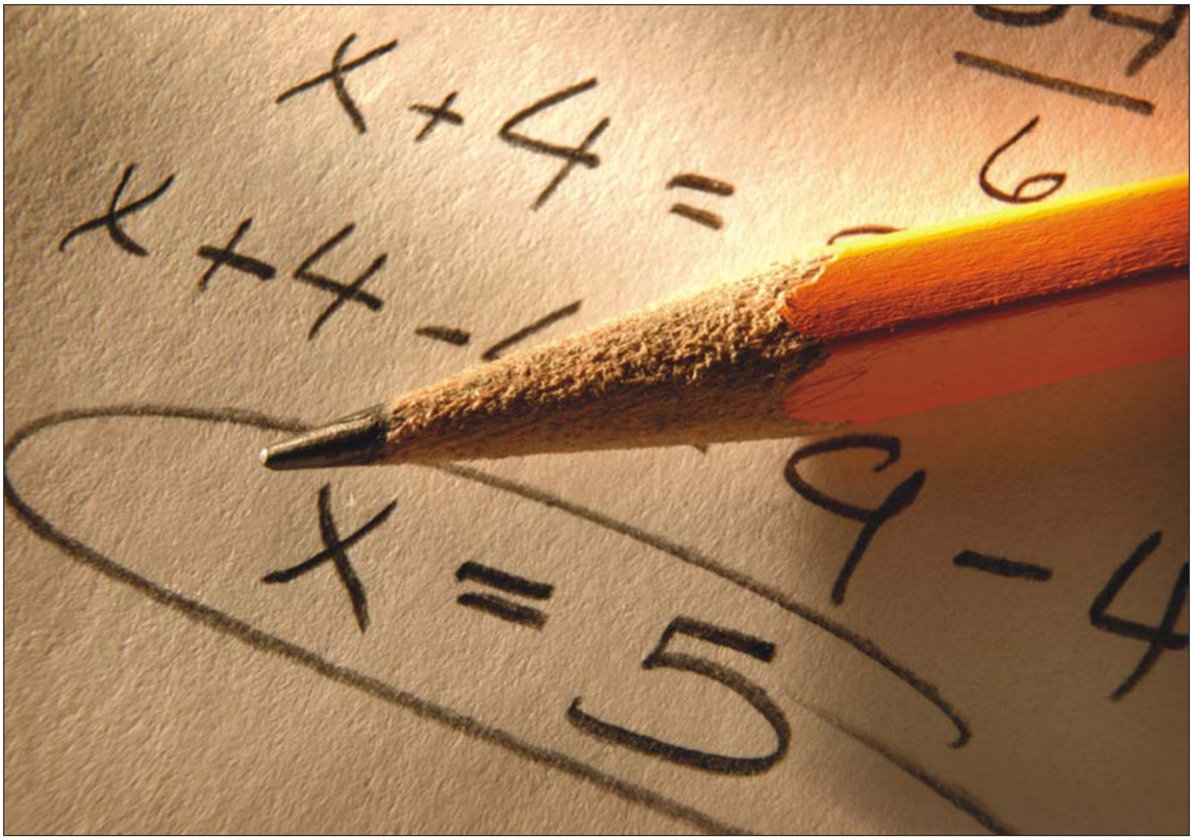

Asian Journal of Algebra

ISSN 1994-540X

Asian Journal of Algebra 4 (1): 12-22, 2011
ISSN 1994-540X / DOI: 10.3923/aja.2011.12.22

Pre A*-Algebra as a Semilattice

[1]Y. Praroopa and [2]J. Venkateswara Rao
[1]Andhra Loyola Institute of Engineering and Technology, Vijayawada-8, A.P., India
[2]Mekelle University, Main Campus, P.O. Box No. 231, Mekelle, Tigrai, Ethiopia

Corresponding Author: Y. Praroopa, Andhra Loyola Institute of Engineering and Technology, Vijayawada-8, A.P., India

ABSTRACT

This paper is a study on algebraic structure of Pre A*-algebra. First we define partial ordering on Pre A*-algebra. We prove if A is a Pre A*-algebra then (A, ≤) is a poset. We define a semilattice on Pre A*-algebra. We prove Pre A*-algebra as a semilattice. Next we prove some theorems on semilattice over a Pre A*-algebra. We define distributive and modular semilattices on Pre A*-algebra We define complement, relative complement of an element in Pre A*-algebra. We define complemented semilattice, relatively complemented semilattices in Pre A*-algebra. We give some examples of these semilattices in Pre A*-algebra. We define weakly complemented, semi-complemented, uniquely complemented semilattices in Pre A*-algebra. We prove some theorems on these semilattices in Pre A*-algebra.

Key words: Pre A*-algebra, semilattice, complemented semilattice

INTRODUCTION

The study lattice theory had been made by Birkhoff (1948). In a drafted paper "The Equational theory of Disjoint Alternatives", around 1989, Manes (1989) introduced the concept of Ada $(A, \wedge, \vee)$ $(-)^{I}$, $(-)_{\pi}$, 0, 1, 2) which however differs from the definition of the Ada. While the Ada of the earlier draft seems to be based on Boolean algebras, the latter concept is based on C-algebras $(A, \wedge, \vee)$ $(-)^{\sim}$ introduced by Guzman and Squier (1990).

In 1994, Koteswara Rao (1994) firstly introduced the concept of A* algebra $(A, \wedge, \vee, *, (-)^{\sim}, (-)_{\pi}, 0, 1, 2)$ and studied the equivalence with Ada, C-algebra, and Ada and its connection with 3-ring, Stone type representation and introduced the concept of A*-Clone and the if -then-else structure over A*-algebra and ideal of A*-algebra. We introduce Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$ analogous to C-algebra as a product of A*-algebra. Recently Pre A*-algebra had been studied by Chandrasekhara Rao *et al*. (2007), Rao and Satyanarayana (2010), Rao and Rao (2010) and Rao and Praroopa (2011).

Definition 1: An algebra $(A, \vee, \wedge$ $(-)^{\sim})$ satisfying:

(a) $(x^{\sim})^{\sim} = x, \forall x \in A$
(b) $x \wedge x = x, \forall x \in A$
(c) $x \wedge y = y \wedge x, \forall x, y \in A$
(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$
(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$
(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$
(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y \in A$

is called a Pre A*-algebra.

Example: 3 = {0, 1, 2} with ∨, ∧ $(-)^{\sim}$ defined below is a Pre A*-algebra.

Example of a Pre A*-algebra:

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Example: 2 = {0, 1} with ∨, ∧ $(-)^{\sim}$ defined below is a Pre A*-algebra

Example of a Pre A*-algebra:

∧	0	1
0	0	0
1	0	1

∨	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note: The elements 0, 1, 2 in examples satisfy the following laws:

(a) $2^{\sim} = 2$
(b) $1 \wedge x = x, \forall x \in 3$ ('1' the identity for ∧)
(c) $1^{\sim} = 0$
(d) $2 \wedge x = 2, \forall x \in 3$
(e) $0 \vee x = x, \forall x \in 3$ ('0' is the identity for ∨)

Note:

(i) (2, ∧, ∧ $(-)^{\sim}$) is a Boolean algebra. So every Boolean algebra is a Pre A* algebra
(ii) The identities Def. 1a and Def. 1d imply that the varieties of Pre A*-algebras satisfies all the dual statements of Def. 1a to Def. 1g

PRE A*-ALGEBRA AS A SEMILATTICE

Definition 2: Let A be a Pre A*-algebra. Define ≤ on A by $x \le y$ if and only if $x \wedge y = y \wedge x = x$, ∀ x, y∈A.

The defined ≤ is said to be partial ordering on Pre A*-algebra A.

Lemma 1: If A is a Pre A*-algebra then (A, ≤) is a Poset.

Proof: Since $x \wedge x = x$, $x \le x$ for all x∈A.

Therefore, ≤ is reflexive.

Suppose that x, y, z∈A, $x \le y$ and $y \le z$.

Then we have $y \wedge x = x \wedge y = x$ and $z \wedge y = y \wedge z = y$.

Now $x = x \wedge y = x \wedge y \wedge z = x \wedge z$

$x \wedge z = z \wedge x = x$

Therefore, $x \le z$

This shows that ≤ is transitive.

Suppose that x, y∈A, $x \le y$ and $y \le x$.

$$y \wedge x = x \wedge y = x \text{ and } y \wedge x = x \wedge y = y$$

This shows that $x = y$.
Therefore, $\leq$ is antisymmetric.
Hence $(A, \leq)$ is poset

Semi lattice in a Pre A*-algebra

Definition 3: A non-empty subset S of a Pre A*-algebra A equipped with a binary operation $\wedge(\vee)$ is said to be a semi lattice, if the following semi lattice axioms are satisfied:

(i) $\wedge(\vee)$ is associative
i.e., $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in S$
(ii) $\wedge(\vee)$ is commutative
i.e., $a \wedge b = b \wedge a, \forall a, b \in S$
(iii) $\wedge(\vee)$ is idempotent
i.e., $a \wedge a = a, \forall a \in S$

Theorem 1: In Pre A*-algebra A $(S, \wedge)$ and $(S, \vee)$ are semi lattices.

Proof: In Pre A*-algebra, $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in A$ (Def. 1e)
$a \wedge b = b \wedge a, \forall a, b \in A$ (Def. 1c)
and $a \wedge a = a, \forall a \in A$ (Def. 1b)
Hence, $(S, \wedge)$ is a semi lattice.
By the duality in A $(S, \vee)$ is a semi lattice.

Theorem 2: In Pre A*-algebra A, the class of semi lattices can be equationally defined as the class of all semi group satisfying the commutative and idempotent laws.

Proof: Let $(S, \wedge, \vee)$ be a semi lattice in a Pre A*-algebra A. By the definition of semi lattice we have $\wedge(\vee)$ is associative. i.e., $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in S$

(i) $\wedge(\vee)$ is commutative
i.e., $a \wedge b = b \wedge a, \forall a, b \in S$
(ii) $\wedge(\vee)$ is idempotent
i.e., $a \wedge a = a, \forall a \in S$

Hence, $(S, \wedge)$ as well as $(S, \vee)$ is a semi group satisfying commutative and idempotent laws.
Therefore, $(S, \wedge (\vee))$ is a semigroup satisfying the commutative and idempotent laws.

Converse: $(S, \wedge)$ as well as $(S, \vee)$ is a semi-group satisfying commutative and idempotent laws.
By the definition of Pre A*-algebra A:

- $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in A$ (Def. 1e)
- $a \wedge b = b \wedge a, \forall a, b \in A$ (Def. 1c)
- $a \wedge a = a, \forall a \in A$ (Def. 1b)

Hence, $\wedge(\vee)$ is associative, commutative and idempotent.

Hence, $(S, \wedge(\vee))$ is a Semilattice in a Pre A*-algebra A.

Pre A*-algebra as a semilattice

Theorem 3: Let A be a Pre A*-algebra. Then A is a semilattice.

Proof: Since A is a Pre A*-algebra:

- Then $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $\forall$ a, b, c$\in$A by (Def. 1e)
- $a \wedge b = b \wedge a$, $\forall$ a, b $\in$A by (Def. 1c)
- $a \wedge a = a$, $\forall$ a$\in$A by (Def. 1b)

Hence, A is a semilattice

Note: We can also define Pre A*-algebra as follows:

An algebra $(A, \wedge, \vee, \tilde{})$ is said to be Pre A*-algebra where A is non-empty set with 1 and $\wedge$, $\vee$ are binary operations $\tilde{}$ is a unary operation satisfying:

- $(A, \wedge)$ is a semilattice
- $x^{\sim\sim} = x$, $\forall$, x A,
- $(x\wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$, $\forall$ x, yA
- $x \wedge (y\vee z) = (x \wedge y) \vee (x \wedge z)$, $\forall$ x, y, z$\in$A
- $x \wedge y = x \wedge (x^{\sim} \vee y)$, $\forall$ x, y, z A

Theorem 4: Let A be a Pre A*-algebra. Then in a semilattice A, define $x \le y$ if and only if $x\wedge y = x$. Then $(A, \le)$ is an ordered set in which every pair of elements has a greatest lower bound. Conversely, given an ordered set P with that property, define $x \wedge y$ = g.l.b. (x, y).

Then $(P, \wedge)$ is a semilattice.

Proof: Let $(A, \wedge)$ be a semilattice and define$\le$ as above. First we check that $\le$ is a partial order:

(i) $x\wedge x = x$ implies $x \le x$
(ii) If $x \le y$ and $y \le x$, then $x = x\wedge y = y\wedge x = y$
(iii) If $x \le y \le z$, then $x\wedge z = (x\wedge y) \wedge z = x \wedge (y\wedge z) = x\wedge y = x$, so $x \le z$

Since $(x\wedge y) \wedge x = x \wedge (x\wedge y) = (x\wedge x) \wedge y = x\wedge y)$, we have $x\wedge y \le x$ similarly $x\wedge y \le y$. Thus $x \wedge y$ is a lower bound for {x, y}.

To see that it is the greatest lower bound, suppose $z \le x$ and $z \le y$. Then $z \wedge (x\wedge y) = (z\wedge x) \wedge y = z\wedge y = z$, so$z \le x \wedge y$

Converse: suppose$(P, \le)$ is an ordered set.

define $x\wedge y$ = g.l.b. (x, y).

Since $(P, \le)$ is an ordered set:

(i) $x \leq x$ implies $x \wedge x = x$
(ii) $x \leq y$ and $y \leq x$, then $x \wedge y = y \wedge x$
(iii) $z \leq x$ and $z \leq y$ implies $z \wedge (x \wedge y) = (z \wedge x) \wedge y$

Hence $(P, \wedge)$ is a semilattice.

Distributive semilattice in a Pre A*-algebra

Definition 4: Let A be a Pre A*-algebra Then L is said to be distributive semilattice if any elements a, b, c in L we have the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a, b, c \in L$$

Example: 2 = { 0, 1} is the distributive semilattice in a Pre A*-algebra

Theorem 5: Let A be a Pre A*-algebra. Then A is a distributive semilattice

Proof: Since A is a Pre A*-algebra, for any elements a, b, c in A we have the distributive law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a, b, c \in A$$

Lemma 2: Let A be a Pre A*-algebra. Then in the Poset $(A, \leq)$:

$$\text{If } a \leq b \Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c), \forall a, b, c \in A$$

Proof: Define $\leq$ in A as $a \leq b$ $a \wedge b = a$ (i.e., $a \vee b = b$)
Suppose $a \leq b$ then $b \wedge a = a$
Now $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c)$
$= a \vee (b \wedge c)$ (by Def. 1f)

Modular semilattice in a Pre A*-algebra

Definition 5: Let A be a Pre A*-algebra. Then L is said to be a modular semilattice if:

$$x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z), \forall x, y, z \in A$$

Example: 3 = { 0, 1, 2 } is the modular semilattice in a Pre A*-algebra

Theorem 6: Let A be a Pre A*-algebra. Then A is a modular semilattice.

Proof: Since A is a Pre A*-algebra,
By lemma 2:

$$x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z), \forall x, y, z \in A$$

Hence, A is a modular semilattice.

Complement of an element in a Pre A*-algebra

Definition 6: Let A be a Pre A*-algebra with least element α, greatest element β. Then $a \in A$ is said to be complement if there exists $x \in A$:

$$\text{Such that } a \wedge x = \alpha,\ a \vee x = \beta$$

Note: Since $\wedge$, $\vee$ are commutative in a Pre A*-algebra, we have if x is a complement of a then a is also a complement of x.

Hence, a, x are complements to one another.

Complemented semilattice in a Pre A*-algebra

Definition 7: Let A be a Pre A*-algebra then A is said to be Complemented semilattice if each element has a complement in it.

Example: The semilattice shown in this Fig. 1 is a complemented semilattice:

- Here every element has a complement but these are not unique
- Here b, c are complements of a
- Here a, c are complements of b
- Here a, b are complements of c

Example: This is example of a semilattice which is not complemented.

In Fig. 2, the elements a, e c, d have complements but the element b has no complement.

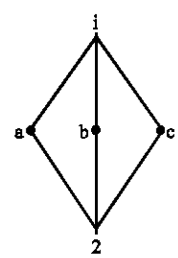

Fig. 1: Complemented semilattice

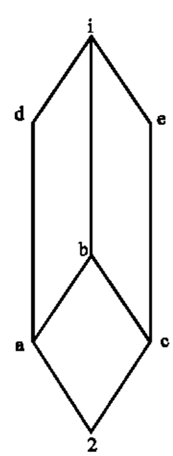

Fig. 2: Example of a semilattice which is not complemented

Theorem 7: Let A be a Pre A*-algebra. Then A is a Complemented semilattice.

Proof: Since A is a Pre A*-algebra, A is a semilattice.
Since each element in A has a complement in it, hence A is a Complemented semilattice.

Theorem 8: Let A be a Pre A*-algebra. Then A is a Complemented distributive semilattice

Proof: Since A is a Pre A*-algebra, Then A is a Complemented semilattice also distributive semilattice.

Unique complement of an element in a Pre A*-algebra
Definition 8: Let A be a Pre A*-algebra Then $a \in A$ is said to be unique complement if a has exactly one complement in A.

Uniquely complemented semilattice in a Pre A*-algebra
Definition 9: Let A be a Pre A*-algebra. Then A is said to be uniquely complemented semilattice if each element in A has unique complement in A.

Relative complement in a Pre A*-algebra
Definition 10: Let A be a Pre A*-algebra. Let $[a, b] \in A$ and u is an element of [a, b]. An element x of A is said to be relative complement of u in [a, b] if $u \wedge x = a$, $u \vee x = b$

Note: If x is a relative complement of u in [a, b] then we have:

$$x \in [a, b] \text{ and } x \text{ is complement of } u \text{ in } [a, b]$$

Relatively complemented semilattice in a Pre A*-algebra
Definition 11: Let A be a Pre A*-algebra. Then A is said to be relatively complemented semilattice, if for any triplet of elements a, b, u such that $a \le u \le b$ there exists at least one complement of u in [a, b] i.e., every interval of A is a complemented semilattice of A.

Example: The semilattice shown in Fig. 1 is an example of a semilattice which is complemented as well as relatively complemented.

Note: Let A be a Pre A*-algebra hen every bounded relatively complemented semilattice in A is complemented but converse is not true i.e., a complemented semilattice in a Pre A*-algebra A may or may not be relatively complemented semilattice.

Example: The semilattice shown in Fig. 3 is complemented but not relatively complemented.

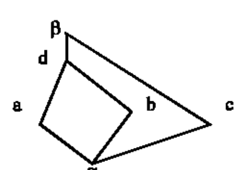

Fig. 3: Example of semilattice which is not relatively complemented

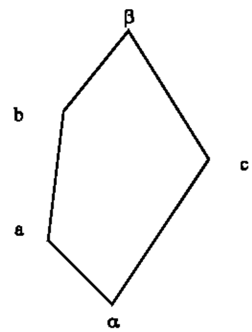

Fig. 4: Example of semilattice which is not section complemented

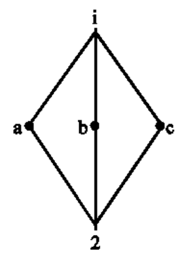

Fig. 5: Section complemented semilattice

Since [α, b], [a, β] are not complemented semilattices since a has no complements in [α, b]

Example: The semilattice shown in Fig. 3 is not relatively complemented.

Since [a, β] ={a, d, β} is not a complemented semilattice since a has no complement hence it is not relatively complemented.

Section complemented semilattice in a Pre A*-algebra

Definition 12: Let A be a Pre A*-algebra with least element α. Then A is said to be section complemented semilattice if every interval of the form [α, a] is a complemented semilattice of A.

i.e., for each pair of elements a, u with $u \le a$ there exists an element $x \in A$ such that $u \wedge x = \alpha$, $u \vee x = a$.

Example: The semilattice shown in this Fig. 4 is not section complemented because [α, b] is a not complemented semilattice.

Example: Figure 5 example of a semilattice which is section complemented.

Theorem 9: Let A be a Pre A*-algebra. Then every relatively complemented semilattice in A is section complemented.

Proof: Since A is a Pre A*-algebra, if L is a relatively complemented semilattice then by the definition L every interval of A is a complemented semilattice of A.

Hence, L is section complemented semilattice.

Note: Let A be a Pre A*-algebra every relatively complemented semilattice in A is Section complemented but converse is not true.

Semi-complement of an element in a Pre A*-algebra

Definition 13: Let A be a Pre A*-algebra with least element α and u∈A. An element x∈A is said to be semi-complement of u if $u \wedge x = \alpha$ (x is not equal to α)

Definition 14: Let A be a Pre A*-algebra with least element α and u∈A. Then all Semi-complements of an element u forms a poset U. If this poset has a maximal element x_0 then x_0 is called maximal semicomplement of u i.e., if there exists x∈A such that $u \wedge x = \alpha$ and $x_0 \leq x$ implies $x = x_0$

Definition 15: Let A be a Pre A*-algebra with least element α. The semi-complements of an element other than the least element α is called a proper semi-complement in A. If in addition the proper semi-complement is maximal then it is called maximal proper semi-complement in A.

Semi-complemented semilattice in a Pre A*-algebra

Definition 16: Let A be a Pre A*-algebra with least element α. Then A is said to be Semi-complemented semilattice if every inner element (other than least and greatest elements in A) has at least one proper semi-complement.

Weakly complemented semilattice in a Pre A*-algebra

Definition 17: Let A be a Pre A*-algebra with least element α. Then A is said to be weakly complemented semilattice if any pair of elements a, b (a<b) of A a has semi complement, that is however, not a semi complement of b i.e., x is semi complement of a but not semi complement of b.

Example: The semilattice shown in Fig. 6 is example of a semilattice which is not weakly complemented.

This is not weakly complemented since a<b and c is semi complement of both a and b.

Theorem 10: Let A be a Pre A*-algebra. Then every weakly complemented semilattice in A is semi-complemented.

Proof: Let A be a Pre A*-algebra with least element α, greatest element β

Let L be any weakly complemented semilattice in A

Claim: Lis semi-complemented

Let a∈L be an inner element i.e.:

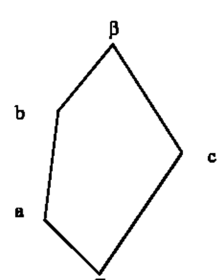

Fig. 6: Example of a semilattice which is not weakly complemented

$$a \neq \alpha,\ a \neq \beta$$

- ⇒ a is not a maximal element
- ⇒ ∃b∈L such that a<b

Since L is weakly complemented semilattice in A, we have that there exists semi-complement x of a which is not a semi-complement of b
i.e.:

$$a \wedge x = \alpha \Rightarrow b \wedge x \neq \alpha$$

then x is proper semi-complement of a and hence L is semi-complemented. Hence every weakly complemented semilattice in A is semi-complemented.

Absorption law in Pre A*-algebra

$$\text{If } a \wedge b \leq b \text{ then } (a \wedge b) \vee b = b$$

Theorem 11: Let A be a Pre A*-algebra with least element α. Then every section complemented semilattice in A is weakly complemented.

Proof: Let A be a Pre A*-algebra with least element α.
And L be section complemented semilattice in A.

Claim: L is weakly complemented.
Let a, b∈L such that a<b
Now [α, b] is a complemented semilattice of L
Since L is section complemented and a∈[α, b]⇒∃ x∈L such that:

$$a \wedge x = \alpha,\ a \vee x = b$$

Consider b ∧ x = (a ∨ x) ∧ x
= x (by absorption law in A)
When:

$$x = \alpha,\ b = a \vee \alpha \Rightarrow b = a \vee (a \wedge x) = a$$

Therefore, b = a which is a contradiction to a<b
Thus x≠α
Hence L is weakly complemented semilattice.

Theorem 12: Let A be a Pre A*-algebra with least element α, greatest element **β**. Then every uniquely complemented semilattice in A is weakly complemented.

Proof: Let A be a Pre A*-algebra with least element α, greatest element **β** and L be any uniquely complemented semilattice in A.

Claim: L is weakly complemented.

Let a, b∈L such that there exists unique complement $a^{\sim}$ of a such that:

$$a \wedge a^{\sim} = \alpha,\ a \vee a^{\sim} = \beta$$

i.e., $a^{\sim}$ is semi-complement of a

since a<b we have $b \vee a^{\sim} > a \vee a^{\sim} = \beta$:

$$\Rightarrow_b \vee a^{\sim} > \beta$$

since β is greatest element in A, $\beta >_b \vee a^{\sim}$:

$$\Rightarrow_b \vee a^{\sim} = \beta$$

$b \wedge a^{\sim} \neq \alpha$ (suppose if $b \wedge a^{\sim} = \alpha$ then $a^{\sim}$ is complement of both a and b which is a contradiction to our assumption that L is uniquely complemented semilattice in a Pre A*-algebra).

Therefore, $a \wedge a^{\sim} = \alpha$, $b \wedge a^{\sim} \neq \alpha$

Hence L is weakly complemented semilattice in A.

REFERENCES

Birkhoff, G., 1948. Lattice Theory. American Mathematical Society, Colloquium Publications, New York.

Chandrasekhara Rao, K., J. Venkateswara Rao and P. Koteswara Rao, 2007. Pre A*-algebras. J. Inst. Mathe. Comput. Sci. Math. Ser., 20: 157-164.

Guzman, F. and C.C. Squier, 1990. The algebra of conditional logic. Algebra Univ., 27: 88-110.

Koteswara Rao, P., 1994. A*-Algebras and if-then-else structures. Ph.D. Thesis, Acharya Nagarjuna University, Andhra Pradesh, India.

Manes, E.G., 1989. The equational theory of disjoint alternatives. Personal Communication to Prof. N.V. Subrahmanyam.

Rao, J.V. and A. Satyanarayana, 2010. Semilattice structure on pre A*-algebras. Asian J. Sci. Res., 3: 249-257.

Rao, J.V. and P.K. Rao, 2010. Subdirect representations in A*-Algebras. Asian J. Math. Stat., 3: 249-253.

Rao, J.V. and Y. Praroopa, 2011. Logic circuits and gates in pre A*-algebra. Asian J. Applied Sci., 4: 89-96.