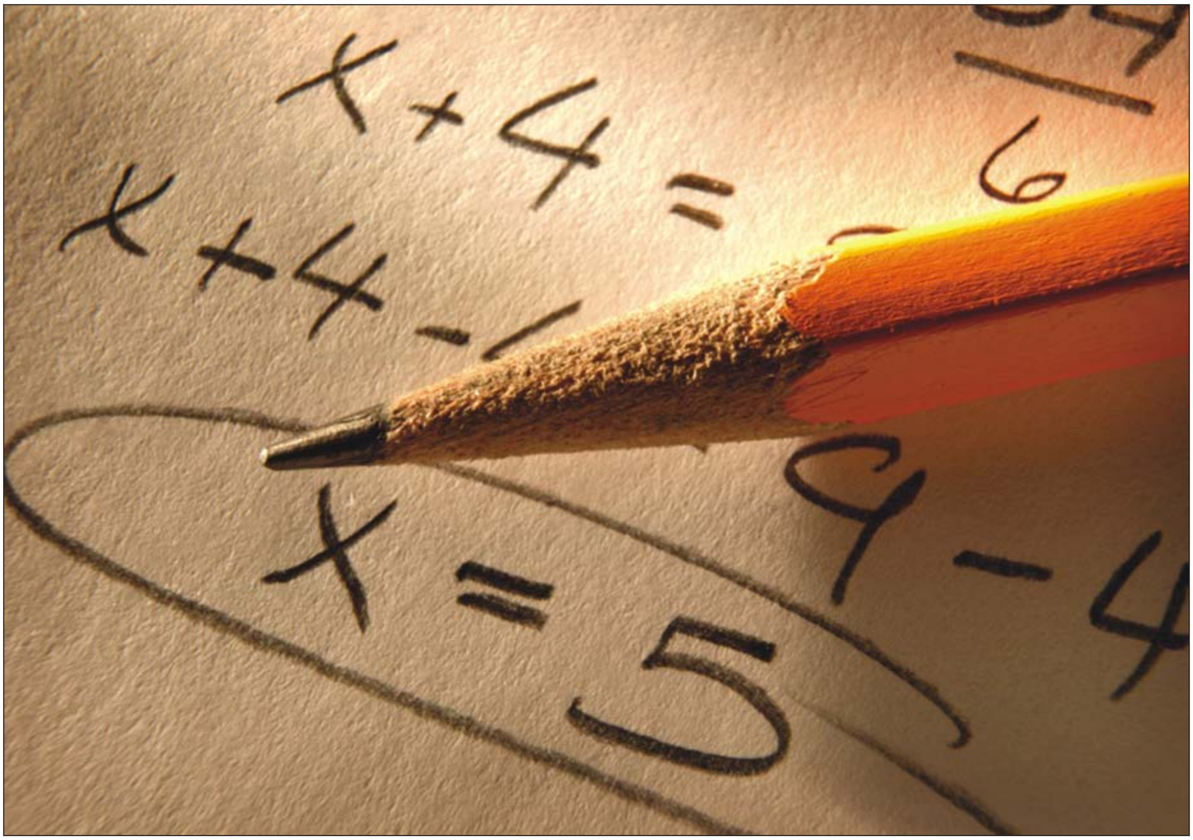

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Dependent Elements in Prime and Semiprime Gamma Rings

[1]Kalyan Kumar Dey, [1]Akhil Chandra Paul and [2]Isamiddin S. Rakhimov
[1]Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh
[2]Department of Mathematics, FS and Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, Malaysia

Corresponding Author: Kalyan Kumar Dey, Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh

ABSTRACT

Let M be a Γ-ring. An element $a \in M$ is called a dependent element on a mapping $F: M \to M$ if $F(x)\alpha a = a\alpha x$ holds for all $x \in M$, $\alpha \in \Gamma$. In this study, we determine the characterizations of dependent elements on certain mappings on prime and semiprime Γ-rings by taking a certain assumption $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z \in M$, $\alpha,\beta \in \Gamma$. For the case of semiprime Γ-ring M, we also prove that the mapping $\sigma+\tau$ is a free action if σ and τ are automorphisms of M.

Key words: Derivation, free action

INTRODUCTION

We consider a Γ-ring due to Barnes (1966). The center of M is denoted by Z(M). For any $x, y \in M$, the notation $[x, y]_\alpha$ and $(x, y)_\alpha$ will denote $x\alpha y - y\alpha x$ and $x\alpha y + y\alpha x$, respectively. We know that $[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x, z]_\alpha\beta y + x[\beta, \alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha \beta z + y[\beta, \alpha]_x z$, for all $x,y,z \in M$ and for all $\alpha,\beta \in \Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z \in M$, $\alpha,\beta \in \Gamma$. Using the assumption (*) the identities $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z$, for all $x,y,z \in M$ and for all $\alpha,\beta \in \Gamma$ are used extensively in our results. A Γ-ring M is prime if $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$ and is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. An additive mapping $D: M \to M$ is called a derivation in case $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all pairs $x, y \in M$, $\alpha \in \Gamma$. A derivation D is inner in case there exists $a \in M$ such that $D(x) = [a, x]_\alpha$ holds for all $x \in M$, $\alpha \in \Gamma$. An additive mapping $T: M \to M$ is called a left centralizer in case $T(x\alpha y) = T(x)\alpha y$ is fulfilled for all pairs $x, y \in M$, $\alpha \in \Gamma$. For any fixed element $a \in M$, the mapping $T(x) = a\alpha x$, $x \in M$, $\alpha \in \Gamma$, is a left centralizer. In case M has the identity element $T: M \to M$ is a left centralizer if and only if T is of the form $T(x) = a\alpha x$, $x \in M$, $\alpha \in \Gamma$, where $a \in M$ is a fixed element. For a semiprime Γ-ring M, a mapping $T: M \to M$ is a left centralizer if and only if $T(x) = q\alpha x$ holds for all $x \in M$, $\alpha \in \Gamma$, where q is an element of Martindale right ring of quotients Q_r. An additive mapping $T: M \to M$ is said to be a right centralizer in case $T(x\alpha y) = x\alpha T(y)$ holds for all pairs $x, y \in M$, $\alpha \in \Gamma$. In case M has the identity element $T: M \to M$ is both left and right centralizer if and only if $T(x) = a\alpha x$, $x \in M$, $\alpha \in \Gamma$, where $a \in Z(M)$ is a fixed element. In case M is a semiprime Γ-ring with extended centroid C a mapping $T: M \to M$ is both left and right centralizer in case T is of the form $T(x) = p\alpha x$, $x \in M$, $\alpha \in \Gamma$, where $p \in C$ is a fixed element. An element $a \in M$ is said to be an element dependent on a mapping $F: M \to M$ if $F(x)\alpha a = a\alpha x$ holds for all $x \in M$, $\alpha \in \Gamma$. A mapping $F: M \to M$ is called a free action in case zero is the only element dependent on F. It is easy to see that in semiprime Γ-rings there are no nonzero nilpotent dependent elements (Ozturk and Jun, 2000, 2001).

The purpose of this study is to investigate dependent elements of some mappings related to derivations and automorphisms on prime and semiprime Γ-rings.

An additive mapping F: M→M, where M is an arbitrary ring, is called a generalized derivation in case $F(x\alpha y) = F(x)\alpha y+x\alpha D(y)$ holds for all pairs $x, y\in M$, $\alpha\in\Gamma$, where D: M→M is a derivation. It is easy to see that F is a generalized derivation if and only if F is of the form F = D+T, where D is a derivation and T a left centralizer.

In classical ring theories, dependent elements were implicitly used by Kallman (1969) to extend the notion of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann (1936). Several other authors have studied dependent elements in operator algebras (Neumann, 1940).

By the same motivations as in the classical ring theories, we study dependent elements in Γ-rings. In this study, we investigate dependents of some mappings related to derivations and automorphisms on prime and semiprime Γ-rings.

DEPENDENTS ELEMENTS OF PRIME AND SEMIPRIME Γ-RINGS

We will need the following two lemmas.

Lemma 1 (Chakraborty and Paul, 2007; Lemma 3.10): Let M be a 2-torsion-free semiprime Γ-ring and let $a,b\in M$. If, for all $x\in M$, the relation $a\alpha x\beta b+b\alpha x\beta a = 0$ holds, then $a\alpha x\beta b = b\alpha x\beta a = 0$ is fulfilled for all $x\in M$, $\alpha, \beta\in\Gamma$.

Lemma 2 (Ozturk and Jun, 2000; Lemma 3.6): Let M be a prime Γ-ring with extended centroid C and let $a,b\in M$ be such that $a\alpha x\beta b = b\alpha x\beta a$ holds for all $x\in M$, $\alpha,\beta\in\Gamma$. If $a\neq 0$, then there exists $p\in C$ such that $b = p\alpha a$.

Theorem 3: Let M be a semiprime Γ-ring satisfying the condition (*) and let D and G be derivations of M into itself. In this case the mapping $x\to D(D(x))+G(x)$ is a free action.

Proof: We have the relation:

$$F(x)\alpha a = a\alpha x,\ x\in M,\ \alpha\in\Gamma \tag{1}$$

where, F(x) stands for D(D(x))+G(x). A routine calculation shows that the relation:

$$F(x\alpha y) = F(x)\alpha y+x\alpha F(y)+2D(x)\alpha D(y) \tag{2}$$

holds for all pairs $x, y\in M$, $\alpha\in\Gamma$. Putting $x\beta a$ for x in Eq. 1 and using Eq. 2, we obtain $F(x)\beta a\alpha a+x\beta F(a)\alpha a+2D(x)\beta D(a)\alpha a = a\alpha x\beta a$, $x\in M$, $\alpha, \beta\in\Gamma$, which reduces because of (1) to:

$$2D(x)\alpha D(a)\beta a+x\alpha a\beta a = 0,\ x\in M,\ \alpha,\beta\in\Gamma \tag{3}$$

Putting, in the above relation, $y\delta x$ for x and applying Eq. 3 we obtain $2D(y)\delta x\alpha D(a)\beta a = 0$, $x, y\in M$, $\alpha,\beta,\delta\in\Gamma$, whence it follows, putting D(x) for x, that:

$$2D(y)\delta D(x)\alpha D(a)\beta a = 0,\ x,y\in M,\ \alpha,\beta,\delta\in\Gamma \tag{4}$$

Multiplying relation Eq. 3 from the left by D(y) and applying the above relation we obtain $D(y)\delta x\alpha a\beta a = 0$, $x,y\in M$, $\alpha,\beta,\delta\in\Gamma$, which gives, for $x = D(a)$ and $y = a$:

$$D(a)\delta D(a)\alpha a\beta a = 0,\ \alpha,\beta,\delta\in\Gamma \tag{5}$$

Multiplying relation (3) from the right by a, putting $x = a$ in Eq. 3 and applying the above relation we obtain $a\alpha a\beta a\delta a = 0$, which means that also $a = 0$. The proof of the theorem is complete.

Theorem 4: Let $F: M\rightarrow M$ be a generalized derivation, where M is a semiprime Γ-ring satisfying the condition (*) and let $a\in M$ be an element dependent on F. In this case $a\in Z(M)$.

Proof: We have the relation:

$$F(x)\alpha a = a\alpha x,\ x\in M,\ \alpha\in\Gamma \tag{6}$$

Let x be $x\beta y$ in the above relation. Then we have:

$$(F(x)\beta y+x\beta D(y))\alpha a = a\alpha x\beta y,\ x,y\in M,\ \alpha,\beta\in\Gamma \tag{7}$$

Using the fact that F can be written in the form $F = D+T$, where T is a left centralizer, we can replace $D(y)\alpha a$ by $F(y)\alpha a\rightarrow T(y)\alpha a$ in Eq. 7, which gives, because of Eq. 6:

$$F(x)\beta y\alpha a+[x,a]_\alpha\beta y-x\beta T(y)\alpha a = 0,\ x,y\in M,\ \alpha,\beta\in\Gamma \tag{8}$$

Let y be $y\delta F(x)$ in Eq. 8. We have:

$$F(x)\beta y\delta F(x)\alpha a+[x,a]_\alpha\beta y\delta F(x)-x\beta T(y)\delta F(x)\alpha a = 0,\ x,y\in M,\ \alpha,\beta,\delta\in\Gamma \tag{9}$$

which reduces, according to Eq. 6, to:

$$F(x)\beta y\delta a\alpha x+[x,a]_\alpha\beta y\delta F(x)-x\beta T(y)\delta a\beta x = 0,\ x,y\in M,\ \alpha,\beta,\delta\in\Gamma \tag{10}$$

Right multiplication of Eq. 8 by x gives:

$$F(x)\beta y\alpha a\delta x+[x,a]_\alpha\beta y\delta x-x\beta T(y)\alpha a\delta x = 0,\ x,y\in M,\ \alpha,\beta,\delta\in\Gamma \tag{11}$$

Subtracting Eq. 11 from Eq. 10 we arrive at:

$$[x,a]_\alpha\beta y\delta(F(x)-x) = 0,\ x,y\in M,\ \alpha,\beta,\delta\in\Gamma \tag{12}$$

Right multiplication of the above relation by a gives, because of Eq. 6, $[x,a]_\alpha\beta y\delta[x,a]_\alpha = 0$, $x,y\in M$, $\alpha,\beta,\delta\in\Gamma$, whence it follows that $[x,a]_\alpha = 0$, $x\in M$, $\alpha\in\Gamma$. The proof of the theorem is complete.

Corollary 5: Let M be a semiprime Γ-ring satisfying the condition (*) and let $a,b\in M$ be fixed elements. Suppose that $c\in M$ is an element dependent on the mapping $x\rightarrow a\alpha x+x\alpha b$, for all $\alpha\in\Gamma$. In this case $c\in Z(M)$.

Proof: A special case of Theorem 4, since it is easy to see that the mapping $x \rightarrow a\alpha x + x\alpha b$ is a generalized derivation.

In the theory of operator algebras the mappings $x \rightarrow a\alpha x + x\alpha b$, which we met in the above corollary, are considered as an important class of the so-called elementary operators (i.e., mappings of the form $x \rightarrow (\sum_{i=1}^{n} a_i \alpha_i x \beta_i b_i)$, α_i, $\beta_i \in \Gamma$.

Theorem 6: Let M be a prime Γ-ring satisfying the condition (*) with extended centroid C and let $a,b \in M$ be fixed elements. Suppose that $c \in M$ is an element dependent on the mapping $x \rightarrow a\alpha x\beta b$, for all $\alpha,\beta \in \Gamma$. In this case the following statements hold:

- $b\alpha c \in Z(M)$
- $a\alpha b\beta c = c$
- $c = p\alpha a$ for some $p \in C$

Proof: We will assume that $a \neq 0$ and $b \neq 0$ since there is nothing to prove in case $a = 0$ or $b = 0$. We have:

$$(a\alpha x\beta b)\delta c = c\beta x,\ x \in M,\ \alpha,\beta,\delta \in \Gamma \tag{13}$$

Let x be $x\gamma y$ in Eq. 13. Then:

$$(a\alpha x\gamma y\beta b)\delta c = c\delta x\gamma y,\ x,y \in M,\ \alpha,\beta,\delta,\gamma \in \Gamma \tag{14}$$

According to Eq. 13 one can replace $c\delta x$ by $(a\alpha x\beta b)\delta c$ in the above relation. Then we have:

$$a\alpha x\beta[b\delta c,y]_\gamma = 0,\ x,y \in M,\ \alpha,\beta,\delta,\gamma \in \Gamma \tag{15}$$

which gives $b\delta c \in Z(M)$, which makes it possible to rewrite relation Eq. 13 in the form:

$$(a\alpha b\delta c - c)\beta x = 0,\ x \in M,\ \alpha,\beta,\delta \in \Gamma \tag{16}$$

whence it follows that:

$$a\alpha b\delta c = c.\ \alpha,\delta \in \Gamma \tag{17}$$

Putting $x\lambda a$ for x in relation in Eq. 13 we obtain, because of Eq. 17:

$$a\alpha x\lambda c = c\alpha x\lambda a,\ x \in M,\ \alpha,,\gamma \in \Gamma \tag{18}$$

whence it follows, according to Lemma 2, that $c = p\alpha a$ for some $p \in C$, $\alpha \in \Gamma$. The proof of the theorem is complete.

Corollary 7: Let M be a prime Γ-ring satisfying the condition (*) with the identity element and extended centroid C and let $\sigma(x) = a\alpha x\beta a\text{-}1$, $x \in M$, $\alpha,\beta \in \Gamma$ be an inner automorphism of M. An element $b \in M$ is an element dependent on σ if and only if $b = p\alpha a$ for some $p \in C$.

Proof: According to Theorem 6 any element dependent on σ is of the form $p\alpha a$ for some $p\in C$, $\alpha\in\Gamma$. It is trivial to see that any element of the form $p\alpha a$, where $p\in C$, $\alpha\in\Gamma$, is an element dependent on σ.

We proceed to our next result.

Theorem 8: Let M be a 2-torsion-free prime Γ-ring satisfying the condition (*) and let $a, b\in M$ be fixed elements. Suppose that $c\in M$ is an element dependent on the mapping $x \rightarrow a\alpha x\beta b + b\alpha x\beta a$, for all $\alpha,\beta\in\Gamma$. In this case the following statements hold:

- $a\alpha c\in Z(M)$ and $b\alpha c\in Z(M)$
- $(a\alpha b+b\alpha a)\beta c = c$
- $c\alpha c\in Z(M)$

Proof: Similarly, as in the proof of Theorem 6, we will assume that $a \rightarrow 0$ and $b \rightarrow 0$. We have the relation:

$$(a\alpha x\beta b + b\alpha x\beta a)\ c = c\delta x,\ x\in M,\ \alpha,\beta,\delta\in\Gamma \tag{19}$$

Let x be $x\lambda y$ in the above relation. Then we have:

$$(a\alpha x\lambda y\beta b + b\alpha x\lambda y\beta a)\delta c = c\delta x\lambda y,\ x,y\in M,\ \alpha,\beta,\delta,\lambda\in\Gamma \tag{20}$$

Right multiplication of relation in Eq. 19 by y gives:

$$(a\alpha x\beta b+b\alpha x\beta a)\lambda c\lambda y = c\delta x\lambda y,\ x,y\in M,\ \alpha,\beta,\delta,\lambda\in\Gamma \tag{21}$$

Subtracting Eq. 21 from Eq. 20 we arrive at:

$$a\alpha x\beta[y,b\delta c]_\lambda + b\alpha x\beta[y,a\delta c]_\lambda = 0,\ x,y\in M,\ \alpha,\beta,\delta,\lambda\in\Gamma \tag{22}$$

Putting $c\gamma x$ for x in the above relation we arrive at:

$$a\alpha c\gamma x\beta[y,b\delta c]_\lambda + b\alpha c\gamma x\beta[y,a\delta c]_\lambda = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma \tag{23}$$

Now, multiplying the above relation first from the left by y, then putting $y\lambda x$ for x in Eq. 23 and finally subtracting the relations so obtained from one another, we arrive at:

$$[y,a\delta c]_\lambda\gamma x\beta[y,b\delta c]_\lambda + [y,b\delta c]_\lambda\gamma x\beta[y,a\delta c]_\lambda = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma \tag{24}$$

Suppose that $a\delta c\rightarrow Z(M)$. In this case we have $[y,a\delta c]_\lambda \rightarrow 0$ for some $y\in M$. Then it follows from relation Eq. 24 and Lemma 1 that $[y,b\delta c]_\lambda = 0$, which reduces relation (22) to $b\gamma x\beta[y,a\delta c]_\lambda = 0$, $x,y\in M$, $\beta,\delta,\gamma,\lambda\in\Gamma$, which means (recall that b is different from zero) that $[y,a\delta c]_\lambda = 0$, contrary to the assumption. We have therefore $a\delta c\in Z(M)$. Now relation Eq. 22 reduces to $a\alpha x\beta[y,b\delta c]_\lambda = 0$, $x,y\in M$, $\alpha,\beta,\delta,\lambda\in\Gamma$, whence it follows that $bc\in Z(M)$. Since $a\delta c$ and $b\delta c$ are in $Z(M)$, one can write relation in Eq. 19 in the form $((a\beta b + b\beta a)\delta c - c)\lambda x = 0$, $x\in M$, $\alpha,\beta,\delta,\gamma,\lambda\in\Gamma$, which gives:

$$(a\beta b + b\beta a)\delta c = c \tag{25}$$

Putting x = c in relation in Eq. 19 we obtain:

$$2\,(a\delta c)\,\beta\,(b\delta c) = c\delta c \tag{26}$$

Since, $a\delta c$ and $b\delta c$ are both in Z(M) it follows from the above relation that $cc\in Z(M)$. The proof of the theorem is complete.

Theorem 9: Let M be a 2-torsion-free prime Γ-ring satisfying the condition (*) with extended centroid C and let $a,b\in M$ be fixed elements. In this case the mapping $x\rightarrow a\alpha x\beta b \rightarrow b\beta x\alpha a$ is a free action for all $\alpha,\beta\in\Gamma$.

Proof: Again we assume that $a\neq 0$ and $b\neq 0$. Besides, we will also assume that a and b are C-independent, otherwise the mapping $x\rightarrow a\alpha x\beta b\text{-}b\alpha x\beta a$, $\alpha,\beta\in\Gamma$ would be zero. We have the relation:

$$(a\alpha x\beta b\text{-}b\alpha x\beta a)\delta c = c\delta x,\ x\in M,\ \alpha,\beta,\delta\in\Gamma \tag{27}$$

Let x be $x\gamma y$ in the above relation. Then we have:

$$(a\alpha x\gamma y\beta b\text{-}b\alpha x\gamma y\beta a)\delta c = c\delta x\gamma y,\ x,y\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \tag{28}$$

Right multiplication of relation Eq. 27 by y gives:

$$(a\alpha x\beta b\ \text{-}b\alpha x\beta a)\delta c\gamma y = c\delta x\gamma y,\ x,y\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \tag{29}$$

Subtracting Eq. 29 from Eq. 28 we arrive at:

$$a\alpha x\beta[y,b\delta c]_\gamma\text{-}b\alpha x\beta[y,a\delta c]_\gamma = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \tag{30}$$

Putting $c\lambda x$ for x in the above relation we arrive at:

$$a\alpha c\lambda x\beta[y,b\delta c]_\gamma \rightarrow b\alpha c\lambda x\beta[y,a\delta c]_\gamma = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma \tag{31}$$

Now, multiplying first the above relation from the left by y, then putting yx for x in Eq. 31 and finally subtracting the relations so obtained from one another, we arrive at:

$$[y,a\delta c]_\gamma\lambda x\beta[y,b\delta c]_\gamma - [y,b\delta c]_\gamma\lambda x\beta[y,a\delta c]_\gamma = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma \tag{32}$$

Suppose that $a\delta c\rightarrow Z(M)$. In this case there exists $y\in M$ such that $[y,a\delta c]_\gamma \rightarrow 0$. Now it follows from the above relation and from Lemma 2 that:

$$[y,b\delta c]_\gamma = p_y\beta[y,a\delta c]_\gamma \tag{33}$$

holds for some $p_y \in C$. According to Eq. 33 one can replace $[y,b\delta c]_\gamma$ by $p\alpha y\beta[y,a\delta c]_\gamma$ in Eq. 30, which gives:

$$(b-p_y\alpha a)\beta x\beta[y,a\delta c]_\gamma = 0,\ x\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma \tag{34}$$

Since $[y,a\delta c]_\gamma \rightarrow 0$ it follows from the above relation that b = p y a, contrary to the assumption that a and b are C-independent. We have therefore proved that $ac\in Z(M)$. Using this fact relation (30) reduces to:

$$a\alpha x\beta[y,\ b\delta c]_\gamma = 0,\ x,y\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \tag{35}$$

whence it follows (recall that $a\neq 0$) that $b\delta c\in Z(M)$. Since a c and b c are both in Z(M), one can rewrite relation in the form $((a\beta b-b\beta a)\delta c-c)\gamma x = 0,\ x\in M,\ \alpha,\beta,\delta,\gamma,\lambda\in\Gamma$. which gives:

$$(a\beta b-b\beta a)\delta c = c \tag{36}$$

Putting x = c in relation in Eq. 27 and using the fact that $b\delta c$ is in Z (M), we obtain:

$$a\beta[b,c]_\gamma\delta c = -c\ c \tag{37}$$

From relation in Eq. 36 one obtains, using the fact that $a\delta c\in Z(M)$:

$$a\beta[b,c]_\gamma = c \tag{38}$$

Right multiplication of the above relation by c gives:

$$a\beta[b,c]_\gamma\delta c = c\delta c \tag{39}$$

Comparing relations in Eq. 37 and 39 one obtains $c\delta c = 0$, since M is 2-torsion-free. Now it follows that c = 0, which completes the proof of the theorem.

Theorem 10: Let M be a semiprime Γ-ring satisfying the condition (*) and let σ and τ be automorphisms of M. In this case the mapping σ+τ is a free action.

Proof: We have the relation:

$$(\sigma(x) + \tau(x))\alpha a = a\alpha x,\ x\in M,\ \alpha\in\Gamma \tag{40}$$

Let x be $x\beta y$ in the above relation. Then:

$$(\sigma(x)\beta\sigma(y) + \tau(x)\beta\tau(y))\alpha a = a\alpha x\beta y,\ x,y\in M,\ \alpha,\beta\in\Gamma \tag{41}$$

Replacing first $a\alpha x$ by $(\sigma(x) + \tau(x))\alpha a$ in the above relation and then $a\beta y$ by $(\sigma(y)+\tau(y))\beta a$, we arrive at:

$$(\sigma(x)\beta\sigma(y) + \tau(x)\beta\tau(y))\alpha a = (\sigma(x) + \tau(x))\alpha\sigma(y) + \tau(y))\beta a, \ x,y\in M, \ \alpha,\beta\in\Gamma \quad (42)$$

which reduces to:

$$\sigma(x)\beta\tau(y)a + \tau(x)\beta\sigma(y)\alpha a = 0, \ x,y\in M, \ \alpha,\beta\in\Gamma \quad (43)$$

The substitution $z\delta x$ for x in the above relation gives:

$$\sigma(z)\delta\sigma(x)\beta\tau(y)\alpha a + \tau(z)\delta\tau(x)\beta\sigma(y)\alpha a = 0, \ x,y,z\in M, \ \alpha,\beta,\delta\in\Gamma \quad (44)$$

Left multiplication of (43) by $\sigma(z)$ gives:

$$\sigma(z)\delta\sigma(x)\beta\tau(y)\alpha a + \sigma(z)\delta\tau(x)\beta\sigma(y)\alpha a = 0, \ x,y,z\in M, \ \alpha,\beta,\delta\in\Gamma \quad (45)$$

Subtracting Eq. 44 from Eq. 45, we arrive at $(\sigma(z)-\tau(z))\delta\tau(x)\beta\sigma(y)\alpha a = 0$, $x,y,z\in M$, $\alpha,\beta,\delta\in\Gamma$. We therefore have:

$$(\sigma(z)-\tau(z))\delta x\beta y\alpha a = 0, \ x,y,z\in M, \ \alpha,\beta,\delta\in\Gamma \quad (46)$$

Putting $x = a$ and $x\delta$ $(\sigma(z)-\tau(z))$ for y in the above relation, we obtain $(\sigma(z)-\tau(z))\delta a\beta x\delta(\sigma(z)- \tau(z))\alpha a = 0$, $x,z\in M$, $\alpha,\beta,\delta\in\Gamma$, whence it follows that:

$$\sigma(z)\delta a = \tau(z)\delta a, \ z\in M, \ \delta\in\Gamma \quad (47)$$

According to Eq. 47 one can replace $\tau(y)\alpha a$ by $\sigma(y)\alpha a$ in Eq. 43, which gives $(\sigma(x)+\tau(x))\beta\sigma(y)\alpha a = 0$, $x,y\in M$, $\alpha,\beta\in\Gamma$. We therefore have:

$$(\sigma(x)+\tau(x))\beta y\alpha a = 0, \ x,y\in M, \ \alpha,\beta\in\Gamma \quad (48)$$

Putting $y = a$ in the above relation and replacing $(\sigma(x)+\tau(x))\beta a$ by $a\beta x$, we obtain $a\beta x\alpha a = 0$, $x\in M$, $\alpha,\beta\in\Gamma$ which gives $a = 0$. The proof of the theorem is complete.

The mapping σ-τ is a special case of the so-called (σ, τ)-derivations. An additive mapping $D: M\to M$, where M is an arbitrary ring, is an (σ,τ)-derivation if $D(x\alpha y) = D(x)\alpha\sigma(y)+\tau(x)\alpha D(y)$ holds for all pairs $x,y\in M$, $\alpha\in\Gamma$, where σ and τ are automorphisms of M.

Theorem 11: Let M be a semiprime Γ-ring satisfying the condition (*) and let $D: M\to M$ be an (σ, τ)-derivation. In this case D is a free action.

Proof: We have the relation:

$$D(x)\alpha a = a\alpha x, \ x\in M, \ \alpha\in\Gamma \quad (49)$$

Putting $x\beta y$ for x in the above relation we obtain:

$$D(x)\beta\sigma(y)\alpha a+\tau(x)\beta D(y)\alpha a = a\alpha x\beta y, \ x,y\in M, \ \alpha,\beta\in\Gamma \quad (50)$$

According to Eq. 49 one can replace $D(y)\alpha a$ by $a\alpha y$ in the above relation, which gives:

$$D(x)\beta\sigma(y)\alpha a+(\tau(x)\alpha a-a\alpha x)\beta y = 0,\ x,y\in M,\ \alpha,\beta\in\Gamma \quad (51)$$

Putting $y\delta z$ for y in Eq. 51 we obtain:

$$D(x)\beta\sigma(y)\delta\sigma(z)\alpha a+(\tau(x)\alpha a-a\alpha x)\beta y\delta z = 0,\ x,y,z\in M,\ \alpha,\beta,\delta\in\Gamma \quad (52)$$

On the other hand, right multiplication of Eq. 51 by z gives:

$$D(x)\beta\sigma(y)\alpha a\delta z+(\tau(x)\alpha a-a\alpha x)\beta y\delta z = 0,\ x,y,z\in M,\ \alpha,\beta,\delta\in\Gamma \quad (53)$$

Subtracting (53) from (52) we obtain $D(x)\beta\sigma(y)\delta(\sigma(z)\alpha a-a\alpha z) = 0$, $x,y,z \in M$, $\alpha,\beta,\delta\in\Gamma$. In other words, we have:

$$D(x)\beta y\delta(\sigma(z)\alpha a-a\alpha z) = 0,\ x,y,z\in M,\ \alpha,\beta,\delta\in\Gamma \quad (54)$$

The substitution $a\gamma y$ for y in the above relation gives, because of Eq. 49:

$$a\beta x\gamma y\delta(\sigma(z)\alpha a-a\alpha z) = 0,\ x,y,z\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \quad (55)$$

Putting $z\alpha x$ for x in the above relation we obtain:

$$a\beta z\alpha x\gamma y\delta(\sigma(z)\alpha a-a\alpha z) = 0,\ x,y,z\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \quad (56)$$

Left multiplication of Eq. 55 by $\sigma(z)$ gives:

$$\sigma(z)\alpha a\beta x\gamma y\delta(\sigma(z)\alpha\ a-a\alpha z) = 0,\ x,y,z\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \quad (57)$$

Subtracting Eq. 56 from Eq. 57 and multiplying the relation so obtained from the right hand side by x, we arrive at:

$$(\sigma(z)\alpha a-a\alpha z)\beta x\gamma y\delta(\sigma(z)\alpha a-a\alpha z)\beta x = 0,\ x,y,z\in M,\ \alpha,\beta,\delta,\gamma\in\Gamma \quad (58)$$

which gives first:

$$(\sigma(z)\alpha a-a\alpha z)\ x = 0,\ x,z\in M,\ \alpha\in\Gamma \quad (59)$$

and then:

$$\sigma(z)\alpha a-a\alpha z = 0,\ z\in M,\ \alpha\in\Gamma \quad (60)$$

Putting $D(x)\alpha a$ instead of $a\alpha x$ in Eq. 50 and $a\beta y$ for $D(y)\beta a$, we obtain $D(x)\beta(\sigma(y)\alpha a-a\alpha y)+ \tau(x)\beta a\alpha y = 0$, $x,y\in M$, $\alpha,\beta\in\Gamma$, which reduces because of (60) to $\tau(x)\ a\ y = 0$, $x,y\in M$, whence it follows that $a = 0$. The proof of the theorem is complete.

Corollary 12: Let M be a semiprime Γ-ring satisfying the condition (*) and let σ and τ be automorphisms of M. In this case the mappings σ-τ and aασ-τβa, where a∈M, α,β∈Γ, is a fixed element, are free actions on M.

Proof: According to Theorem 11 there is nothing to prove, since the mappings σ-τ and aασ-τβa are (σ, τ)-derivations.

Corollary 13: Let M be a semiprime Γ-ring satisfying the condition (*), let D: M→M be a derivation and let σ be an automorphism of M. In this case the mappings x→D(σ(x)), x→σ(D(x)), x→D(α(x)) +σ(D(x)) and x→D(σ(x))-σ(D(x)) are free actions.

Proof: A special case of Theorem 11, since all mappings are (σ, σ)-derivations. For our next result we need the following lemma.

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