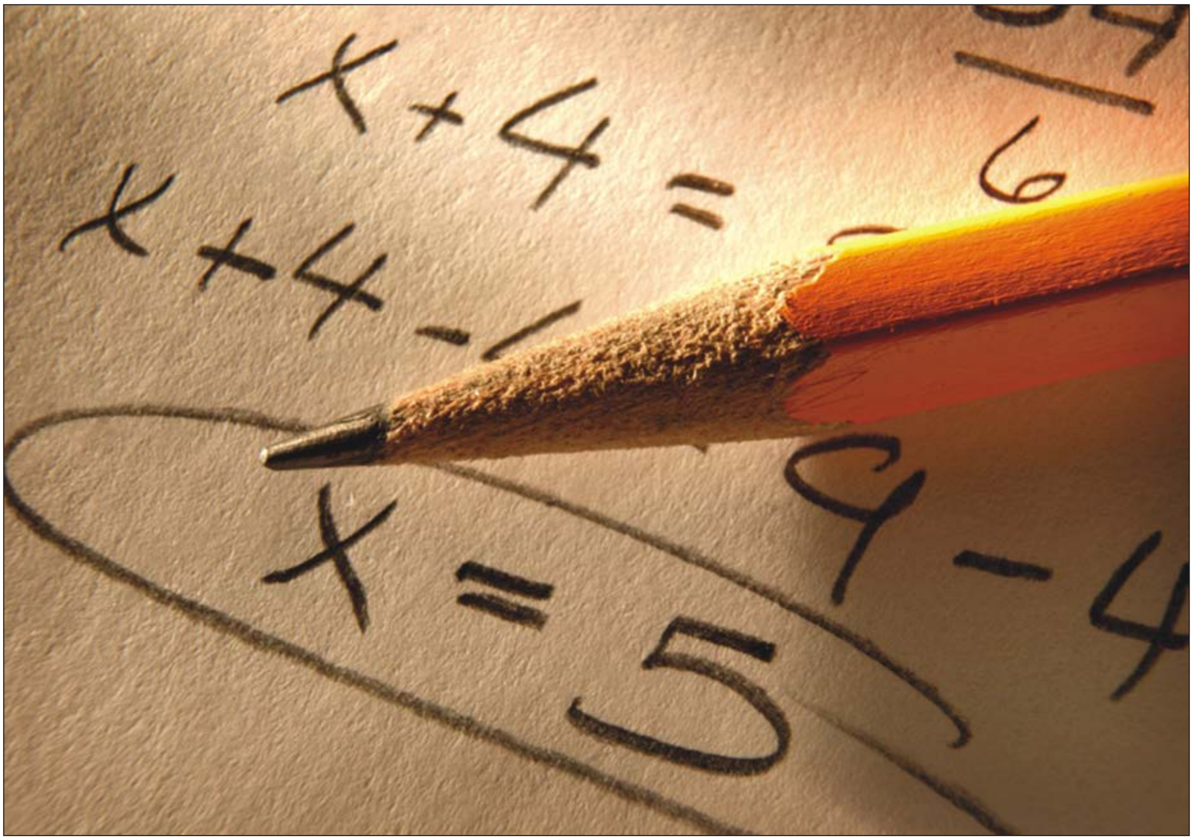

Asian Journal of Algebra

ISSN 1994-540X

Asian Journal of Algebra 3 (2): 53-58, 2010
ISSN 1994-540X / DOI: 10.3923/aja.2009.11.16

About Catalan-Mihailescu Theorem*

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Abstract: (MSC = 11) more than one century ago, the Belgian mathematician Eugene Catalan has formulated a famous conjecture. It became a theorem in 2004. The theorem stipulates that the following equation $Y^p = 1+X^q$ has only one solution, which is $3^2 = 1+2^3$ when X>1, Y>1, p>1, q>1 all integers. We prove in this research firstly that Catalan equation is equivalent to the following equation $Y^{q-p} = X^{p-1}$. After a little change of the data of the problem, we prove also that Catalan equation implies two other equations. Those equations allow to define convergent sequences. It is the Algebraic-Analytic approach which conducts to the impossibility of Catalan equation for p>2. The equation is simplified to the case p = 2, q = 3. It becomes consequently easy to prove that the only solution of Catalan equation is (X,Y,p,q) = (2,3,2,3).

Key words: Catalan, diophantine equations, algebraic proof

INTRODUCTION

More than one century after the formulation of the famous conjecture by the Belgian mathematician Eugene Catalan, Catalan Conjecture has been proved by Mihailescu (2004). It is no more an open problem, but it is still interesting for the researchers, because P. Mihailescu has opened the door to other proofs. We propose, in this research, a solution which is a variant of Fermat-Catalan conjecture one. It is based on an Algebraic-Analytic Approach that we have developed for Diophantine Equations.

THE PROOF

Let Catalan equation
$Y^p = 1+X^q$
with
p>1, q>1, X>1, Y>1
all integers, we pose
$X^p = a(1+Y^{q-p}),\ X^p + X^q = b(Y^{q-p}+Y^p),\ X^p + X^q = cY^{q-p}+Y^p,$
$X^p + dX^q = Y^{q-p} + dY^p$
$(a-c)(a-b)(b-c)(c-1)(b-1)(a-1)(d-1) = 0 \Rightarrow a = b = c = d = 1$
If we make the hypothesis that
$(a-c)(a-b)(b-c)(c-1)(b-1)(a-1)(d-1) \neq 0$, then

$$Y^p = X^q + 1 \Rightarrow Y^{q-p} = X^p - d = \frac{X^p - a}{a} = \frac{X^p - 1}{c} = \frac{X^p - b + (1-b)X^q}{b}$$

$$= \frac{d-a}{a-1} = \frac{d-1}{c-1} = \frac{a-1}{c-a} = \frac{d-b+(1-b)X^q}{b-1} = \frac{1-b+(1-b)X^q}{b-c}$$

$$\Rightarrow (a-c)X^p = a(1-c) \quad (1)$$

$$\Rightarrow (a-b)X^p = a(b-1)X^q \quad (2)$$

**Originally Published in Asian Journal of Algebra, 2009*

$$\Rightarrow (d-a)(c-a)=(a-1)^2 \Rightarrow dc-ac-ad+2a-1=0 \quad (3)$$

$$(d-a)(c-a)X^p=(a-1)^2X^p=(d-a)a(c-1)=a(a-1)(d-1)$$

$$a=1 \Rightarrow a=b=c=d=1, \text{ else } (a-1)X^p=a(d-1) \quad (4)$$

But (2) $\Rightarrow (a-1)X^p=(b-1)(X^p+aX^q)=a(d-1)$

$$=a(b-1)(Y^{q-p}+Y^p)=a(d-1)$$

$$\Rightarrow d-1=(b-1)(Y^{q-p}+Y^p)=b(Y^{q-p}+Y^p)-(Y^{q-p}+Y^p)=X^p+X^q-Y^p-Y^{q-p}=X^p-1-Y^{q-p}$$

$$\Rightarrow d=X^p-Y^{q-p}=X^p-\frac{X^p-1}{c}=\frac{(c-1)X^p+1}{c}$$

$$\Rightarrow cd-1=(c-1)X^p \quad (5)$$

But (4) $\quad (a-1)X^p=a(d-1)$ and (1)

$$\Rightarrow (a-1)(cd-1)=(ad-a)(c-1)$$

$$\Rightarrow acd-a-cd+1=acd-ac+a-ad \Rightarrow 2a+cd-ac-ad-1=0 \quad (6)$$

$$(c-a)(c-1)X^p=a(c-1)^2=(cd-1)(c-a) \Rightarrow c^2(a-d)-2ac+c+acd=0 \quad (7)$$

$$(c-a)(a-1)X^p=(ad-a)(c-a)=(ac-a)(a-1) \Rightarrow acd+2a^2-a^2d-a^2c-a=0 \quad (8)$$

Three Eq. (6-8) for three unknown a, c, d, with the evident solution $a=d=c=1 \Rightarrow a=b=c=d=1$

The initial hypothesis is false, the only solution is a = b = c = 1

Then

$$Y^p=1+X^q \Rightarrow Y^q=Y^p(X^p-1)$$

$$Y^q=Y^p(X^p-1)>Y^p \Rightarrow q>p$$

We conclude $1<X^p-1=Y^{q-p}<X^q+1=Y^p \Rightarrow p<q<2p$

$$Y^p=X^q+1>X^p+1>X^p \Rightarrow Y>X$$

But

$$\forall p,q; \exists r \mid r=p+q$$

then

$$Y^p=1+X^{r-p} \Rightarrow X^pY^p=X^r+X^p$$

it is Fermat-Catalan equation, it has no solution for p>2 and r>2. But r>2, the only solution is p = 2, which means $p=2 \Rightarrow 2p>q>p \Rightarrow 4>q>2 \Rightarrow q=3$. Let us prove it

$$X^r=(XY)^p-X^p$$

We pose

$$u=X^{2r}; x=X^r(XY)^p; y=-X^rX^p; z=-X^p(XY)^p$$

They verify

$$u=X^rX^r=X^r((XY)^p-X^p)=x+y$$

$$\frac{1}{z}=\frac{1}{-X^p(XY)^p}=\frac{X^{2r}}{-X^pX^r(XY)^pX^r}=\frac{u}{xy}=\frac{x+y}{xy}=\frac{1}{x}+\frac{1}{y}$$

We deduce

$$(x+y)z=xy$$

$$xz=y(x-z)=yx_2$$

$$yz=x(y-z)=xy_2$$

$$x_2y_2=z^2$$

$$x=x_2+z=x_2+\sqrt{x_2y_2}=\sqrt{x_2}(\sqrt{x_2}+\sqrt{y_2})$$

$$y=y_2+z=y_2+\sqrt{x_2y_2}=\sqrt{y_2}(\sqrt{x_2}+\sqrt{y_2})$$

The process is available until infinity, for I-1

$$x_{i-1} = \sqrt{x_i}(\sqrt{x_i} + \sqrt{y_i})$$
$$y_{i-1} = \sqrt{y_i}(\sqrt{x_i} + \sqrt{y_i})$$
$$x_{i-1} + y_{i-1} = (\sqrt{x_i} + \sqrt{y_i})^2$$
$$\frac{1}{z_i} = \frac{1}{x_i} + \frac{1}{y_i}$$

Lemma 1

x_i, y_i have the following expressions

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1}$$
$$y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1}$$

Proof of Lemma 1

$$x = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{x_2}(x + y)^{\frac{1}{2}} \Rightarrow x_2 = \frac{x^2}{x + y}$$
$$y = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{y_2}(x + y)^{\frac{1}{2}} \Rightarrow y_2 = \frac{y^2}{x + y}$$

It is verified for 2, we suppose it verified for I

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{x_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$
$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{y_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$
$$x_{i+1} = \frac{{x_i}^2}{x_i + y_i} = x^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})$$
$$= x^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$
$$y_{i+1} = \frac{{y_i}^2}{x_i + y_i} = y^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})$$
$$= y^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

It is proved by induction

$$\text{But } (x - y) \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = x^{2^{i-1}} - y^{2^{i-1}}$$

Lemma 2

For $x \neq y$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y) = X^r \frac{(XY)^{p2^{i-1}}}{(XY)^{p2^{i-1}} - X^{p2^{i-1}}}((XY)^p + X^p)$$
$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y) = X^r \frac{(X)^{p2^{i-1}}}{(XY)^{p2^{i-1}} - X^{p2^{i-1}}}((XY)^p + X^p)$$

Lemma 3

x_i-y_i = x-y

Lemma 4

The solution of lemmas 2, 3, 4 is
xy(x-y) = 0

Proof of Lemma 4

$$x_i - x_{i+1} = y_i - y_{i+1} = \sqrt{x_{i+1}y_{i+1}}$$
$$\forall i \geq 1$$
$$\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) = x - x_2 - x_2 + x_3 + ... + (-1)^i(x_{i-1} - x_i)$$
$$= x - 2x_2 + 2x_3 - ... + 2(-1)^i x_{i-1} + (-1)^{i+1} x_i$$
$$= y - y_2 - y_2 + y_3 + ... + (-1)^i(y_{i-1} - y_i)$$
$$= y - 2y_2 + 2y_3 - ... + 2(-1)^i y_{i-1} + (-1)^{i+1} y_i$$
$$\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) = 2\sum_{j=1}^{j=i} ((-1)^{j+1}x_j) - x - (-1)^{i+1}x_i = 2\sum_{j=1}^{j=i} ((-1)^{j+1}y_j) - y - (-1)^{i+1}y_i$$
$$= 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j) + x + (-1)^{i+1}x_i = 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}y_j) + y + (-1)^{i+1}y_i$$
$$\Rightarrow \sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) - 2\sum_{j=1}^{j=i} ((-1)^{j+1}x_j) + x_i + (-1)^{i+1}x = x_i - x + (-1)^{i+1}(x - x_i) = (x_i - x)(1 + (-1)^i)$$
$$= y_i - y + (-1)^{i+1}(y - y_i) = (y_i - y)(1 + (-1)^i) = \sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) - 2\sum_{j=1}^{j=i} ((-1)^{j+1}y_j) + y_i + (-1)^{i+1}y$$

and

$$\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) = 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j) + x + (-1)^{i+1}x_i = 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}y_j) + y + (-1)^{i+1}y_i$$
$$\Rightarrow \sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) - x_i - (-1)^{i+1}x - 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j)$$
$$= x - x_i - (-1)^{i+1}(x - x_i) = (x - x_i)(1 + (-1)^i)$$
$$= y - y_i - (-1)^{i+1}(y - y_i) = (y - y_i)(1 + (-1)^i)$$
$$= \sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) - y_i - (-1)^{i+1}y - 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}y_j)$$
$$\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) = 2\sum_{j=1}^{j=i} ((-1)^{j+1}x_j) - x - (-1)^{i+1}x_i = 2\sum_{j=1}^{j=i} ((-1)^{j+1}y_j) - y - (-1)^{i+1}y_i$$
$$= 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j) + x + (-1)^{i+1}x_i = 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}y_j) + y + (-1)^{i+1}y_i$$
$$= \sum_{j=1}^{j=i} ((-1)^{j+1}x_j) + \sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j) = \sum_{j=1}^{j=i} ((-1)^{j+1}y_j) + \sum_{j=2}^{j=i-1} ((-1)^{j+1}y_j)$$
$$\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) + x_i + (-1)^{i+1}x - 2\sum_{j=1}^{j=i} ((-1)^{j+1}x_j) = x_i - x + (-1)^{i+1}(x - x_i) = -(x - x_i)(1 + (-1)^i)$$
$$= -\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) + x_i + (-1)^{i+1}x + 2\sum_{j=2}^{j=i-1} ((-1)^{j+1}x_j)$$
$$= y_i - y + (-1)^{i+1}(y - y_i) = -(y - y_i)(1 + (-1)^i)$$
$$= -\sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) + y_i + (-1)^{i+1}y + 2\sum_{j=2}^{j=i-1} ((-1)y_j)$$
$$= \sum_{j=2}^{j=i} ((-1)^j\sqrt{x_j y_j}) + x_i + (-1)^{i+1}x - 2\sum_{j=1}^{j=i} ((-1)^{j+1}x_j)$$

$$=\frac{1}{2}(x_i+y_i)+\frac{(-1)^{i+1}}{2}(x+y)-\sum_{j=2}^{j=i}((-1)^j\sqrt{x_jy_j})+\sum_{j=2}^{j=i-1}((-1)^{j+1}y_j)+\sum_{j=2}^{j=i-1}((-1)^{j+1}x_j)$$
$$=\frac{1}{2}(x_i+y_i)+\frac{(-1)^{i+1}}{2}(x+y)-\sum_{j=1}^{j=i}((-1)^{j+1}x_j)-\sum_{j=2}^{j=i-1}((-1)^{j+1}x_j)+\sum_{j=2}^{j=i-1}((-1)^{j+1}y_j)+\sum_{j=2}^{j=i-1}((-1)^{j+1}x_j)$$
$$=\frac{1}{2}(x_i+y_i)+\frac{(-1)^{i+1}}{2}(x+y)+\sum_{j=2}^{j=i-1}((-1)^{j+1}y_j)-\sum_{j=1}^{j=i}((-1)^{j+1}x_j)$$
$$=\frac{1}{2}(x_i+y_i)+\frac{(-1)^{i+1}}{2}(x+y)+\sum_{j=2}^{j=i-1}((-1)^{j+1}y_j)-\sum_{j=2}^{j=i-1}((-1)^{j+1}x_j)-x-(-1)^{i+1}x_i$$
$$=\frac{1}{2}(x_i+y_i-2x)+\frac{(-1)^{i+1}}{2}(x+y-2x_i)+\sum_{j=2}^{j=i-1}((-1)^{j+1}(y_j-x_j))$$
$$=\frac{1}{2}(x_i+y_i-2x)+\frac{(-1)^{i+1}}{2}(x+y-2x_i)+\sum_{j=2}^{j=i-1}((-1)^{j+1}(y-x)$$
$$=\frac{1}{2}(x_i+y_i-2x)+\frac{(-1)^{i+1}}{2}(x+y-2x_i)+\frac{1}{2}(1+(-1)^{i+1})(y-x)$$
$$=\frac{1}{2}(x_i+y_i-2x)+\frac{1}{2}(y_i-x_i)+\frac{(-1)^{i+1}}{2}(x+y-2x_i)+\frac{(-1)^{i+1}}{2}(y-x)$$
$$=(y_i-x)+(-1)^{i+1}(y-x_i)=x_i-x+(-1)^{i+1}(x-x_i)$$
$$\Rightarrow(y_i-x)+(-1)^{i+1}(y-x_i)-(x_i-x+(-1)^{i+1}(x-x_i))=0$$
$$=y_i-x_i+(-1)^{i+1}(y-x)=(y-x)(1+(-1)^{i+1})=0$$
$$\forall i\geq 2$$
$$\Rightarrow y-x=0$$

We conclude that

$$x_i=x_{i+1}=x_{i+1}+\sqrt{x_{i+1}y_{i+1}}\Rightarrow y_{i+1}=0\Rightarrow y=0$$
$$y_i=y_{i+1}=y_{i+1}+\sqrt{x_{i+1}y_{i+1}}\Rightarrow x_{i+1}=0\Rightarrow x=0$$
$$x_i>x_{i+1},y_i>y_{i+1}\Rightarrow x=y$$

We have proved the lemma 4.

And there is no solution, it means that p = 2, effectively, the expression of the sequences for p = 1, is

$$x_i=\frac{x^{2^{i-1}}}{x^{2^{i-1}}-y^{2^{i-1}}}(x-y)=\frac{(XY)^{2^{i-2}.2}}{(XY)^{2^{i-2}.2}-X^{2^{i-2}.2}}X^r(XY+X)=\frac{(XY)^{2^{i-3}.4}}{(XY)^{2^{i-3}.4}-X^{2^{i-3}.4}}X^r(XY+X)$$

As there is an infinity of solutions for p = 1, the expressions of the sequences imply the existence of solutions for p = 2 and does not guarantee at all the existence of the sequences for p>2 and I = 2. Then, p = 2 and q = 3 is the only solution.

The equation becomes

$$Y^2=X^3+1=(X+1)(X^2-X+1)$$
$$Y^q=Y^3=Y^p(X^p-1)=Y^2(X^2-1)\Rightarrow Y=(X-1)(X+1)$$
$$Y^p=Y^2=(X-1)^2(X+1)^2=X^q+1=X^3+1=(X+1)(X^2-X+1)$$
$$\Rightarrow(X-1)^2(X+1)=X^2-X+1\Rightarrow(X^2-2X+1)(X+1)=X^2-X+1$$
$$\Rightarrow X^3-X^2-X+1=X^2-X+1\Rightarrow X^2-X-1=X-1\Rightarrow X-1=1\Rightarrow X=2$$
$$\Rightarrow Y^2=X^3+1=2^3+1=9\Rightarrow Y=3$$

The only solution of Catalan equation is effectively
$(X,Y,p,q)=(2,3,2,3)$

CONCLUSION

Catalan equation has effectively only one solution an elementary proof exists. It seems that many open problems of number theory can be solved by the same way. How ? We showed one solution.

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