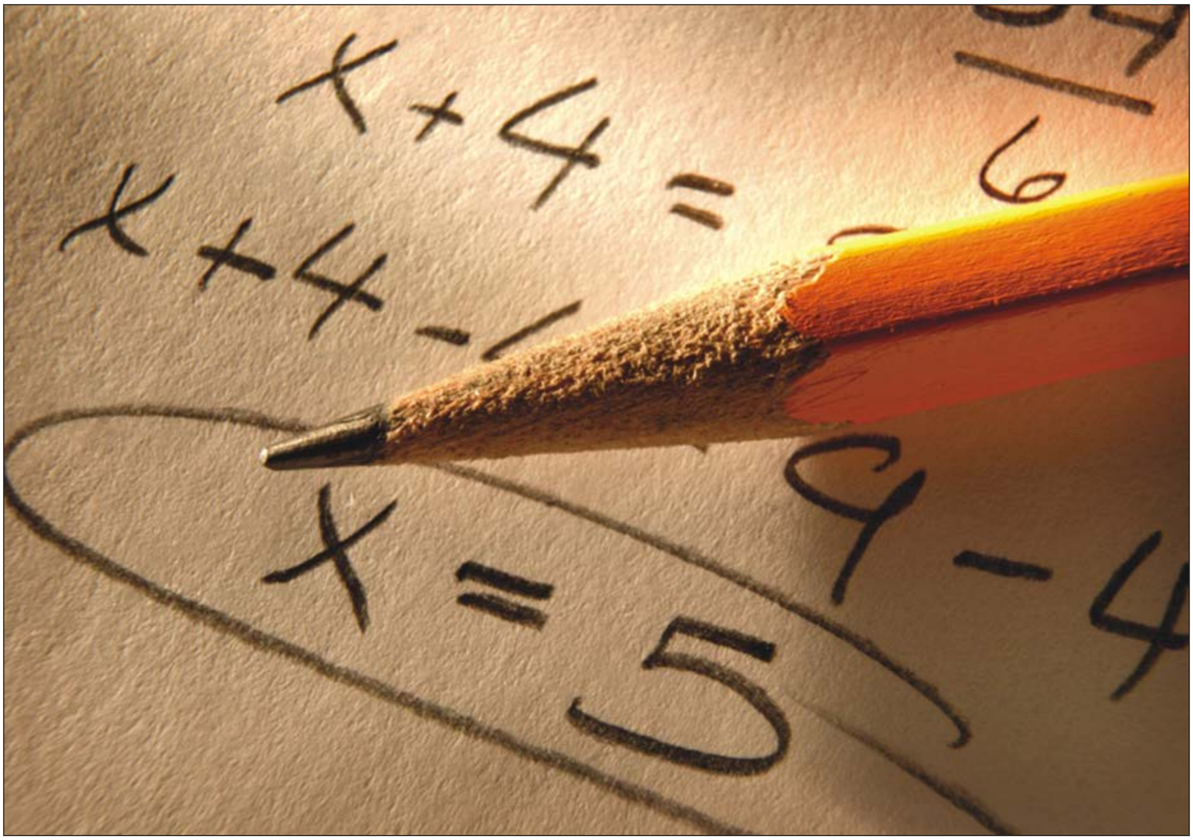

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Weakly C*-Normal Subgroups and p-Nilpotency of Finite Groups*

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Abstract: A subgroup H is called to be weakly c*-normal in G if there exists a subnormal subgroup K such that G = HK and H ∩ K is s-quasi normal embedded in G.The following result is established: Let G be a group such that G is S_4-free. Also let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every minimal subgroup of P of order p or 4 (when p = 2) is weakly c*-normal in $N_G(P)$ and when p = 2 P is quaternion-free, then G is p-nilpotent.The main result is established and a generalization of some authors'.

Key words: Sylow p-subgroups, weakly c*-normal subgroups p-nilpotent, minimal subgroup

INTRODUCTION

A subgroup is quasi normal in G if for every subgroup K of G such that HK = KH, by Ore (1937) which is a generalization of normality. A subgroup H of G is s-quasi normal if H permutes with all Sylow subgroups of G, by Kegel (1962) and extensive studied by Deskins (1963). A subgroup H is c*-normal in G if there exists a normal subgroup K of G such that G = HK and H ∩ K is s-quasi normal embedded in G, by Wei and Wang (2007). Recently, Liu (2009) established some results on the base of weakly c*-normal subgroups of finite groups.

SOME DEFINITION AND PRELIMINARIES

Lemma 1

Suppose that U is s-quasi normally embedded in a group G, $H \le G$ and $K \trianglelefteq G$. Then:

- If $U \le H$, then U is s-quasi normally embedded in H
- If UK is s-quasi normally embedded in G, then UK/K is s-quasi normally embedded in G/K
- $K \le H$ and H/K is s-quasi normally embedded in G/K, then H is s-quasi normally embedded in G

Proof

Lemma 1 by Ballester-Bolinches and Pearaza-Anguilera (1998).

Lemma 2

Let G be a group. Then we have,

- If H is weakly c*-normal in G and $H \le$ weakly c*-normal in K
- If $N \trianglelefteq G$ and $N \le H$. Then H is H is weakly c*-normal in G if and ouly if H/N is H is weakly c*-normal in G/N
- Let be a set of primes. H is a -subgroup of G and N a normal -subgroup of G. If H is weakly c*-normal in G. Then HN/N is weakly c*-normal in G/N

Proof

Lemma 2.2 by Liu (2009).

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Lemma 3

Let G be a group, K an s-quasi normal subgroup of G, P a Sylow p-subgroup of K, where, p is prime number of |G|. If $P \leq O_p(G)$ or $K_G = 1$, then P is s-quasi normal in G.

Proof

Lemma 2.5 by Wei and Wang (2007).

Lemma 4

Let G be a group and P a s-quasi normal p-subgroups of G, where, P is a prime number of |G|, then $O^p(G) \leq N_G(P)$.

Proof

Lemma 2.2 by Li *et al*. (2003).

Lemma 5

Let G be a group and p a prime dividing |G| with (|G|, p-1) = 1.Then:

- If N is normal in G of order p, then N is in Z(G)
- If G has a cyclic Sylow p-subgroup, then G is p-nilpotent
- If $M \leq G$ and |G:M| = p, then $M \triangleleft G$

Proof

Lemma 2.8 by Wei and Wang (2007).

Lemma 6

Let P be a p-subgroup of a group G and N a normal π'-subgroup of G for some prime p. If A is a minimal subgroup of P and A is weakly c*-normal in $N_G(P)$, then AN/N is weakly c*-normal in $N_G(P)N/N$.

Proof

If A is normal in G, then AN/N is weakly c*-normal in $N_G(P)N/N$. If A is not normal in G, then by hypotheses, there exists a subgroup K of $N_G(P)$ such that $N_G(P) = AK$ and $A \cap K = 1$. Obviously $N_G(P)N/N = (AN/N)(KN/N)$. If $(AN/N)\cap(KN/N) \neq 1$, then $K \leq AN$ and therefore $N_G(P)N/N = KN/N$. By comparing the order, there has a contradiction. So, AN/N is weakly c*-normal in $N_G(P)N/N$.

Lemma 7

Let p be the smallest prime dividing |G| and P a Sylow p-subgroup of G. If every minimal subgroup of P is AN/N is weakly c*-normal in G and when p = 2 either every cyclic subgroup of P is weakly c*-normal in G or P is quaternion-free, then G is p-nilpotent.

Proof

Suppose that the result is false and Let G be a minimal counter example. By lemma 2(1), the hypotheses is inherited by subgroups. Therefore, G is minimal non-p-nilpotent group. By Ito (Robinson, 2003, Theorem 10.3.3) G is a minimal non-nilpotent. Then G is of order $p^\alpha q^\beta$, where, q is a prime, $q \neq p$, P is normal in G and any Sylow q-subgroup Q of G is cyclic. Moreover, P is of exponent p if p is odd and exponent at most 4 if p = 2 (Robinson, 2003).

Let A be a minimal subgroup of P. Then by hypotheses, there exists a subgroup K of G such that G = AK and $A \cap K$ is s-quasi normally embedded in G. If A is not normal in G, then K is a maximal subgroup of G of index p. Since, P is the smallest prime divisor of G. This leads that K is normal in G and so the Sylow q-subgroup of K are normal in G, thus G is nilpotent, a contradiction. So, A is normal in G, this leads that A is in the center of G. If either p is odd, or p = 2 and every cyclic subgroup of P is weakly c*-normal in G, then G is p-nilpotent by Ito'lemma. Then let B = $<b>$ be a subgroup of P of order 4. Then by hypotheses, there exists a subnormal subgroup K such that G = BK and $B \cap K$ is s-quasinormally embedded in G. If $|G:K| = 4$, then $K<b^2>$ is a subgroup of index 2 and therefore is normal in G. This implies that the Sylow q-subgroup are normal in G. Then G is nilpotent. A contradiction. If $|G:K| = 2$, then K is normal in G, we also get a contradiction. Then B is normal in G. If $B \neq P$, then, since G is a minimal non-nilpotent group and the exponent of P is at most 4, we have $P \leq C_G(Q)$ and $G = P \times Q$ is nilpotent, another contradiction. The lemma is proved.

MAIN RESULTS

Theorem 1

Let G be a group such that G is S_4-free. Also let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every minimal subgroup of P of order p or 4(when p=2) is weakly c*-normal in $N_G(P)$ and when p = 2 P is quaternion-free, then G is p-nilpotent.

Proof

Assume that the theorem is false and let G be a counter example of minimal order. Then:

(1) $O_{P'}(G) = 1$

If $O_{P'}(G) \neq 1$, then we can chose a minimal normal subgroup N of G such that $N \leq O_{P'}(G)$. Now consider the quotient group G/N. Obviously PN/N is a Sylow p-subgroup of G/N. By lemma 6, AN/N is weakly c*-normal in $N_G(P)N/N$. The minimality of G implies that G/N is p-nilpotent and hence G is p-nilpotent, a contradiction. Thus $O_{P'}(G) = 1$.

(2) For every subgroup M of G satisfying $P \leq M < G$, M must be p-nilpotent. In particular, $N_G(P)$ is p-nilpotent

If $N_G(P) = G$, then, by lemma 7, G is p-nilpotent. Hence, $N_G(P) < G$. Since $N_G(P) \cap M \leq N_M(P) \leq N_G(P)$, M satisfying the hypotheses of our theorem. The minimal choice of G implies that M is p-nilpotent.

(3) G is solvable. Furthermore, P is a maximal subgroup of G and a Hall p'-subgroup of G is an elementary abelian q-subgroup Q for some prime q

Since, G is not p-nilpotent, by Frobenius' theorem (Robinson, 2003), theorem 10.3.2), there exists a subgroup H of P such that $N_G(H)$ is not p-nilotent. So by (2) we think that $N_G(H)$ is not p-nilpotent but $N_G(K)$ is p-nilpotent for every subgroup K of P such that $K < K \leq P$. Now we show $N_G(H) = G$. Suppose that $N_G(H) < G$. Then, we $H < P^* \leq P$ for some Sylow p-subgroup P* of $N_G(H)$. Since every minimal subgroups of P* of order p or 4 is weakly c*-normal in $N_G(P)$. On the other hand, by the choice of H, $N_G(P^*)$ is p-nilpotent and so $N_{N_G(H)}(P^*)$ is p-nilpotent. This implies that $N_G(H)$ satisfying the hypotheses of our theorem for its Sylow p-subgroup P* of $N_G(H)$. Now, the choice of G implies that $N_G(H)$ is p-nilpotent, a contradiction. Hence, $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for every subgroup K of P with $O_p(G) \leq K \leq P$. Now, by Frobenius' theorem (Robinson, 2003), theorem 10.3.2), $G/O_p(G)$ is p-nilpotent and hence G is p-nilpotent. By the odd order thorem, G is solvable.

Let $T/O_p(G)$ be a chief factor of G. Then $T/O_p(G)$ is an elementary abelian q-group for some prime $q \neq p$ and there exists a Sylow q-subgroup Q of T such that $T = QQ_p(G)$. It is clear that PT = PQ. If PT<G, then, by (2), PT is p-nilpotent and so $Q \leq C_G(O_p(G))$, which contradicts the fact $C_G(O_p(G)) \leq O_p(G)$ since, G is solvable. Hence, G = PQ and Q is a Hall p'-subgroup of G. The minimality of $T/O_p(G)$ implies that $P/O_p(G)$ is a maximal subgroup of $G/O_p(G)$ and therefore P is a maximal subgroup of G.

(4) $G = O_p(G)L$, where, L is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-subgroup $\langle a \rangle$, $a^p \in Z(L)$ and the action of a on Q is irreducible

Let $P_1/O_p(G)$ be a normal p-complement of $P/O_p(G)$. By Schur-Zassenhaus' theorem we have $D = O_p(G)Q$.

Let $P_1/O_p(G)$ be a maximal subgroup of $P/O_p(G)$. Then $N(G)P_1 = P$ or G. If $N(G)P_1 = P$, then $N(H)P_1 = P_1$, where, H = P D = PQ. Then H satisfying the hypotheses of the theorem. Since the minimality of G, we have that H is p-nilpotent. Then $O_p(G)Q = O_p(G) \times Q$ and Q is normal in G,a contradiction. Hence, $P1 \trianglelefteq G$. So, $O_p(G) = P_1$ and $P/O_p(G)$ is a cyclic group. On the other hand, by the Frattini argument, $G = O_p(G)N_G(Q)$. Since, P is not normal in G, so we assume that $G = O_p(G)L$, where, $L = \langle a \rangle \rhd\langle Q$ is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-subgroup $\langle a \rangle$. Since, $|P:O_p(G) = p|$ and $O_p(G) \cap N_G(Q) \trianglelefteq N_G(Q)$, $a_p \in Z(L)$. Also, since P is a maximal subgroup of G, $O_p(G)Q/O_p(G)$ is a minimal normal subgroup of $G/O_p(G)$ and therefore the action of a (by conjugation) on Q is irreducible.

(5) If $\Omega_1(O_p(G)) \cap \langle a \rangle = 1$, then $[\Omega_1(O_p(G)),Q] = 1$

Set $G_1 = \Omega_1(O_p(G))L$. Obviously, $\Omega_1(O_p(G))$ is an elementary abelian and characteristic in $O_p(G)$. Since, for any $1 \neq x \in \Omega_1(O_p(G))$, $\langle x \rangle$is normal in G, $\langle x \rangle\langle a \rangle = \langle a \rangle\langle x \rangle$. Hence $x^a \in \Omega_1(O_p(G)) \cap (\langle x \rangle\langle a \rangle)$. This implies that a induces a power automorphism of p-power order in the elementary abelian p-group $\Omega_1(O_p(G))$. Thus $[\Omega_1(O_p(G), a] = 1$. If there exists an element $1 \neq x \in \Omega_1(O_p(G))$ and an element $1 \neq g \in Q$ such that $x^g = x_1 \neq x$, then $x^{a^{-1}ga} = x_1$ and therefore $x^{a^{-1}gag^{-1}} = x$. It follows that $\langle \Omega_1(O_p(G)), \langle a \rangle, a^{-1} gag^{-1} \rangle \in C_{G1}$. Since, the action of a on Q is irreducible, $Q\Omega_1(O_p(G))/\Omega_1(O_p(G))$ is a minimal normal subgroup of $G_1/O_p(G)$ and $\Omega_1 O_p(G)) \langle a \rangle$ is a maximal subgroup in G_1. Thus, $CG_1(x) = \Omega_1(O_p(G)) \langle a \rangle$ or G_1. But $1 \neq a^{-1}gag^{-1} \in Q$. Hence $CG_1(x) = G_1$, in contradiction to $x^g \neq x$. So $[\Omega_1(O_p(G)), Q] = 1$.

(6) The final contradiction

We consider the following two cases:

Case 1

p>2 or p = 2 and P is quaternion-free. Set $G_1 = \Omega_1(O_p(G))L$. If $\Omega_1(O_p(G)) \cap \langle a \rangle = 1$, then by (5) $[\Omega_1(O_p(G)), Q] = 1$.

Assume that $\Omega_1(O_p(G)) \cap \langle a \rangle = \langle a^{pa} \rangle$. Then $\langle a^{pa} \rangle$ is a cyclic group with order p and $\langle a^{pa} \rangle \leq Z(G_1)$ since $a^p \leq Z(L)$. Consider th quotient $G_1/\langle a^{pa} \rangle$. It is clear that $(\Omega_1(O_p(G))/\langle a^{pa} \rangle) \cap (\langle a \rangle/\langle a^{pa} \rangle) = 1$and every subgroup K of $\Omega_1(O_p(G))$ of order p is weakly c*-normal in $N_G(P)$ by hypotheses. Then there exists a subnormal subgroup H such that $N_G(P) = HK$ and $H \cap K$ is s-quasi normally embedded in G. Let W denote $H \cap K$. Then W is a Sylow p-subgroup of some s-quasi normal subgroup M of G and so W is normal in M with $Q \leq M$. Since MP = PM, $WQ = W \times Q$. So, $[O_p(G), Q] = 1$ and Q is normal in G, a contradiction.

Case 2

$p = 2$ and every cyclic subgroup of with order 2 or 4 is weakly c*-normal in $N_G(P)$. Let $G_2 = \Omega_2(O_p(G))L$. If $\Omega_1(O_p(G)) \cap \langle a\rangle = 1$, then by (5) $[\Omega_1(O_p(G)), Q] = 1$. Now assume that $\Omega_1(O_p(G)) \cap \langle a\rangle = A = \langle c\rangle$ is a cyclic group of order 2. It is clear that $A \le Z(\Omega_1(O_p(G)))$. Let $x \in \Omega_2(O_p(G))$ with order 4. By hypothesis, $\langle x\rangle$ is weakly c*-normal in $N_G(P)$. There exists a subnormal K of $N_G(P)$ such that $N_G(P) = \langle x\rangle K$ and $\langle x\rangle \cap K$ is s-quasi normally embedded in G. Set $W = \langle x\rangle \cap K$, then there exists a s-quasi normal subgroup H of G such that W is a Sylow p-subgroup of H. If W = H, then W is s-quasi normal in G, so WQ = QW is p-nilpotent by lemma 5 and by lemma 4 Q is normal in G, a contradiction. If H=G, there has nothing to prove. So, we have $W<H<G$. If G = PH, then G = PH = PQ. Therefore, $Q<H$ and $PQ<G$, a contradiction. Then $PH<G$ and by (2) PH is p-nilpotent. Let Q* be the normal p-complement of PH, PH = PQ* = Q*P and so WQ* = Q*W = $W \times Q^*$. So, W is normal in G, by (1) G/W is p-nilpotent, then G is p-nilpotent, contradiction.

Remark 1

The hypothesis that G is S_4-free can't be removed. Let $G = S_4$, P the Sylow p-subgroup. Then $P = N_G(P)$ and every minimal subgroup of P is weakly c*-normal in $N_G(P)$, But G is not 2-nilpotent.

Remark 2

The hypothesis that P is quaternion-free can not be removed. Let $A = \langle a, b: a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$ be a quaternion group, then A has an automorphism of order 3. Let $G = \langle \alpha\rangle \rhd \langle A$, clear then every element of G with order 2 lies in that center of G and is weakly c*-normal in $N_G(P)$, Bu G is not 2-nilpotent.

Corollary 1

Let G be a finite group, p a prime dividing the order of G such that $(|G|,p-1) = 1$. If there exists a normal subgroup N of G such that G/N is p-nilpotent and every subgroup of prime and order 4 of G is s-quasi normally embedded in G, then G is p-nilpotent.

Proof

Theorem 4.1 by Li *et al*. (2005).

CONCLUSION

Let G be a group such that G is S_4-free. Also let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every minimal subgroup of P of order p or 4 (when $p = 2$) is weakly c*-normal in $N_G(P)$ and when $p = 2$ P is quaternion-free, then G is p-nilpotent.

REFERENCES

Ballester-Bolinches, A. and M.C. Pedraza-Aguilera, 1998. Sufficient conditions for supersolvability of finite groups. J. Pure Appl. Algebra, 127: 113-128.

Deskins, W.E., 1963. On quasi normal subgroups of finite groups. Math. Z., 82: 125-132.

Kegel, O., 1962. Sylow-gruppen and subnormalteiler endlincher gruppen. Math. Z., 78: 205-221.

Li, Y., Y. Wang and H. Wei, 2003. The influence of p-quasinormality of some subgroups of a finite group Archiv der Mathematic, 81: 245-252.

Li, Y., Y. Wang and H. Wei, 2005. On p-nilpotency of finite groups with some subgroups p-quasi normally embedded. Acta Math. Hungar., 108: 283-298.

Liu, S., 2009. On the p-nilpotency of finite groups, to appear. J. Math. Res.

Ore, O., 1937. Structure and group theory I. Duke Math. J., 3: 149-174.

Robinson, D.J.S., 2003. A Course in the Theory of Groups. 2nd Edn., Springer-Valag, New York.

Wei, H. and Y. Wang, 2007. On C*-normality and its properties. J. Group Theory, 10: 211-223.