fuzzy space. Indeed this approach is a generalization of the notion fuzzy groups based on fuzzy spaces. A correspondence relation between intuitionistic fuzzy groups and both fuzzy and ordinary groups is obtained, also a relation between intuitionistic fuzzy groups and classical intuitionistic fuzzy subgroups is obtained and studied.

Key words: Intuitionistic fuzzy space, intuitionistic fuzzy binary operation, intuitionistic fuzzy group

INTRODUCTION

The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. The notion of intuitionistic fuzzy set was first defined by Atanassov (1986) as a generalization of Zadeh's (1965) fuzzy set. After the concept of intuitionistic fuzzy set was introduced, several papers have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics. The difficulty in such generalizations lies in how to pick out the rational generalization from the large number of available approaches. The study of fuzzy groups was first started with the introduction of the concept of fuzzy subgroups by Rosenfeld (1971). Anthony and Sherwood (1979) redefined fuzzy subgroups using the concept of triangular norm. In his remarkable paper Dib (1994) introduced a new approach to define fuzzy groups using his definition of fuzzy space which serves as the universal set in classical group theory. Dib (1994) remarked the absence of the fuzzy universal set and discussed some problems in Rosenfeld's approach. In the case of intuitionistic fuzzy mathematics, there were some attempts to establish a significant and rational definition of intuitionistic fuzzy group. Zhan and Tan (2004) defined intuitionistic fuzzy subgroup as a generalization of Rosenfeld's fuzzy subgroup. By starting with a given classical group they define intuitionistic fuzzy subgroup using the classical binary operation defined over the given classical group. In this study, to overcome the problems that will occur due to the absence of the concept of intuitionistic fuzzy universal set, we introduce the notion of intuitionistic fuzzy group based on the notion of intuitionistic fuzzy space and intuitionistic fuzzy function defined by Fathi and Salleh (2008a, b), which will serve as a universal set in the classical case.

PRELIMINARIES

Here, we will recall some of the fundamental concepts and definitions required in the sequel.

Let $L = I \times I$, where $I = [0, 1]$. Define a partial order on L, in terms of the partial order on I, as follows: For every $(r_1, r_2), (s_1, s_2) \in L$:

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- $(r_1, r_2) \leq (s_1, s_2)$, if $r_1 \leq s_1$, $r_2 \leq s_2$, whenever $s_1 \neq s_2$
- $(0, 0) = (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$

Thus the cartesian product $L = I \times I$ is a distributive, not complemented lattice. The operation of infimum and supremum in L are given respectively by:
$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$ and $(r_1, r_2) \vee (s_1, s_2) = (r_1 \vee s_1, r_2 \vee s_2)$.

Definition 1 (Atanassov, 1986)
Let X be a nonempty fixed set. An intuitionistic fuzzy set A is an object having the form:

$$A \ \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where, the functions μ_A: $X \rightarrow I$ and ν_A: $X \rightarrow I$ denote the degree of membership and the degree of noumembership respectively of each element $x \in X$ to the set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Remark 1
The intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X will be denoted by $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ or simply $A = \left(x, \underline{A}(x), \overline{A}(x)\right)$ where, $\underline{A}(x) = \mu_A(x)$ and $\overline{A}(x) = \nu_A(x)$.

The support of the intuitionistic fuzzy set $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ in X is the subset A of X defined by:

$$A_\circ = \{x \in X : \underline{A}(x) \neq 0 \text{ and } \overline{A}(x) \neq 1\}$$

Definition 2
Let X be a nonempty set. An intuitionistic fuzzy space (simply IFS) denoted by (X, I , I) is the set of all ordered triples (x, I, I), where (x, I, I) = {(x, r, s): r, s∈I with $r+s \leq 1$ and x∈X} the ordered triplet (x, I, I) is called an intuitionistic fuzzy element of the intuitionistic fuzzy space (X, I, I) and the condition r, s∈I with $r+s \leq 1$ will be referred to as the 'intuitionistic condition.

Therefore an intuitionistic fuzzy space is an (ordinary) set with ordered triples. In each triplet the first component indicates the (ordinary) element while the second and the third components indicate its set of possible membership and noumembership values respectively.

Definition 3
Let U_0 be a given subset of X. An intuitionistic fuzzy subspace U of the IFS (X, I, I) is the collection of all ordered triples $(x, \underline{u}_x, \overline{u}_x)$, where, x∈U and $\underline{u}_x, \overline{u}_x$ are subsets of I such that u_x contains at least one element beside the zero element and $\underline{u}_x$ contains at least one element beside the unit. If $x \notin U_\circ$, then $\underline{u}_x = 0$ and $\overline{u}_x = 1$. The ordered triple $(x, \underline{u}_x, \overline{u}_x)$ will be called an intuitionistic fuzzy element of the intuitionistic fuzzy subspace U. The empty fuzzy subspace denoted by ϕ is defined to be:

$$\phi = \{(x, I, I) : x \in \phi\}$$

Remark 2
For the sake of simplicity throughout this paper by saying an intuitionistic fuzzy space X we mean the intuitionistic fuzzy space (X, I, I).

Let X be an intuitionistic fuzzy space and A be an intuitionistic fuzzy subset of X. The fuzzy subset A *induces* the following intuitionistic fuzzy subspaces:

The lower-upper intuitionistic fuzzy subspace induced by A:

$$H_{lu}(A)=\{(x,[0,\underline{A}(x)],[\overline{A}(x),1]):x\in A_0\}$$

The upper-lower intuitionistic fuzzy subspace induced by A:

$$H_{ul}(A)=\{(x,\{0\}\cup[\underline{A}(x),1],[0,\overline{A}(x)]\cup\{1\}):x\in A_0\}$$

The finite intuitionistic fuzzy subspace induced by A is denoted by:

$$H_0(A)=\{(x,\{0,\underline{A}(x)\},\{\overline{A}(x),1\}):x\in A_0\}$$

Definition 4

Let X and Y be any two intuitionistic fuzzy spaces. The intuitionistic fuzzy cartesian product denoted by (X, I, I)⊠(Y, I, I) is defined as follows:

$$(X,I,I)\boxtimes(Y,I,I)=\{((x,y),(r_1,r_2),(s_1,s_2)):x\in X,y\in Y\text{and}(r_1,r_2),(s_1,s_2)\in I\times I\}$$

Definition 5

An intuitionistic fuzzy relation ρ from an IFS X to an IFS Y is a subset of the intuitionistic fuzzy cartesian product (X, I, I)⊠(Y, I, I). An intuitionistic fuzzy relation from an IFS X into itself is called an intuitionistic fuzzy relation in the IFS X.

From the above definition we note that an intuitionistic fuzzy relation from an IFS X to an IFS Y is simply a collection of intuitionistic fuzzy subsets of (X, I, I)⊠(Y, I, I), also the intuitionistic fuzzy cartesian product (X, I, I)⊠(Y, I, I) is itself an intuitionistic fuzzy relation.

Definition 6

An intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is an intuitionistic fuzzy relation F from the X to Y satisfying the following conditions:

- For every x∈X with r, r, s∈I, there exists a unique element y∈Y with w, z∈I; such that ((x, y), (r, w), (s, z)) for some A∈F
- If $((x, y),(r_1, w_1), (s_1, z_1))\in A\in F$ and $((x, y'), (r_2, w_2), (s_2, z_2))\in B\in F$ then y = y'
- If $((x, y),(r_1, w_1), (s_1, z_1))\in A\in F$, and $((x, y'), (r_2, w_2), (s_2, z_2))\in B\in F$ then $(r_1>r_2)$ implies $(w_1>w_2)$ and $(s_1>s_2)$ implies $(z_1>z_2)$
- If $((x, y),(r_1, w_1), (s_1, z_1))\in A\in F$, then r = 0 implies w = 0, s = 1 implies z = 0 and r = 1 implies w = 1, s = 0 implies z = 1

Thus conditions 1 and 2 imply that there exists a unique (ordinary) function from X to Y, namely F: X→Y and that for every x∈X there exists unique (ordinary) functions from I to I, namely $\underline{f}_x,\overline{f}_x:I\to I$. On the other hand conditions 3 and 4 are respectively equivalent to the following conditions:

- $\underline{f}_x,\overline{f}_x$ are nondecreasing on I.
- $\underline{f}_x(0)=0=\overline{f}_x(1)$ and $\underline{f}_x(1)=1=\overline{f}_x(0)$

That is, an intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is a function F from X to Y characterized by the ordered triple:

$$\left(F(x),\{\underline{f}_x\}_{x\in X},\{\overline{f}_x\}_{x\in X}\right)$$

where, F(x) is a function from x to Y and $\{\underline{f}_x\}_{x\in X},\{\overline{f}_x\}_{x\in X}$ are family of functions from I to I satisfying the conditions (I) and (ii) such that the image of any intuitionistic fuzzy subset A of the IFS X under F is the intuitionistic fuzzy subset $F^{(A)}$ of the IFS Y defined by:

$$F(A)y = \begin{cases} \left(\bigvee_{x\in F^{-1}(y)} \underline{f}_x(\mu_A(x)), \bigwedge_{x\in F^{-1}(y)} \overline{f}_x(\nu_A(x))\right) & \text{if } F^{-1}(y) \neq \phi \\ (0,1) & \text{if } F^{-1}(y) = \phi \end{cases}$$

We will call the functions $\underline{f}_x, \overline{f}_x$ the comembership functions and the cononmembership functions, respectively. The intuitionistic fuzzy function F will be denoted by:

$$F = \left(F, \underline{f}_x, \overline{f}_x\right)$$

INTUITIONISTIC FUZZY GROUPS

Here, we introduce the concept of intuitionistic fuzzy group and study some of its properties. First we start by defining intuitionistic fuzzy binary operation on a given IFS.

Definition 7

An intuitionistic fuzzy binary operation F on an IFS (X, I, I) is an intuitionistic fuzzy function F: X×X→X with comembership functions $\underline{f}_{xy}$ and cononmembership functions $\overline{f}_{xy}$ satisfying:

- $\underline{f}_{xy}(r,s) \neq 0$ iff $r\neq 0$ and $s\neq 0$ and $\overline{f}_{xy}(w,z) \neq 1$ iff $w\neq 1$ and $z\neq 1$
- $\underline{f}_{xy}, \overline{f}_{xy}$ are onto. That is, $\underline{f}_{xy}(I\times I) = I$ and $\overline{f}_{xy}(I\times I) = I$

Thus for any two intuitionistic fuzzy elements (x, I, I), (y, I, I) of the IFS X and any intuitionistic fuzzy binary operation $F = \left(F,\underline{f}_{xy},\overline{f}_{xy}\right)$ defined on an IFS X. The action of the intuitionistic fuzzy binary operation $F = \left(F,\underline{f}_{xy},\overline{f}_{xy}\right)$ over the IFS X is given by:

$$(x,I,I)\ F(y,I,I) = F((x,I,I),(y,I,I)) = (F(x,y),\underline{f}_{xy}(I\times I),\overline{f}_{xy}(I\times I)) = (F(x,y),I,I)$$

An intuitionistic fuzzy binary operation is said to be uniform if the associated comembership and cononmembership functions are identical. That is, if $\underline{f}_{xy} = \underline{f}, \overline{f}_{xy} = \overline{f}$ for all x, y ∈X. A left semiuniform (right semiuniform) fuzzy binary operation is an intuitionistic fuzzy binary operation having identical comembership functions (cononmembership functions).

Definition 8

An intuitionistic fuzzy groupoid, denoted by ((X, I, I), F), is an IFS (X, I, I) together with an intuitionistic fuzzy binary operation F defined over it. A uniform (left semiuniform, right semiuniform) intuitionistic fuzzy groupoid is an intuitionistic fuzzy groupoid with uniform (left semiuniform, right semiuniform) intuitionistic fuzzy binary operation.

Theorem 1

Associated to each intuitionistic fuzzy groupoid ((X, I, I), F) where, $F = \left(F,\underline{f}_{xy},\overline{f}_{xy}\right)$ are two fuzzy groupoids, namely ((X, I), $\underline{F}$) and $\left((X,I),\overline{F}\right)$ where, $\underline{F}$ (F, $\underline{f}_{xy}$) and $\overline{F} = (F, 1-\overline{f}_{xy})$ which are isomorphic to the intuitionistic fuzzy groupoid ((X, I, I), F) by the correspondence (x, I, I)↔(x, I).

To each intuitionistic fuzzy groupoid ((X, I, I), F) there is an associated (ordinary) groupoid (X, F) which is isomorphic to the intuitionistic fuzzy groupoid by the correspondence (x, I, I)↔x.

Definition 9

The ordered pair (U; F) is said to be an intuitionistic fuzzy subgroupoid of the intuitionistic fuzzy groupoid ((X, I, I), F) iff U is an intuitionistic fuzzy subspace of the intuitionistic fuzzy space X and U is closed under the intuitionistic binary operation F.

Definition 10

An intuitionistic fuzzy semigroup is an intuitionistic fuzzy groupied that is associative. An intuitionistic fuzzy monoid is an intuitionistic fuzzy semigroup that admits an identity.

After defining the concepts of intuitionistic fuzzy groupoid, intuitionistic fuzzy semigroup and intuitionistic fuzzy monoid we introduce now the notion of intuitionistic fuzzy group.

Definition 11

An intuitionistic fuzzy group is an intuitionistic fuzzy monoid in which each intuitionistic fuzzy element has an inverse. That is, an intuitionistic fuzzy groupoid ((G, I, I), F) is an intuitionistic fuzzy group iff the following conditions are satisfied:

- For any choice of $(x,I,I),(y,I,I),(z,I,I)\in((G,I,I),F)\big((x,I,I)\,F\,(y,I,I)\big)F\,(z,I,I)=(x,I,I)\,F\big((y,I,I)F(z,I,I)\big)$ (associativity)
- There exists an intuitionistic fuzzy element (e, I, I) ∊(C, I, I) such that for all (X, I, I) in ((G, I, I), F) : (e, I, I)F(x, I, I) = (x, I, I)F(e, I, I) = (x, I, I) (existence of an identity)
- For every intuitionistic fuzzy element (x, I, I) in ((G, I, I), F) there exists an intuitionistic fuzzy element (x^{-1}, I, I) in ((G, I, I), F) such that: $(x, I, I)F(x^{-1}, I, I) = (x^{-1}, I, I) = (e, I, I)$ (existence of an inverse)

An intuitionistic fuzzy group ((G, I, I), F) is called an abelian (commutative) intuitionistic fuzzy group if and only if for all (x, I, I), (y, I, I)∊((G, I, I), F)(x, I, I)F(y, I, I) = (y, I, I)F(z, I, I).

By the order of an intuitionistic fuzzy group we mean the number of intuitionistic fuzzy elements in the intuitionistic fuzzy group. An intuitionistic fuzzy group of infinite order is an infinite intuitionistic fuzzy group.

Remark 3

Throughout this study by saying an intuitionistic fuzzy group we mean an intuitionistic fuzzy group based on intuitionistic fuzzy space and by saying a classical intuitionistic fuzzy group we mean an intuitionistic fuzzy subgroup based on Rosenfeld's approach.

Theorem 2

Associated to each intuitionistic fuzzy group ((G, I, I), F) where, $F=(F,\underline{f}_{xy},\overline{f}_{xy})$ two fuzzy groups, namely $((G,I),\underline{F}),\big((G,I),\overline{F}\big)$ where, $\underline{F}=(F,\underline{f}_{xy})$ and $\overline{F}=(F,1-\overline{f}_{xy})$ which are isomorphic to the intuitionistic fuzzy group ((G, I, I), F) by the correspondence (x, I, I)↔(x, I).

To each intuitionistic fuzzy group ((G, I, I), F) there is an associated (ordinary) group (G, F) which is isomorphic to the intuitionistic fuzzy group by the correspondence (x I, I)↔x.

As a result of Theorem 2 (which we will refer to as the associativity theorem) a sufficient and necessary condition for intuitionistic fuzzy group is given in the following corollary:

Corollary 1

Let (X, I, I) be an intuitionistic fuzzy space and let $F = (F, \underline{f}_{xy}, \bar{f}_{xy})$ be an intuitionistic fuzzy binary operation defined over (X, I, I). ((X, I, I), F) defines an intuitionistic fuzzy group iff ((X, I, $\underline{F}$)) and $((X, I), \bar{F})$ are both fuzzy groups.

Example 1

Consider the set G = {a}. Define the intuitionistic fuzzy binary operation $F = (F, \underline{f}_{xy}, \bar{f}_{xy})$ over the intuitionistic fuzzy space (G, I, I) such that: F (a, a) = a and $\underline{f}_{aa}(r,s) = r \wedge s, \bar{f}_{aa}(r,s) = r \vee s$.

Thus, the intuitionistic fuzzy space (G, I, I) together with F define a (trivial) intuitionistic fuzzy group ((G, I, I), F).

Consider the set Z_3 = {0, 1, 2}. Define the intuitionistic fuzzy binary operation $F = (F, \underline{f}_{xy}, \bar{f}_{xy})$ over the intuitionistic fuzzy space (Z_3, I, I) as follows: F (x, y) = $x +_3 y$, where, +3 refers to addition modulo 3 and $\underline{f}_{xy}(r,s) = r \cdot s, \bar{f}_{xy}(r,s) = r \cdot s$. Then (($Z_3$, I, I), F) is an intuitionistic fuzzy group.

The next theorem follows directly from the definition of intuitionistic fuzzy group and the associativity theorem.

Theorem 3

For any intuitionistic fuzzy group ((G, I, I), F), the following are true:

- The intuitionistic fuzzy identity element is unique
- The inverse of each intuitionistic fuzzy element (x, I, I)∈((G, I, I), F) is unique
- $((x^{-1})^{-1}$, I. I) = (x, I, I)
- For all (x, I, I), (y, I, I)∈((G, I, I), F), ((x. I, I)F(y, I, I))$^{-1}$ = (y^{-1}, I, I)F(x^{-1}, I, I)
- For all (x, I, I)F(y, I, I), (z, I, I)∈((G, I, I), F)

If $(x,I,I)F(y,I,I) = (z,I,I)F(y,I,I)$, then $(x,I,I) = (z,I,I)$
if $(y,I,I)F(x,I,I) = (y,I,I)F(z,I,I)$, then $(x,I,I) = (z,I,I)$

Definition 12

Let S be an intuitionistic fuzzy subspace of the IFS (G, I, I). The ordered pair (S; F) is an intuitionistic fuzzy subgroup of the IFG ((G, I, I), F), denoted by (S; F)≤((G, I, I), F), if (S; F) defines an IFG under the intuitionistic fuzzy binary operation F.

Obviously, if (S; F) is an intuitionistic fuzzy subgroup of ((G, I, I), F) and (T; F) is an intuitionistic fuzzy subgroup of (S; F), then (T; F) is an intuitionistic fuzzy subgroup of (G, I, I), F). Also if (G, I, I), F) is an IFG with an intuitionistic fuzzy identity (e, I, I) then both {(e, I, I), F} and ((G, I, I), F) are (trivial) intuitionistic fuzzy subgroups of ((G, I, I), F).

Theorem 4

Let $S = \{(x, \underline{s}_x, \bar{s}_x : x \in S_o\}$ be an intuitionistic fuzzy subspace of the intuitionistic fuzzy space (G, I, I). Then (S; F) is an intuitionistic fuzzy subgroup of the IFG ((G, I, I), F) iff:

- (S_o; F) is an (ordinary) subgroup of the group (G, F)
- $\underline{s}_x \underline{f}_{xy} \underline{s}_y = \underline{s}_{xFy}$ and $\bar{s}_x \bar{f}_{xy} \bar{s}_y = \bar{s}_{xFy}$, for all x, y∈$S_o$

Proof

If (1) and (2) are satisfied, then:

- The intuitionistic fuzzy subspace S is closed under F: Let $(x, \underline{s}_x, \bar{s}_x), (y, \underline{s}_y, \bar{s}_y)$ be in S then

$$(x,\underline{s}_x,\overline{s}_x)F(y,\underline{s}_y,\overline{s}_y) = F((x,\underline{s}_x,\overline{s}_x),(y,\underline{s}_y,\overline{s}_y)) = (F(x,y),\underline{f}_{xy}(\underline{s}_x,\underline{s}_y),\overline{f}_{xy}(\overline{s}_x,\overline{s}_y))) = ((xFy),\underline{s}_{xFy},\overline{s}_{xFy})$$

- (S; F) satisfies the conditions of intuitionistic fuzzy group:
  - Let $(x,\underline{s}_x,\overline{s}_x),(y,\underline{s}_y,\overline{s}_y)$ and $(z,\underline{s}_z,\overline{s}_z)$ be in S then:

$$\big((x,\underline{s}_x,\overline{s}_x)F(y,\underline{s}_y,\overline{s}_y)\big)F(z,\underline{s}_z,\overline{s}_z) = (F(x,y),\underline{f}_{xy}(\underline{s}_x,\underline{s}_y),\overline{f}_{xy}(\overline{s}_x,\overline{s}_y)))F(z,\underline{s}_z,\overline{s}_z)$$
$$= ((xFy)Fz,\underline{s}_{(xFy)Fz},\overline{s}_{(xFy)Fz}) = (xF(yFz),\underline{s}_{xF(yFz)},\overline{s}_{xF(yFz)}) = (x,\underline{s}_x,\overline{s}_x)F\big((y,\underline{s}_y,\overline{s}_y)F(z,\underline{s}_z,\overline{s}_z)\big)$$

  - Since (S_0; F) is an (ordinary) subgroup of the group (G, F) then S_0 contains the identity e. That is, $(e,\underline{s}_e,\overline{s}_e)\in S$ thus:

$$(x,\underline{s}_x,\overline{s}_x)F(e,\underline{s}_e,\overline{s}_e) = (F(x,e),\underline{f}_{xe}(\underline{s}_x,\underline{s}_e),\overline{f}_{xe}(\overline{s}_x,\overline{s}_e))) = (xFe,\underline{s}_{xFe},\overline{s}_{xFe})$$
$$= (eFx,\underline{s}_{eFx},\overline{s}_{eFx}) = (e,\underline{s}_e,\overline{s}_e)F(x,\underline{s}_x,\overline{s}_x) = (x,\underline{s}_x,\overline{s}_x)$$

Similarly we can show that $(e,\underline{s}_e,\overline{s}_e)F(x,\underline{s}_x,\overline{s}_x) = (x,\underline{s}_x,\overline{s}_x)$.

  - Again, since (S_0, F) is an (ordinary) subgroup of the group ((G, I, I), F) then S_0 contains the inverse element x^{-1} for each x∈s, that is, for each $(x^{-1},\underline{s}_x^{-1},\overline{s}_x^{-1})\in S$, then:

$$(x,\underline{s}_x,\overline{s}_x)F(x^{-1},\underline{s}_{x^{-1}},\overline{s}_{x^{-1}}) = (F(x,x^{-1}),\underline{f}_{xx^{-1}}(\underline{s}_x,\underline{s}_{x^{-1}}),\overline{f}_{xx^{-1}}(\overline{s}_x,\overline{s}_{x^{-1}}))) = (xFx^{-1},\underline{s}_{xFx^{-1}},\overline{s}_{xFx^{-1}})$$
$$= (x^{-1}Fx,\underline{s}_{x^{-1}Fx},\overline{s}_{x^{-1}Fx}) = (x^{-1},\underline{s}_{x^{-1}},\overline{s}_{x^{-1}})F(x,\underline{s}_x,\overline{s}_x) = (e,\underline{s}_e,\overline{s}_e)$$

Similarly we can show that $(x^{-1},\underline{s}_{x^{-1}},\overline{s}_{x^{-1}})F(x,\underline{s}_x,\overline{s}_x) = (e,\underline{s}_e,\overline{s}_e)$.

By (I) and (ii) we conclude that (S; F) is an intuitionistic fuzzy subgroup of ((G, I, I), F).

Conversely if (S; F) is an intuitionistic fuzzy subgroup of ((G, I, I), F) then (1) holds by the associativity theorem. Also $\underline{s}_x\underline{f}_{xy}\underline{s}_y = \underline{f}_{xy}(\underline{s}_x\times\underline{s}_y) = \underline{s}_{xFy}$ being onto over the partial ordered sub-lattice $\underline{s}_x\times\underline{s}_y$ of the vector lattice I×I.

Example 2

- Let ((G, I, I), F) be defined as in Example 2(1). Consider the fuzzy subspace $S = \{(a,[0,\alpha],[\beta,1])\}$ such that 0<α, β<1. Then (S, F) defines an intuitionistic fuzzy subgroup of ((G, I, I), F). If we consider S' = {(α, [0, γ], [δ, 1])} such that 0<γ, δ<1 and α≠γ, β≠δ then (S', F) defines an intuitionistic fuzzy subgroup of ((G, I, I), F), where, S≠S'. That is, a trivial intuitionistic fuzzy group can have more than one intuitionistic fuzzy subgroup, which is different from the case of ordinary groups, since an ordinary trivial group can only have one subgroup, namely the group itself.
- Let (Z_3, I, I), F) be defined as in Example 2(2). Consider the fuzzy subspace $S' = \{(a,[0,\gamma],[\delta,1])\}$ such that $0<\gamma, \delta<1$. Then (Z, F) is not an intuitionistic fuzzy subgroup of ((G, I, I), F), since Z is not closed under F. That is:

$$(0, [0, \alpha], [\beta, 1])F(1, [0, \gamma], [\delta, 1]) = (1, [0, \alpha\cdot\gamma], [\beta\cdot\delta, 1])\neq Z$$

If α, γ = 1 and β, δ = 0, then Z {(0, I, I), (1, I, I)} together with F defines an intuitionistic fuzzy subgroup of ((G, I, I), F).

INTUITIONISTIC FUZZY SUBGROUPS INDUCED BY INTUITIONISTIC FUZZY SUBSETS

Here, we introduce intuitionistic fuzzy subgroups induced by intuitionistic fuzzy subsets and then we obtain a relationship between intuitionistic fuzzy subgroups and classical fuzzy subgroups.

Let A be an intuitionistic fuzzy subset of the set G and let H_{lu} (A), H_{ul} (A) and H_o (A) be intuitionistic fuzzy subspaces induced by the fuzzy subset A. For these intuitionistic fuzzy spaces we can re-state Theorem 3.13 in the following manner.

Theorem 5

(H_{lu} (A), F), (H_{ul} (A), F) and (H_o, (A), F) are intuitionistic fuzzy subgroups of ((G, I, I), F) iff

- $xFy \in A_o$ for all $x, y \in A_o$
- $\underline{f}_{xy}(\underline{A}(x), \underline{A}(y)) = \underline{A}(xFy)$ and $\overline{f}_{xy}(\overline{A}(x), \overline{A}(y)) = \overline{A}(xFy)$

Definition 13

An intuitionistic fuzzy subset A of G induces intuitionistic fuzzy subgroups of ((G, I, I), F) iff (H_{lu} (A), F), (H_{ul} (A), F) and (H_o (A), F) are intuitionistic fuzzy subgroups.

Let ((G, I, I), F) with $F = (F, \underline{f}_{xy}, \overline{f}_{xy})$ be a uniform intuitionistic fuzzy group with $\underline{f}_{xy}, \overline{f}_{xy}$ having the t-norm properties, then we have the following theorem:

Theorem 6

- Every intuitionistic fuzzy subset A of G which induces intuitionistic fuzzy subgroups is a classical fuzzy subgroup of (G, F).
- If (S, F) is an ordinary subgroup of the group (G, F) then every intuitionistic fuzzy subset A of G for which $A_o = S$ induces an intuitionistic fuzzy subgroup of ((G, I, I), P) where, $P = \{P, \underline{p}_{xy}, \overline{p}_{xy}\}$ with P = F and $\underline{p}_{xy}, \overline{p}_{xy}$ are suitable comembership and cononmembership functions, respectively.

Proof

- If the intuitionistic fuzzy subset A induces intuitionistic fuzzy subgroups of ((G, I, I), P) then by Theorem 4.1 we have $\underline{f}_{xy}(\underline{A}(x), \underline{A}(y)) = \underline{A}(xFy)$ and $\overline{f}_{xy}(\overline{A}(x), \overline{A}(y)) = \overline{A}(xFy)$, for all $\underline{A}(x) \neq 0 \neq \underline{A}(y)$ and $\overline{A}(x) \neq 1 \neq \overline{A}(y)$

That is, if the intuitionistic fuzzy subset A induces intuitionistic fuzzy subgroups of ((G, I, I), F) then it satisfies the inequalities $\underline{f}_{xy}(\underline{A}(x), \underline{A}(y)) \leq \underline{A}(xFy)$ and $\overline{f}_{xy}(\overline{A}(x), \overline{A}(y)) \geq \overline{A}(xFy)$, for all $x, y \in G$. Therefore, A is a classical intuitionistic fuzzy subgroup.

- Let (S, F) be an ordinary subgroup of the group (G, F) and let A be an intuitionistic fuzzy subset of G for which $A_o = S$ and let $\underline{p}_{xy}, \overline{p}_{xy}$ be given t-norms. Define the intuitionistic fuzzy group ((G, I, I), F) as follows:

$$P = \{P, \underline{p}_{xy}, \overline{p}_{xy}\}$$

where, P = F and $\underline{p}_{xy}(r_1, r_2) = \underline{\psi}_{xy}(\underline{f}(r_1, r_2)), \overline{p}_{xy}(s_1, s_2) = \overline{\psi}_{xy}(\overline{f}(s_1, s_2))$ such that:

If $\underline{f}(\underline{A}(x), \underline{A}(y)) = 0$, then $\underline{\psi}_{xy}(t) = t$ for all $t \in I$. If $\underline{f}(\underline{A}(x), \underline{A}(y)) \neq 0$, then:

$$\underline{\psi}_{xy}(t)=\begin{cases}\dfrac{\underline{A}(xFy)}{\underline{f}(\underline{A}(x),\underline{A}(y))}t & \text{if } t\le \underline{f}(\underline{A}(x),\underline{A}(y))\\ 1+\dfrac{1-\underline{A}(xFy)}{1-\underline{f}(\underline{A}(x),\underline{A}(y))}(t-1) & \text{if } t\ge \underline{f}(\underline{A}(x),\underline{A}(y))\end{cases}$$

and if $\bar{f}(\bar{A}(x),\bar{A}(y))=1$, then $\bar{\psi}_{xy}(k)=k$ for all k∈I. If $\bar{f}(\bar{A}(x),\bar{A}(y))\neq 1$, then:

$$\bar{\psi}_{xy}(k)=\begin{cases}1+\dfrac{\bar{A}(xFy)}{\bar{f}(\bar{A}(x),\bar{A}(y))}(k-1) & \text{if } k\le \bar{f}(\bar{A}(x),\bar{A}(y))\\ \dfrac{\bar{A}(xFy)}{\bar{f}(\bar{A}(x),\bar{A}(y))}k & \text{if } k\ge \bar{f}(\bar{A}(x),\bar{A}(y))\end{cases}$$

It is clear that $\underline{\psi}_{xy}(r_1,r_2),\bar{\psi}_{xy}(s_1,s_2):x,y\in G$ are continuous comembership and cononmembership functions, respectively. Moreover $\underline{\psi}_{xy}(r_1,r_2)=0$ iff $r_1=0$ or $r_2=0$ and $\bar{\psi}_{xy}(s_1,s_2)=1$ iff $s_1=1$ or $s_2=0$. Hence P is an intuitionistic fuzzy binary operation on G.

Now, based on the property of the given t-norms $\underline{f}(r_1,r_2),\bar{f}(s_1,s_2)$ and the construction of $\underline{p}_{xy}(r_1,r_2),\bar{p}_{xy}(s_1,s_2)$ we notice that $\underline{f}(\underline{A}(x),\underline{A}(y))\neq 0$ whenever both $\underline{A}(x)$, $\underline{A}(y)$ have nonzero values and $\bar{f}(\bar{A}(x),\bar{A}(y))\neq 1$ whenever both $\bar{A}(x),\bar{A}(y)$ are not equal to one. That is:

$$\underline{p}_{xy}(\underline{A}(x),\underline{A}(y))=\underline{\psi}_{xy}\left(\underline{f}(\underline{A}(x),\underline{A}(y))\right)=\underline{A}(xFy),$$
$$\bar{p}_{xy}(\bar{A}(x),\bar{A}(y))=\bar{\psi}_{xy}\left(\bar{f}(\bar{A}(x),\bar{A}(y))\right)=\bar{A}(xFy)$$

Thus, by Theorem 4.1 and the assnmption that (S, F) is an ordinary subgroup A induces an intuitionistic fuzzy subgroup of the intuitionistic fuzzy group ((G, I, I), P).

Corollary 2

Every classical intuitionistic fuzzy subgroup A of (G, F) induces intuitionistic fuzzy subgroups relative to some intuitionistic fuzzy group (G, P).

CONCLUSION

In this study, we have generalized the study initiated in (Dib, 1994) about fuzzy groups to the context of intuitionistic fuzzy groups. In the absence of the intuitionistic fuzzy universal set, formulation of the intrinsic definition for an intuitionistic fuzzy subgroup is not evident. In this paper we define the notion of an intuitionistic fuzzy group using the notion of an intuitionistic fuzzy space. The use of intuitionistic fuzzy space as a universal set corrects the deviation in the definition of intuitionistic fuzzy subgroups. This concept can be considered as a new formulation of the classical theory of intuitionistic fuzzy groups.

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