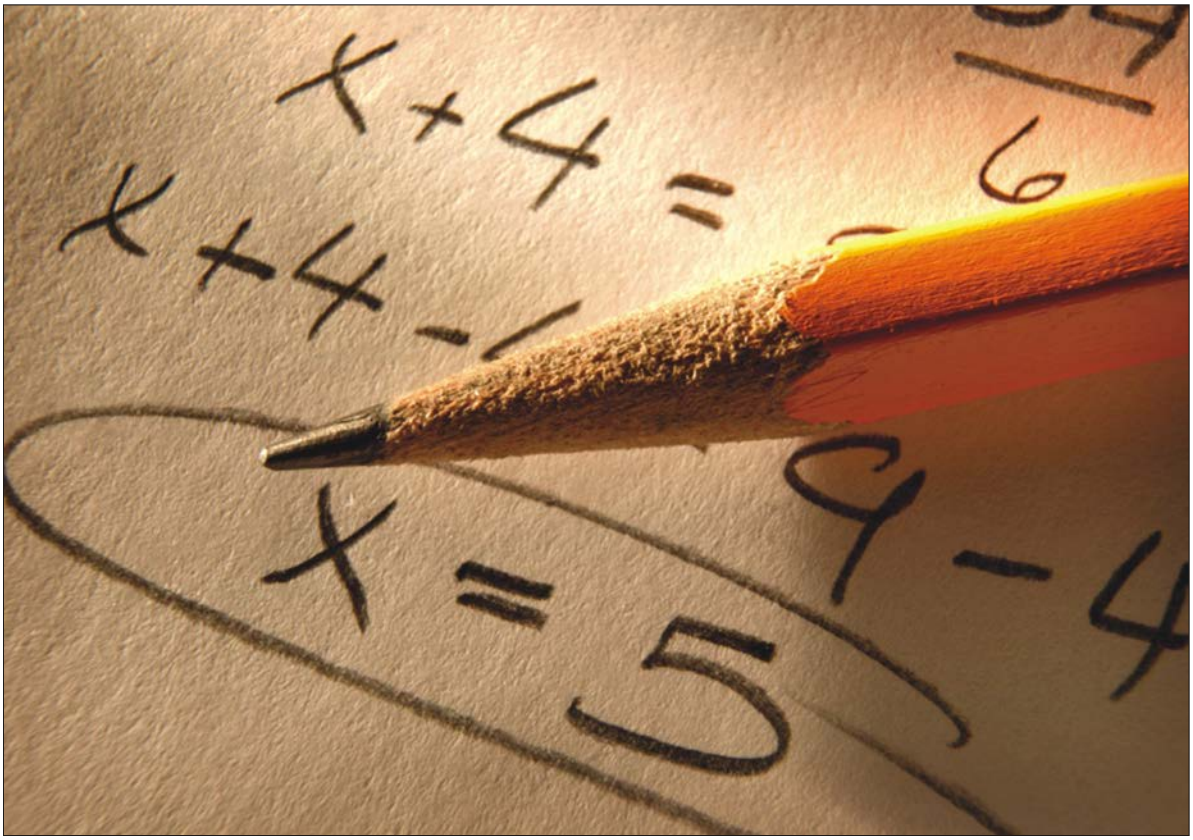


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# On the Hyper Marginal Groups and Perfect Groups in an Arbitrary Variety of Groups

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## ABSTRACT

Let $\vartheta$ be a variety of groups defined by a set V of laws. In this study, for infinite groups, we first continue the hyper marginal series of infinite ordinal numbers via transfinite recursion and by this idea introduce the hyper marginal subgroup of an arbitrary group then with define the first hyper marginal groups, extend the Grun Lemma in an arbitrary varity $\vartheta$ of groups and show that all of perfect groups in variety $\vartheta$ ($\vartheta$- perfect groups) are first hyper marginal groups. In the end of this investigation, we compute hyper marginal subgroup of the quotient group $\frac{G}{N}$ and direct product of a finite collection of perfect groups.

**Key words:** Variety, $\vartheta$-perfect, hyper, marginal, Grun lemma

## INTRODUCTION

Let F be a free group on a countably infinite set $\{x_1, x_2,...\}$ and let V be a non-empty subset of F. If $v = x_{i_1}^{l_1} x_{i_2}^{l_2} ... x_{i_n}^{l_n} \in V$ and $g_1, g_2,...g_1$ are elements of a group G, we define the value of word v at $(g_1, g_2,..., g_n)$ to be $v(g_1, g_2,..., g_n) = g_1^{l_1} g_2^{l_2} ... g_n^{l_n}$.

The subgroup of G generated by all values in G of words in V is called verbal subgroup of G determined by V, i.e:

$$V(G) = \langle \{v(g_1,g_2,...,g_n) | v \in V, g_i \in G, 1 \le i \le n\} \rangle \tag{1}$$

Marginal subgroup of G determined by V is defined as:

$$V^*(G) = \{g \in G | v(g_1,...,g_i g,...,g_n) = v(g_1,...,g_n), g_i \in G, 1 \le i \le n\} \tag{2}$$

A variety generated by the set V is the class of all groups G such that V (G) = 1. Let $N \triangleleft G$ we defined:

$$[NV^*G] = \langle \{v(g_1,...,g_i a,...,g_n) v(g_1,...,g_n)^{-1} | v \in V, a \in N, g_i \in G, 1 \le i \le n\} \rangle \tag{3}$$

A group G is called $\vartheta$-perfect at variety $\vartheta$ if G = V (G). Neumann (1970) and Hekster (1989) developed the concepts like isologism and stemgroup in variety and they provided some preliminary properties and notions concerning verbal and marginal subgroups, for example he showed that for an arbitrary group G in variety $\vartheta$, G = $V^*$ (G) if and only if V (G) = 1 or if $N \triangleleft G$ then

$$\frac{V^*(G)N}{N} \leq V^*\left(\frac{G}{N}\right)$$

**Definition 1:** Let $\vartheta$ be a variety of groups, we define the hyper $\vartheta$- marginal (or briefly hyper marginal) series of G to be:

$$1 = V_0^*(G) \leq V^*(G) = V_1^*(G) \leq \ldots \leq V_n^*(G) \leq \ldots$$

where:

$$\frac{V_n^*(G)}{V_{n-1}^*(G)} = V^*\left(\frac{G}{V_{n-1}^*(G)}\right), \text{ for } n \geq 0$$

For infinite groups, one can continue the hyper marginal series of infinite ordinal numbers via transfinite recursion for a limit ordinal $\lambda$, define

$$V_\lambda^*(G) = \bigcup_{\alpha<\lambda} V_\alpha^*(G)$$

The limit of this processes (the union of a higher marginal subgroups) is called the hyper marginal subgroup of G and denoted by $V^*_H$ (G). The group G is called hyper marginal if $V^*_H$ (G) = (G). In the particular case, if $\vartheta$ be the variety of abelian groups, defined by the set V = {[$x_1$, $x_2$]}, then the last series is the hyper center series and the hyper marginal subgroup of G is the hyper center subgroup of G, such that Baer (1953) studied many aspects of them like that the hyper center subgroup of G which denoted by H (G) is a characteristic subgroup and smallest normal subgroup of G such that

$$H\left(\frac{G}{H(G)}\right) = 1$$

**Example 1:** In the variety of abelian groups, the quaternion group $Q_8$, the dihedral group of degree 4, $D_8$ and set of integral numbers are hyper marginal (hyper center) groups.

**Definition 2:** Let $\vartheta$ be a variety of groups, if for an arbitrary group G, $V^*$ (G)$\neq$ 1 and

$$V^*\left(\frac{G}{V^*(G)}\right) = 1$$

then $V^*_n$ (G) = $V^*$ (G), for all natural number n and so on $V^*_H$ (G) = $V^*$ (G). In this case, the group G is called the first hyper marginal group.

**Lemma 1, (Extension of grun lemma):** Let $\vartheta$ be a variety of groups and G be a $\vartheta$ - perfect group, then

$$V^*\left(\frac{G}{V^*(G)}\right)=1$$

**Proof:** Let

$$gV^*(G)\in V^*\left(\frac{G}{V^*(G)}\right)$$

then for all $g_i \in G$, $1 \le i \le n$ and $v \in V$,

$$v\left(g_1V^*(G),\ldots,g_iV^*(G)gV^*(G),\ldots,g_nV^*(G)\right)=v\left(g_1V^*(G),\ldots,gV^*(G),\ldots,g_nV^*(G)\right)$$

so

$$v(g_1,\ldots,g_ig,\ldots,g_n)v(g_1,\ldots,g,\ldots,g_n)^{-1}\in V^*(G)$$

By definition of verbal subgroup, V (G) = [GV*G], then V (G)⊆V* (G). Using the perfectness of G, implies that G = V* (G) so gV* (G) = V* (G) and this complete the proof.

**Corollary 1, (Grun lemma):** If G be a perfect group, then

$$Z\left(\frac{G}{Z(G)}\right)=1$$

**Proof:** In the previous lemma, set V = {[$x_1$, $x_2$]}.

**Corollary 2:** Let $\vartheta$ be a variety of groups and G be a $\vartheta$ - perfect group, then G is a first hyper marginal group.

**Proof:** According to lemma 1,

$$\frac{V_2^*(G)}{V^*(G)}=1$$

so $V_2^*$ (G) = V* (G). Induction on n, shows that $V^*_n$ (G) = V* (G) and so on $V^*_H$ (G) = V* (G).

**Corollary 3:** Let $\vartheta$ be a variety of groups, then the first hyper marginal group G is hyper marginal group if and only if V (G) = 1.

**Proof:** If G be a first hyper marginal group, then $V^{*}_{H}$ (G) = $V^{*}$ (G). Now if assume G is a hyper marginal group, then G = $V^{*}_{H}$ (G) and so on G = $V^{*}$ (G) and by Hekster (1989) G = $V^{*}$ (G) if and only if V (G) = 1 so verbal subgroup of G is trivial. The converse part is clear.

**Corollary 4:** Let $\vartheta$ be a variety of groups, a $\vartheta$ - perfect group G is hyper marginal group if and only if G be the trivial group.

**Proof:** It is clearly by using previous corollaries.

## RESULTS

Here, we investigate the properties of hyper marginal subgroup of an arbitrary group G and then we compute the hyper marginal subgroups of the quotient group $\frac{G}{N}$ and the direct product of a finite collection of perfect groups.

Some of results of Ramos and Maier (2007) like that if $N \triangleleft G$ then

$$\frac{H(G)N}{N} \leq H\left(\frac{G}{N}\right)$$

are extended to hyper marginal subgroups.

**Lemma 2:** Let $\vartheta$ be a variety of groups and G be an arbitrary group, then $V^{*}_{a}$ (G) is a characteristic subgroup of G, for all ordinal numbers.

**Proof:** One can easily shows that $V^{*}$(G) is a characteristic subgroup of G. It is clearly that if $H \triangleleft N \triangleleft G$, H is a characteristic subgroup of G if and only if $\frac{H}{N}$ is a characteristic subgroup of $\frac{G}{N}$ so since

$$\frac{V_2^*(G)}{V_1^*(G)} = V^*\left(\frac{G}{V_1^*(G)}\right) \overset{c.h}{\leq} \frac{G}{V^*(G)}$$

then $V^{*}_{2}$ (G) is a characteristic subgroup of G. By doing this processes $V^{*}_{2}$ (G) is a characteristic subgroup of G, for all ordinal numbers $\alpha$.

**Proposition 1:** Let $\vartheta$ be a variety of groups, then for an arbitrary group G, $V^{*}_{H}$ (G) is a characteristic subgroup of G.

**Proof:** Let $\varphi \in$Aut (G) and $x \in V^{*}_{H}$ (G), then there exists $\alpha$, such that $x \in V^{*}_{\alpha}$(G). According to Lemma 2 $\varphi(x) \in V^{*}_{a}$(G) so $\varphi$ (x)$\in V^{*}_{H}$(G), then $\varphi$ ($V^{*}_{H}$(G))$\leq V^{*}_{H}$(G) and consequently $V^{*}_{H}$(G) is a characteristic subgroup of G and so $V^{*}_{H}$ (G)$\triangleleft$G.

**Theorem 1:** Let $\vartheta$ be a variety of groups, then for an arbitrary group G, $V^{*}_{H}$ (G) is the smallest normal subgroup of G such that:

$$V^*\left(\frac{G}{V_H^*(G)}\right) = 1$$

**Proof:** According to definition 1, there exists α, such that

$$V^*\left(\frac{G}{V_H^*(G)}\right)=\frac{V_\alpha^*(G)}{V_H^*(G)}$$

On the other hand by the definition $V^*_H(G)$, $V^*_\beta(G) \leq V^*_H(G)$ for all ordinal number **β**, so $V^*_\alpha(G) = V^*_H(G)$, then

$$V^*\left(\frac{G}{V_H^*(G)}\right)=1$$

Now shows that $V^*_H(G)$ is the smallest normal subgroup of G with above property.

Let $N \triangleleft G$ such that

$$V^*\left(\frac{G}{N}\right)=1$$

on the other hand according to Hekster (1989)

$$\frac{V^*(G)N}{N}\leq V^*\left(\frac{G}{N}\right)$$

so $V^*(G)$. Also:

$$1=V^*\left(\frac{G}{N}\right)\cong V^*\left(\frac{\frac{G}{V^*(G)}}{\frac{N}{V^*(G)}}\right)\geq\frac{V^*\left(\frac{G}{V^*(G)}\right)\frac{N}{V^*(G)}}{\frac{N}{V^*(G)}},$$

so

$$\frac{V_2^*(G)}{V^*(G)}=V^*\left(\frac{G}{V^*(G)}\right)\leq\frac{N}{V^*(G)}$$

By doing this processes, $V^*_\alpha(G) \leq N$ for all ordinal number α and so on $V^*_H(G) \leq N$.

**Theorem 2:** Let $\vartheta$ be a variety of groups, G be an arbitrary group and $N \triangleleft G$ then

$$\frac{V_H^*(G)N}{N}\leq V_H^*\left(\frac{G}{N}\right)$$

**Proof:** Let

$$V_H^*\left(\frac{G}{N}\right)=\frac{M}{N}\triangleleft\frac{G}{N}$$

so $M \triangleleft G$. On the other hand:

$$1 = V^*\left(\frac{\frac{G}{N}}{V_H^*\left(\frac{G}{N}\right)}\right) = V^*\left(\frac{\frac{G}{N}}{\frac{M}{N}}\right) = V^*\left(\frac{G}{M}\right)$$

And so on, according to previous theorem, $V^*_H(G) \le M$ then

$$\frac{V_H^*(G)N}{N} \le \frac{MN}{N} = \frac{M}{N} = V_H^*\left(\frac{G}{N}\right)$$

**Theorem 3:** Let $\vartheta$ be a variety of groups, G be an arbitrary group, $N \triangleleft G$ and $N \triangleleft V^*_H(G)$ then

$$V_H^*\left(\frac{G}{N}\right) = \frac{V_H^*(G)}{N}$$

**Proof:** Using the previous theorem implies that

$$\frac{V_H^*(G)}{N} \le V_H^*\left(\frac{G}{N}\right)$$

On the other hand,

$$1 = V^*\left(\frac{G}{V_H^*(G)}\right) \cong V^*\left(\frac{\frac{G}{N}}{\frac{V_H^*(G)}{N}}\right)$$

According to theorem 1,

$$V_H^*\left(\frac{G}{N}\right) \le \frac{V_H^*(G)}{N}$$

In the particular case, it is enough to set $N = V^*_H(G)$.

**Proposition 2:** The homomorphic image of a hyper marginal group is a hyper marginal group.

**Proof:** Let $\vartheta$ be a variety of groups, $G_1, \ldots G_n$ be an epimorphism for any group K. According to theorem 3,

$$V_H^*(K) = V_H^*\left(\frac{G}{\mathrm{Ker}\alpha}\right) = \frac{V_H^*(G)}{\mathrm{Ker}\alpha} = \frac{G}{\mathrm{Ker}\alpha} = K$$

**Proposition 3:** Let $\vartheta$ be a variety of groups, $G_1, \ldots, G_n$ be arbitrary groups, then

$$V_n^*(G_1\times\ldots\times G_m)=V_n^*(G_1)\times\ldots\times V_n^*(G_m)$$

for all natural numbers n and m.

**Proof:** We doing the proof by induction on n. If n = 1, it is clear by definition of marginal subgroup. Assume that statement holds for n = k, then for n = k+1 we have:

$$V^*\left(\frac{G_1\times\ldots\times G_m}{V_k^*(G_1\times\ldots\times G_m)}\right)=\frac{V_{k+1}^*(G_1\times\ldots\times G_m)}{V_k^*(G_1\times\ldots\times G_m)}.$$

According to induction hypothesis

$$V_k^*(G_1\times\ldots\times G_m)=V_k^*(G_1)\times\ldots\times V_k^*(G_m)$$

so

$$V^*\left(\frac{G_1\times\ldots\times G_m}{V_k^*(G_1\times\ldots\times G_m)}\right)=V^*\left(\frac{G_1}{V_k^*(G_1)}\times\ldots\times\frac{G_m}{V_k^*(G_m)}\right).$$

Using definition of marginal subgroup implies that

$$V^*\left(\frac{G_1\times\ldots\times G_m}{V_k^*(G_1\times\ldots\times G_m)}\right)=V^*\left(\frac{G_1}{V_k^*(G_1)}\right)\times\ldots\times V^*\left(\frac{G_m}{V_k^*(G_m)}\right).$$

Now definition 1 implies that

$$V^*\left(\frac{G_i}{V_k^*(G_i)}\right)=\frac{V_{k+1}^*(G_i)}{V_k^*(G_i)}$$

for $1\le i\le m$. Consequently

$$\frac{V_{k+1}^*(G_1\times\ldots\times G_m)}{V_k^*(G_1\times\ldots\times G_m)}=\frac{V_{k+1}^*(G_1)}{V_k^*(G_1)}\times\ldots\times\frac{V_{k+1}^*(G_m)}{V_k^*(G_m)}=\frac{V_{k+1}^*(G_1)\times\ldots\times V_{k+1}^*(G_m)}{V_k^*(G_1)\times\ldots\times V_k^*(G_m)}.$$

Using induction hypothesis again, implies that

$$\frac{V_{k+1}^*(G_1\times\ldots\times G_m)}{V_k^*(G_1\times\ldots\times G_m)}=\frac{V_{k+1}^*(G_1)\times\ldots\times V_{k+1}^*(G_m)}{V_k^*(G_1\times\ldots\times G_m)}$$

so

$$V_{k+1}^*(G_1\times\ldots\times G_m)=V_{k+1}^*(G_1)\times\ldots\times V_{k+1}^*(G_m)$$

and this implies that statement holds for all natural number n.

**Proposition 4:** Let $\vartheta$ be a variety of groups and $G_1$, $G_2$ be two arbitrary groups, then

$$V_H^*(G_1 \times G_2) = V_H^*(G_1) \times V_H^*(G_2).$$

**Proof:** Let $(x,y) \in V_H^*(G_1 \times G_2)$, then there exists an ordinal number $\alpha$ such that $(x,y) \in V_\alpha^*(G_1 \times G_2)$. Previous proposition implies that $x \in V^*_\alpha (G_1)$ and $y \in V^*_\alpha (G_2)$, so

$$(x,y) \in \bigcup_\alpha V_\alpha^*(G_1) \times \bigcup_\alpha V_\alpha^*(G_2) = V_H^*(G_1) \times V_H^*(G_2)$$

then $V_H^*(G_1 \times G_2) \subseteq V_H^*(G_1) \times V_H^*(G_2)$. Now assume that $(x,y) \in V_H^*(G_1) \times V_H^*(G_2)$, then there exists ordinal numbers $\alpha_1$ and $\alpha_2$ such that $x \in V_{\alpha_1}^*(G_1)$ and $y \in V_{\alpha_2}^*(G_2)$. Let $\alpha_1 \leq \alpha_2$, then since $V_{\alpha_1}^*(G_1) \leq V_{\alpha_2}^*(G_1)$ so $x \in V_{\alpha_2}^*(G_1)$ and so on $(x,y) \in V_{\alpha_2}^*(G_1) \times V_{\alpha_2}^*(G_2) = V_{\alpha_2}^*(G_1 \times G_2)$, then

$$(x,y) \in \bigcup_\alpha V_\alpha^*(G_1 \times G_2) = V_H^*(G_1 \times G_2)$$

so $V_H^*(G_1) \times V_H^*(G_2) \subseteq V_H^*(G_1 \times G_2)$ and this complete the proof.

**Theorem 4:** Let $\vartheta$ be a variety of groups and $G_1, \ldots, G_n$ be arbitrary groups, then

$$V_H^*(G_1 \times \ldots \times G_n) = V_H^*(G_1) \times \ldots \times V_H^*(G_n).$$

**Proof:** According to proposition 4 and by induction on n, the proof is trivial.

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