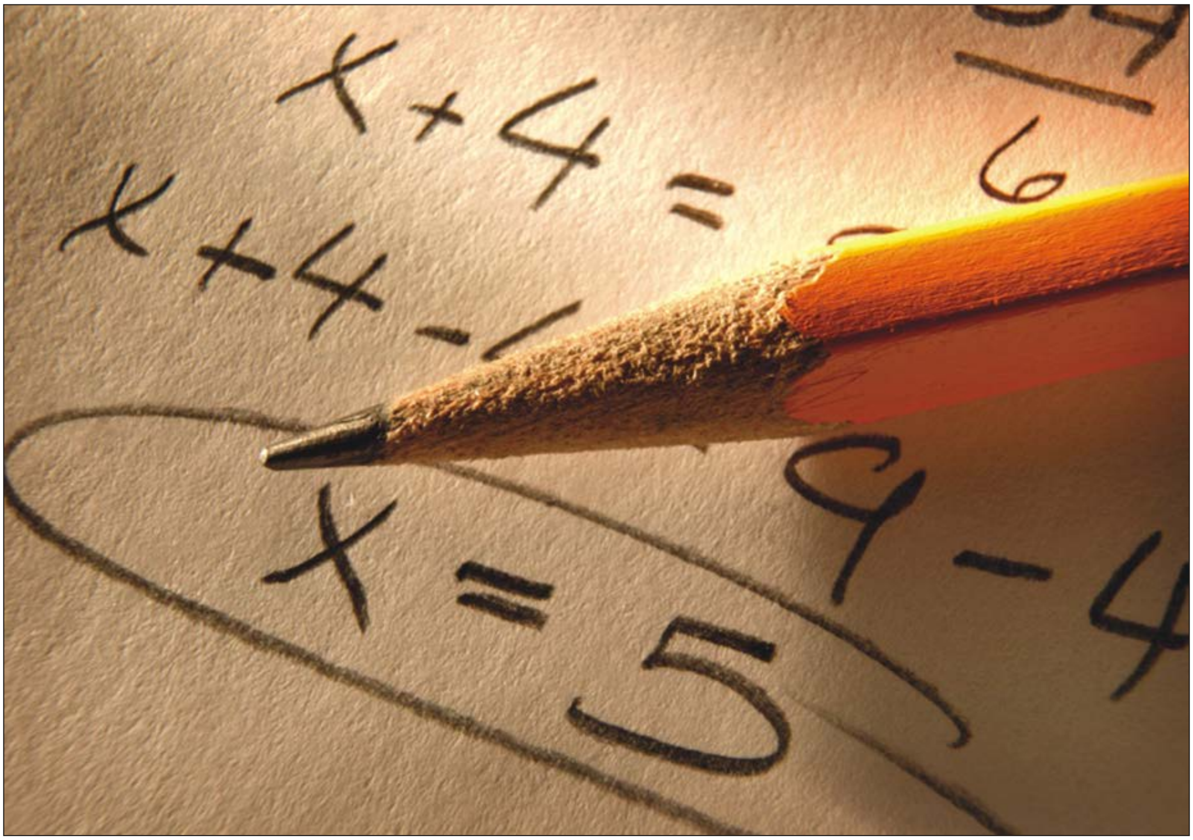

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Quasi Valuation and Valuation Derived from Filtered Ring and their Properties

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ABSTRACT

In this study we show if R is a filtered ring then we can define a quasi valuation. And if R is some kind of filtered ring then we can define a valuation. Then we prove some property and relation for R.

Key words: Filtered ring, quasi valuation ring, valuation ring

INTRODUCTION

In algebra valuation ring and filtered ring are two most important structure (Nishida, 2005; Puninskia *et al*., 2007; Rush, 2007). We know that filtered ring is also the most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some similar results, on the Andreadakis-Johnson filtration of the automorphism group of a free group (Cohen *et al*., 2010), on the depth of the associated graded ring of a filtration (Gopalakrishnan, 1983; Hazewinkel *et al*., 2004). So, as these important structures, the relation between these structure is useful for finding some new structures and if R is a discrete valuation ring then R has many properties that have many usage for example decidability of the theory of modules over commutative valuation domains (Rush, 2007), Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices (Puninskia *et al*., 2007).

In this study we investigate the relation between filtered ring and valuation and quasi valuation ring. We prove that if we have filtered ring then we can find a quasi valuation on it. Continuously we show that if R be a strongly filtered then exist a valuation, similarly we prove it for PID. At the end we explain some properties for them.

PRELIMINARIES

Definition 1: A filtered ring R is a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying in the following conditions:

- $R_0 = R$
- $R_{n+1} \subseteq R_n$ for all $n \geq 0$
- $R_n R_m \subseteq R_{n+m}$ for all n, $m \geq 0$

Definition 2: Let R be a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the following conditions:

- $R_0 = R$
- $R_{n+1} \subseteq R_n$ for all $n \geq 0$
- $R_n R_m = R_{n+m}$ for all n, $m \geq 0$

Then we say R has a strong filtration.

Definition 3: Let R be a ring and I an ideal of R. Then $R_n = I^n$ is called I-adic filtration.

Definition 4: A map f: M→N is called a homomorphism of filtered modules if: (i) f is R-module an homomorphism and (ii) $F(M_n) \subseteq N_n$ for all $n \geq 0$.

Definition 5: A subring R of a filed K is called a valuation ring of K if for every $\alpha \in k$, $\alpha \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$.

Definition 6: Let Δ be a totally ordered abelian group. A valuation ν on R with values in Δ is a mapping ν: $R^* \to \Delta$ satisfying:

- $\nu(ab) = \nu(a) + \nu(b)$
- $\nu(a+b) \geq \text{Min}\{\nu(a), \nu(b)\}$

Definition 7: Let Δ be a totally ordered abelian group. A quasi valuation ν on R with values in Δ is a mapping ν: $R^* \to \Delta$ satisfying:

- $\nu(ab) \geq \nu(a) + \nu(b)$
- $\nu(a+b) \geq \text{Min}\{\nu(a), \nu(b)\}$

Remark 1: R is said to be vaulted ring; $R_v = \{X \in R: \nu(x) \geq 0\}$ and $\nu^{-1}(\infty) = \{x \in R: \nu(x) \infty\}$.

Definition 8: Let K be a filed. A discrete valuation on K is a valuation ν: $K^* \to Z$ which is surjective.

Theorem 1: If R is a UFD then R is a PID (Lam, 1991).

Proposition 1: Any discrete valuation ring is a Euclidean domain (Hazewinkel *et al.*, 2004).

Remark 2: If R is a ring, we will denote by Z(R) the set of zero-divisors of R and by T(R) the total ring of fractions of R.

Definition 9: A ring R is said to be a Manis valuation ring (or simply a Manis ring) if there exist a valuation ν on its total fractions T(R), such that $R = R_v$.

Definition 10: A ring R is said to be a Prüfer ring if each overring of R is integrally closed in T(R).

Definition 11: A Manis ring R_v is said to be ν-closed if $R_v/\nu^{-1}(\infty)$ is a valuation domain (Theorem 2 of Zanardo, 1990).

QUASI VALUATION AND VALUATION DERIVED FROM FILTERED RING

Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n>0}$.

Lemma 1a: Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to Z$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\}$. Then we have $v(\alpha\beta) \geq v(\alpha)+v(\beta)$.

Proof: For any α, $\beta \in R$ with $v(\alpha) = i$ and $v(\beta) = j$, $\alpha\beta \in R_i R_j \subseteq R_{i+j}$. Now let $v(\alpha\beta) = k$ then we have $\alpha\beta \in R_k \setminus R_{k+1}$.

We show that $k \geq i+j$.

Let $k<i+j$ so we have $k+1 \leq i+j$ hence $R_{K+1} \supset R_{i+j}$ then $\alpha\beta \in R_{i+j} \subseteq R_{k+1}$ it is contradiction. So $k \geq i+j$. Now we have $v(\alpha\beta) \geq v(\alpha)+v(\beta)$.

Lemma 2a: Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to Z$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\}$. Then $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$.

Proof: For any α, $\beta \in R$ such that $v(\alpha) = i$ also $v(\beta) = j$ and $v(\alpha+\beta) = k$ so we have $\alpha+\beta \in R_k \setminus R_{k+1}$. Without losing the generality, let $i<j$ so $R_j \subset R_i$ hence $\beta \in R_i$. Now if $k<i$ then $k+1 \leq i$ and $R_i \subset R_{k+1}$ so $\alpha+\beta \in R_i \subset R_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$.

Theorem 1a: Let R be a filtered ring. Then there exist a quasi valuation on R.

Proof: Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to Z$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\}$. Then:

- By lemma (1a) we have $v(\alpha\beta) \geq v(\alpha)+v(\beta)$
- By lemma (2a) we have $v(\alpha+\beta) \geq \min\{v(\alpha)+v(\beta)\}$. So by "Defination 7" R is quasi valuation ring

Proposition 1a: Let R be a strongly filtered ring. Then there exists a valuation on R.

Proof: By theorem (1a) we have $v(\alpha\beta) \geq v(\alpha)+v(\beta)$ and $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha)+v(\beta)$. Let $v(\alpha\beta) > v(\alpha)+v(\beta)$ so $k>i+j$ and it is contradiction. So $v(\alpha\beta) = v(\alpha)+v(\beta)$, then there is a valuation on R.

Corollary 1a: Let R be a strongly filtered ring, then R is a Euclidean domain.

Proof: By proposition (1a) R is a discrete valuation and so by proposition (1) R is a Euclidean domain.

Proposition 2a: Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then there exists a valuation on R.

Proof: By theorem (1a) we have $v(\alpha\beta) \geq v(\alpha)+v(\beta)$ and $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha)+v(\beta)$. Let $v(\alpha\beta) > v(\alpha)+v(\beta)$ so $k>i+j$ then $\alpha\beta \in P^k \subseteq P^{i+j}$ and $k \geq i+j+1$, since P is a prime ideal hence $\alpha \in P^{i+1}$ or $\beta \in P^{j+1}$ and it is contradiction. So $v(\alpha\beta) = v(\alpha)+ v(\beta)$, then there is a valuation on R.

Proposition 3a: Let R be a PID then there is a valuation on R.

Proof: By theorem (1a) and proposition (2a) there is a valuation on R.

Corollary 2a: If R is an UFD then there exists a valuation on R, then R is a Euclidean domain.

Corollary 3a: Let R be a ring and P is a prime ideal of R. If R has a P_adic filtration and $R = \bigcup_{i=0}^{+\infty} P^i$, then R is a Euclidean domain.

Proof: By proposition (2a) R is a discrete valuation and so by proposition (1) R is a Euclidean domain.

Corollary 4a: Let R be a PID then R is a Euclidean domain.

Proof: By proposition (3a) and proposition (1) we have R is a Euclidean domain.

Corollary 5a: Let R be a UFD then R is a Euclidean domain.

Corollary 6a: Let R be a strongly filtered ring. Then R is Manis ring.

Corollary 7a: Let P is a prime ideal of R and $\{P^n\}_{n>0}$ be P-adic filtration. Then R is Manis ring.

Proposition 4a: Let R_v be a Manis ring. If R_v is v-closed, then R_v is Prüfer.

Proof: Proposition 1 of Zanardo (1993).

Proposition 5a: Let R be a strongly filtered ring. Then R is v-closed.

Proof: By proposition (1a) and definition (9) we have R is Manis ring and $R = R_v$.

Now let $\alpha, \beta \in R$ and:

$$v(\alpha) = i \text{ and } v(\beta) = j$$

consequently if:

$$(\alpha + v^{-1}(\infty))(\beta + v^{-1}(\infty)) \in v^{-1}(\infty)$$

Then $i+j \geq \infty$ so $\alpha \in v^{-1}(\infty)$ or $\beta \in v^{-1}(\infty)$. Hence by definition (11) R is v-closed.

Corollary 8a: Let P be a strongly filtered ring. Then R is Prüfer.

Proof: By proposition (6a) R is v-closed so by proposition (4a) R is Prüfer.

Proposition 6a: Let P is a prime ideal of R and $\{P_n\}_{n>0}$ be P-adic filtration. Then R is v-closed.

Proof: By proposition (2a) and definition (9) we have R is Manis ring and $R = R_v$.
Now let $\alpha, \beta \in R$ and:

$$\nu(\alpha) = i \text{ and } \nu(\beta) = j$$

consequently if:

$$(\alpha+\nu^{-1}(\infty))(\beta+\nu^{-1}(\infty))\in\nu^{-1}(\infty)$$

Then $i+j\geq\infty$ so $\alpha\in\nu^{-1}(\infty)$ or $\beta\in\nu^{-1}(\infty)$. Hence by definition (11) R is ν-closed.

Corollary 9a: Let P is a prime ideal of R and $\{P^n\}_{n>0}$ be P-adic filtration. Then R is Prüfer.

Proof: By proposition (6a) R is ν-closed so by proposition (4a) R is Prüfer.

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