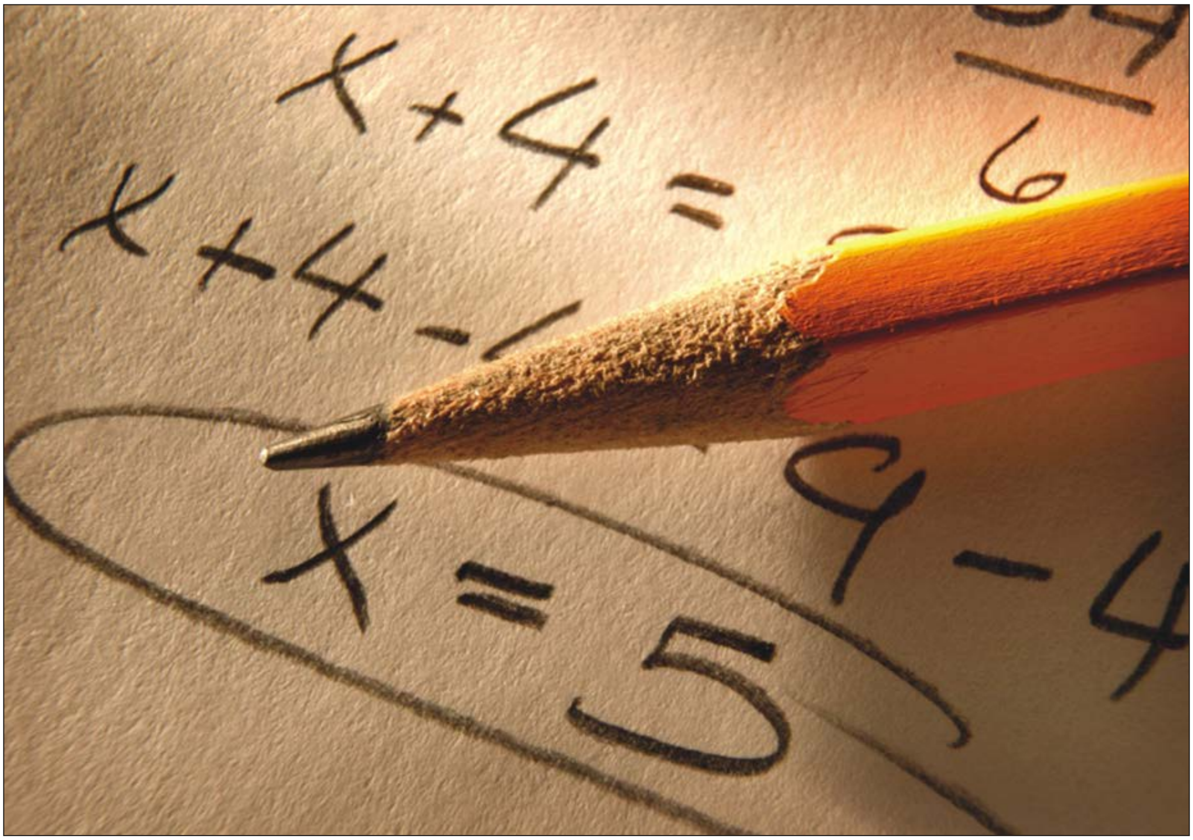

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(-1, 1) and Generalized Kac-Moody Algebras

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ABSTRACT

This study determines all third power-associative (-1, 1) algebras, whose commutator algebras are generalized Kac-Moody algebras.

Key words: (-1, 1) algebras, power-associative, commutator algebras, generalized kac-moody algebras, cartan matrix

INTRODUCTION

Associated with any algebra Θ over a field of char. ≠ 2 are two algebras denoted by Θ^- and Θ^+. There algebras have the same underlying vector space as Θ but are given the products [x, y] = x * y-y * x and x∘y = ½(x * y+y * x), respectively, where * is the multiplication in Θ.

Albert (1948), proposed the problem of classifying all power-associative flexible Lie admissible algebras, whose commutator algebras are semi simple Lie algebras. Without assuming flexibility, Benkart (1984) classified all third power-associative Lie admissible algebras whose commutator algebras are semisimple Lie algebras. Myung (1985) classified all third power-associative Lie admissible algebras associated with the Virasoro algebra and the Witt albebra. Jeong *et al.* (1997) determined all third power-associative Lie admissible algebras whose commentator algebras are Kac-Moody algebras. In this study, we will determine all third power associative (-1, 1) algebras whose commutator algebras are simple generalized Kac-Moody algebras. Thedy (1975) collected information on two natural concepts in a right alternative algebra R, the submodule M generated by all alternators (x, x, y) and a new nucleus called alternative nucleus. He also studied the properties of (-1, 1) rings. Hentzel (1974), characterized the properties of (-1, 1) rings and showed that all simple (-1, 1) rings are associative. He developed several identities for (-1, 1) rings. Benkart (1982), classified all power-associative products * that can be defined on the algebra A of n/sup x/n matrices over a 2, 3-torsion free field F satisfying the condition x*y-y*x = xy-yx for all x, y ∈ A and stated that such product * are automatically Lie-admissible and they are Jordan-admissible too. Myung (1996) studied the properties of Lie-admissible algebras along with Virasoro algebra and determined third power associative Lie-admissible algebras whose commutator algebras are Virasoro algebras. Humphreys (1972), developed the basic general theory of Lie algebras to give a first insight into basics of the structure theory and representation theory of semisimple Lie algebras. It is noted that a 2, 3-torsion free right alternative algebra is a (-1, 1) algebra if it satisfies the identity (x, y, z) + (y, z, x) + (z, x, y) = 0.

MATERIALS AND METHODS

Preliminaries: We begin with some of the basic facts about (-1, 1) algebras. Let (Θ, *) be a (nonassociative) algebra over the complex field $\mathbb{C}$ with multiplication * and let (Θ^-, [,]) be its commutator algebra, where the underlying space is the same as Θ and the bracket operation $[,]:\Theta\times\Theta\rightarrow\Theta$ is defined by [x, y] = x * y-y * x for all x, y ∈ Θ.

An algebra Θ is called a (-1, 1) algebra if it satisfies the identity:

$$(x, y, y) = 0 \quad (1)$$

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad (2)$$

where, (x, y, z) = (xy)z - x(yz) denotes the associator of elements x, y and z. Identity (1) is meant to imply that Θ is right alternative and (2) implies that Θ is Lie-admissible, that is, the commutator algebra Θ^{-} with a product [x, y] = xy - yx is a Lie algebra.

A ring (Θ, *) is third power-associative, if $x * (x * x) = (x * x) * x$ for $x \in R$ (3)

For a (-1, 1) ring (Θ, *), the multiplication * can be written as

$$x * y = \frac{1}{2}[x, y] + x \circ y \quad (4)$$

Equation 3 is equivalent to

$$[x, x \circ y] = 0 \quad (5)$$

for all $x \in R$ which can be linearized to

$$2[x, x \circ y] + [y, x \circ x] = 0 \quad (6)$$

for all $x, y \in R$. Again linearizing (6) we obtain

$$[x, y \circ z] + [y, z \circ x] + [z, x \circ y] = 0 \quad (7)$$

for all $x, y, z \in R$.

The structure of generalized Kac-Moody algebras has been considered by many authors (Kac, 1990). A generalized Kac-Moody algebra is a Lie algebra that is similar to a Kac-Moody algebra, except that it is allowed to have imaginary simple roots.

Generalized Kac-Moody algebras are also sometimes called GKM algebras. The best known example is the monster Lie algebra. Most properties of GKM algebras are straight forward extensions of the usual properties of Kac-Moody algebras. A GKM algebra has an invariant symmetric bilinear form such that $(e_i, f_i) = 1$. There is a character formula for highest weight modules, similar to the Weyl-Kac character formula for Kac-Moody algebras except that it has correction terms for the imaginary simple roots.

A symmetrized Cartan matrix is a (Possibly infinite) square matrix with entries x_{ij} such that:

$$x_{ij} = x_{ji}$$
$$x_{ij} \leq 0, \text{ if } i \neq j$$
$$2x_{ij}/x_{ij} \text{ is an integer if } x_{ii} > 0$$

The universal generalized Kac-Moody algebra with given symmetrized cartan matrix is defined by generator e_i, f_i and h_i and relations:

$$[e_i, f_i] = h_i \text{ if } i \neq j, \text{ otherwise}$$
$$[h_i, e_j] = x_{ij} e_j, \ [h_i, f_j] = -x_{ij} f_j$$
$$[e_i, [e_i, \ldots, [e_i, e_j]]] = [f_i, [f_i, \ldots [f_i, f_j]]] = 0$$
$$\text{for } 1\text{-}2x_{ij}/x_{ii} \text{ applications of } e_i \text{ or } f_i \text{ if } x_{ii}>0$$
$$[e_i, e_j] = [f_i, f_j] = 0 \text{ if } x_{ij} = 0$$

These differ from the relations of a (symmetrizable) Kac-Moody algebra mainly by allowing the diagonal entries of a Cartan matrix to the nonpositive. In other words we allow simple roots to be imaginary, whereas in a Kac-Moody algebra simple roots are always real.

A generalized Kac-Moody algebra is obtained from a universal one by changing the certain matrix by the operations of killing something in the center or taking a central extension or adding outer derivations.

We will use the identities (3), (4), (5), (6) and (7) frequently in determining the third power-associative (-1, 1) multiplications on generalized Kac-Moody algebras and our approach is based on the techniques developed by Benkart (1984), Jeong *et al.* (1997) and Myung (1996).

Let $A = (a_{ij})_{i, j \in I}$ be a generalized Cartan matrix where I is the finite index and g = g (A) be the generalized Kac-Moody algebra associated with A. We will determine all the third power associative (-1, 1) structures on the generalized Kac-Moody algebra g = g (A). When A is of finite type, this problem was settled by Benkart (1984). Hence we will assume that the generalized Cartan matrix is not of finite type.

We start with two technical Lemmas, which support the rest of the results and the proof of these Lemmas can be found in (1997).

Main section

Lemma 1: The algebra of linear functional on K is an integral domain. In particular, if $x \in \phi$ is a root of g and $f: K \rightarrow \mathbb{C}$ is a linear functional on K satisfying $f(h)x(h') + f(h')x(h) = 0$ for all $h, h' \in K$ then f = 0.

Lemma 2: Let $A = (a_{ij})_{i, j \in I}$ be a generalized cartan matrix which is not of finite type. If $a_{ij} \neq 0$ for some indices $i, j \in I$, there exists a root $\beta = \Sigma_{x \in I} c_k x_k \in \phi$ of g such that $c_i>0$, $c_j>0$ and $c_i \neq c_j$.

Remark: Lemma 2 holds for all generalized cartan matrices that are not of type A_n see (1972), (1990).

Let B be an abelian group, K be the vector space over F. Denote by FB the group algebra of B over F. The elements t^c, $c \in B$ from a basis of this algebra and the multiplication is defined by $t^c t^d = t^{c+d}$. We shall write 1 instead of t^0. The tensor product $M = FB \otimes_F K$ is a free left FB- module. Denote an arbitrary element of K by d. For the sake of simplicity we write $t^c\delta$ instead of $c^c \otimes \delta$. We fix a pairing $\phi: K \times B \rightarrow F$ which is F-linear in the first variable and additive in the second one. For convenience we use the following notations:

$$\phi(\delta, c) = <\delta, c> = \delta(c)$$

for arbitrary $\delta \in k$ and $c \in R$.

The following multiplications:

$$[t^c\delta_1, t^d\delta_2]: = t^{c+d} (\delta_1 (d)\delta_2\text{-}\delta_2(c)\delta_1),$$

for arbitrary c, d ∈ R and $\delta_1,\delta_2 \in K$ make g into a Lie algebra, called a generalized Kac-Moody algebra.

Let $g_c = t^c$ k for x ∈ R, in particular $g_o = K$ then $[g_c, g_d] \subset g_{c+d}$ holds for all c, d ∈ B. This means that g is a B-graded Lie algebra. It is clear that $[\delta, t^c\delta_1] = \delta(x)+t^c\delta_1$, hence δ is semisimple. Consequently K is a torus.

Let $B_o = \{x \in B/<\delta, x> = 0, \forall \delta\in K\}$ and $K_o = \{\delta \in K/<\delta, x> = 0, \forall \delta \in B\}$. ϕ is said nondegenerate if $B_o = 0$ and $K_o = 0$.

Lemma 3:

(i) The Cartan sub algebra K is closed under ∘.
(ii) For each x ∈ B/{0}, there exists a linear functional f_x: K→C and a by linear map u_x: K×K→K such that:

$$\delta\circ t^x\delta_x = f_x (\delta)t^x\delta_x+u_x(\delta, \delta_x) \tag{8}$$

for all δ ∈ K, $t^x\delta_x \in g_x$ and the f_x satisfies:

$$f_x (\delta)\delta'(x)+f_x(\delta')\delta(x) = (\delta\circ\delta')(x) \tag{9}$$

for all δ, δ′ ∈ K.

Proof:

(i) By bilinearity and commutativity of ∘ it suffices to show that δ∘δ ∈ K. write:

$$\delta \circ \delta = \delta\text{`}+ \sum_{x\in B/\{0\}} t^x\delta_x$$

It follows from [δ, δ∘δ] = 0 that

$$\delta \circ \delta = \delta\text{`}+ \sum_{\delta(x)=0} t^x\delta_x$$

for $\delta_1 \in K$, write $\delta\circ\delta_1 = \delta'' +\delta\circ\delta = \delta+ \sum_{y\in B/\{0\}} t^y \delta_y$ By (6) we have

$$0 = 2[\delta, \delta\circ\delta_1] + [\delta_1, \delta\circ\delta] = 2 \sum_{y\in B/\{0\}} \delta(y)t^y \delta_y + \sum_{x\in B/\{0\},\delta(x)=0} \delta_1(x)t^x \delta_x$$

$$=2 \sum_{x\in B/\{0\},\delta(x)\neq 0} \delta(x)t^x \delta'_x + \sum_{x\in B/\{0\},\delta(x)=0} \delta_1(x)t^x \delta_x$$

Hence, if δ(x) = 0, then $\delta_1(x)t^x\delta_x = 0$ for all $\delta_1 \in K$ which implies $\delta_x = 0$. Therefore δ∘δ = δ′ ∈ T.

(ii) Write $\delta\circ t^x\delta_x = \delta' + \sum_{y\in B/\{0\}} t^y \delta^{-1} y$ By (6), we have

$$2[\delta,\delta\circ t^x\delta_x] + [t^x\delta_x,\delta\circ\delta] = 2\sum_{y\in B/\{0,0\}}\delta(y)t^y\delta_y^{-1} - (\delta\circ\delta)(x)t^x\delta_x = 0$$

Hence, we have $\delta\circ t^x\delta_x = \delta' + t^x\delta'_x \sum_{y\in B/\{x,0\},\delta(y)=0} t^y\delta_y^{-1}$

Similarly, for $\delta_1 \in K$, we have:

$$\delta\circ t^x\delta_x = \delta' + t^x\delta'' + t^x\delta''_x + \sum_{z\in B/\{x,0\},\delta_1(z)=0} t^z\delta''_z$$

By (7), we have $[\delta, \delta_1\circ t^x\delta_x] + [\delta_1, t^x\delta_x\ \delta] + [t^x\delta_x, \delta\circ\delta_1] = 0$ which yields

$$\sum_{y\in B/\{x,0\},(y)=0}\delta_1(y)t^x\delta'_y + \sum_{z\in B/\{x,0\},\delta_1(z)=0}\delta(z)t^z\delta''_z = 0 \tag{10}$$

By (10), one can easily find that

$$\delta\circ t^x\delta_x \in K + g_c$$

Denote $\delta\circ t^x\delta_x = u_x(\delta, \delta_x) + t^x\delta_{(\delta,\ \delta x)}$ where $u_x(\delta, \delta_x)$ and $\delta_{(\delta,\ \delta x)} \in K$. By:

$$[\delta,\delta'\circ t^x\delta_x] + [\delta', t^x\delta_x \circ\delta] + [t^x\delta_x, \delta\circ\delta'] = 0$$

We can find:

$$\delta(x)t^x\ \delta_{(\delta',\ \delta x)} + \delta'(x)t^x\ \delta_{(\delta,\ \delta x)} = (\delta\circ\delta')(x)t^x\ \delta_x \tag{11}$$

Fixing $\delta' \in K$ so that $\delta'(x) = 1$, we see that if $\delta(x) = 0$, $\delta_{(\delta,\ \delta x)} = (\delta\circ\delta')(x)t^x\delta_x$ and $2\ \delta_{(\delta',\ \delta x)} = (\delta'\circ\delta')\ (x)\delta_x$
Define $f_x(\delta') = \frac{1}{2}(\delta'\circ\delta')(x)$ for all $\delta \in$ Ker x: $= \{\delta \in K/\delta(x) = 0\}$ and $f_x(\delta') = \frac{1}{2}(\delta'\circ\delta')(x)$.

Since, $K = Ker_x \otimes F\delta'$, f_x is well defined. Then $\delta\circ t^x\delta_x = f_x(\delta)t^x\delta_x + u_x(\delta, \delta_x)$ since, $\delta_{(\delta,\ \delta x)} = f_x(\delta)\delta_x$. It follows from (13) that (11) is true. It is clear that u_x: $K\times K \to K$ is a bilinear map.

From the following Lemma we can see that the linear functional f_x are the same for all $x \in B/\{0\}$.

Lemma 4:

(i) For all x, $y \in B/\{0\}$, $f_x = f_y$. We denote it by f
(ii) For all $\delta, \delta' \in K$, $\delta\circ\delta' = f(\delta')\delta + f(\delta)\delta'$

Proof: If dim K = 1, then $K = F\delta$ and for any $x \in B/\{0\}$, $\delta(x) \neq 0$ since, $B_0 = 0$. Hence for $\delta_1, \delta_2 \in K$, we have $\delta_1 = \delta_2$ if $\delta_1(x) = \delta_2(x)$ for some $x \in B/\{0\}$.

from (9), we have $2f_x(\delta)\delta(x) = (\delta\circ\delta)(x)$.

So $f_x(\delta)\delta = \frac{1}{2}\delta\circ\delta$ this means that $f_x = f_y$ for any $y \in B/\{0\}$.

Now we suppose that dim K>1. If $T_x \neq Ty$, then $K = T_x + T_y$ and there exists $\delta_1, \delta_2 \in K$ such that $\delta_1(x) = 1$, $\delta_1(y) = 0$, $\delta_2(x) = 0$, $\delta_2(y) = 1$ and $K = F\delta_1 \otimes F\delta_2(y) \otimes (T_x$ Ç $T_y)$. Let $\delta_3 \in T_x \cap T_y$. For any $\delta, \delta' \in K$, it follows from (9) that

$$f_{x+y}(\delta)\delta'(x+y) + f_{x+y}(\delta')\delta(x+y) = (\delta\circ\delta')(x+y) = f_x(\delta)\delta'(x) + f_x(\delta')\delta(x) + f_y(\delta)\delta'(x) + f_y(\delta')\delta(x) \quad (12)$$

In (12), let $\delta = \delta' = \delta_1$ we have $f_{x+y}(\delta_1) = f_x(\delta_1)$
Let $\delta = \delta_1$, $\delta' = \delta_3$ we have $f_{x+y}(\delta_3) = f_x(\delta_3)$.
Similarly, we have $f_{x+y}(\delta_2) = f_y(\delta_2)$, $f_{x+y}(\delta_3) = f_y(\delta_3)$

Let $\delta = \delta_1$, $\delta' = \delta_2$, we have:

$$f_x(\delta_1) + f_y(\delta_2) = f_{x+y}(\delta_1) + f_{x+y}(\delta_2) = f_x(\delta_2) + f_y(\delta_1) \quad (13)$$

Now let $z = 2x+y$ in (11) we have

$$f_z(\delta)\delta'(z) + f_z(\delta')\delta(z) = 2f_x(\delta)\delta'(x) + 2f_x(\delta')\delta(x) + f_y(\delta)\delta'(y) + f_y(\delta')\delta(y) \quad (14)$$

In (16), let $\delta = \delta' = \delta_1$, we have $f_z(\delta_1) = f_x(\delta_1)$
Similarly we have $f_z(\delta_2) = f_y(\delta_2)$
Let $\delta = \delta_1$, $\delta' = \delta_2$, we show

$$f_x(\delta_1) + 2f_y(\delta_2) = 2f_x(\delta_2) + f_y(\delta_1) \quad (15)$$

From 13 and 15 we know that $f_y(\delta_2) = f_x(\delta_2)$, $f_y(\delta_1) = f_x(\delta_1)$. Hence, $f_x = f_y$. If $T_x = T_y$, we can choose $z \in B/\{0\}$ such that $T_z \neq T_x$. Then $f_x = f_z = f_y$. (ii) follows from first part (i) and 9

Lemma 5: For each $x \in B/\{0\}$, there exists a linear functional λ_x: $K \to F$ and a symmetric bilinear map σ_x: $K\times K \to K$ such that $t^x\delta_1\circ t^x\delta_2 = \lambda_x(\delta_2)\, t^x\delta_1 + \lambda_x(\delta_1)\, t^x\delta_2 + \sigma_x(\delta_1, \delta_2)$ for all $\delta_1, \delta_2 \in K$

Proof: For $x, y \in B/\{0\}$, by (7) and Lemma 3 and Lemma 4 we have

$$[\delta, t^x\delta_1\circ t^y\delta_2] = [\delta\circ t^x\delta_1, t^y\delta_2]+[\delta\circ t^y\delta_2, t^x\delta_1] = [f(\delta)t^x\delta_1 + u_x(\delta, \delta_2), t^y\delta_2] + [f(\delta)t^y\delta_2 + u_y(\delta, \delta_2), t^x\delta_1] = u_x(\delta, \delta_2)(y)t^y\delta_2 + u_y(\delta, \delta_2)(x)t^x\delta_1 \quad (16)$$

For all $d \in K$. It follows that $t^x\delta_1\circ t^y\delta_2 \in K + g_x + g_y$.

Let $x = y$, we have $t^x\delta_1\circ t^x\delta_2 \in K+g_x$. Then $t^x\delta_1\circ t^x\delta_2 = {}^{tx}\delta'+\sigma_x(\delta_1, \delta_2)$ for some symmetric bilinear map σ_x: $K\times K \to K$ and $\delta' \in K$. For any $d \in K$, we have $[\delta, t^x\delta'_1\circ t^x\delta_2] = \delta(x)t^x\delta'$. Using 16 we have $\delta(x)\delta' = u_x(\delta, \delta_1)(x)\delta_2 + u_x(\delta, \delta_2)(x)\delta_1$. Fixing $\delta_0 \in K$ such that

$$\delta_0(x) \neq 0, \text{ let } \lambda_x(\delta_1) = u_x\left(\frac{\delta_0}{\delta_0(x)}, \delta_1\right)(x)$$

for all $\delta_1 \in K$. It is clear that $\lambda_x(\delta_1)$ does not depend on the choice of δ_0 with $\delta_0(x) \neq 0$. So $t^x\delta_1\circ t^y\delta_2 = \lambda_x(\delta_1)\, t^x\delta_2 + \lambda_2(\delta_2)\, t^x\delta_1 + \sigma_x(\delta_1, \delta_2)$.

Lemma 6:

(i) For any $x, y \in B/\{0\}$, there exists a bilinear s_{xy}: K´ K → K such that:

$$t^x\delta_1 \circ t^y\delta_2 = \lambda_y(\delta_2)\, t^x\delta_1 + \lambda_x(\delta_1) t^y\delta_2 + \sigma_{xy}(\delta_1, \delta_2) \text{ for all } \delta_1, \delta_2 \in T$$

(ii) For any $\delta, \delta_1 \in K$, $x \in B/\{0\}$, $\delta \circ t^x\delta_1 = f(\delta)t^x\delta_1 + \lambda_x(\delta_1)\delta$

Proof:

(i) Let $x, y \in B/\{0\}$ and $x \neq y$. For any $\delta_1, \delta_2 \in K$ we know that $t^x\delta_1 \circ t^y\delta_2 = t^x\delta'_1 + t^y\delta'_2 + \sigma_{xy}(\delta_1, \delta_2)$ for some $\delta'_1, \delta'_2, \sigma_{xy}(\delta_1, \delta_2) \in K$. It is clear that σ_{xy} is a bilinear map from K×K to K. For any $\delta \in K$, by $[\delta, t^x\delta_1 \circ t^y\delta_2] = \delta(x)t^x\delta'_1 + \delta(y)t^y\delta'_2$ and (16) we have $\delta(x)\delta = u_y(\delta, \delta_2)(x)\delta_1$ and $\delta(y)\, \delta_2^1 = u_x(\delta, \delta_1)(y)\delta_2$. Choosing $\delta_0, \delta' \in K$ such that $\delta_0(x) = 1, \delta'\circ(y) = 1$, we can define $\lambda^y{}_x(\delta_2) = u_y(\delta_0, \delta_2)(x)$ for all $\delta_2 \in K$ and $\lambda^x{}_y(\delta_1) = u_x(\delta'\circ, \delta_1)(y)$ for all $\delta_1 \in K$. Then $t^x\delta_1 \circ t^y\delta_2 = \lambda^x{}_y(\delta_2)\, t^x\delta_1 + \lambda^x{}_y(\delta_1)t^y\delta_2 + \sigma_{xy}(\delta_1, \delta_2)$. By $[\delta, t^x\delta_1 \circ t^y\delta_2] + [t^x\delta_1, t^y\delta_2 \circ \delta] + [t^y\delta_1, \delta' \circ t^x\delta_1] = 0$,
we have $\lambda^x{}_y(\delta_2)\, \delta(x)t^x\delta_1 + \lambda^x{}_y(\delta_1)\, \delta(y)t^y\delta_2 - u_y(\delta, \delta_2)(x)t^x\delta_1 - u_x(\delta, \delta_1)(y)t^y\delta_2 = 0$. Hence

$$\lambda^x{}_y(\delta_1)\, \delta(y) = u_x(\delta, \delta_1)(y) \tag{17}$$

for all $\delta_1, \delta \in K$. Denote $\lambda^x{}_x = \lambda_x$, we have that 17 holds for all $y \in B/\{0\}$

Now we shall prove that $\lambda^y{}_x = \lambda^z{}_x$ for all $x, y, z \in B/\{0\}$.

If Ker y = Ker z, since $\lambda^x{}_y(\delta_1)\delta - u_x(\delta, \delta_1))\,(y) = 0$, we know $\lambda^x{}_y(\delta_1)\delta - u_x(\delta, \delta_1) \in$ Ker y = Ker z and hence $(\lambda^x{}_y(\delta_1)\delta - u_x(\delta, \delta_1))(z) = 0$. Then $\lambda^x{}_y(\delta_1)\delta(z) = u_x(\delta, \delta_1)(z) = \lambda^z{}_x(\delta_1)\delta(z)$ for all $\delta, \delta_1 \in K$. Hence $\lambda^y{}_x = \lambda^z{}_x$.

If Ker y ≠ Ker z, we can choose $\delta', \delta'' \in K$ such that $\delta'(y) = 1, \delta'(z) = 0, \delta''(y) = 0, \delta''(z) = 1$. By (17), we can show

$$\lambda^y{}_x(\delta_1)\delta(y) + \lambda^z{}_x(\delta_1)\delta(z) = u_x(\delta, \delta_1)(y+z) = \lambda_x{}^{y+z}(\delta_1)\delta(y+z)$$

Let $\delta = \delta'$ and δ'', we show

$$\lambda^y{}_x(\delta_1) = \lambda_x{}^{y+z}(\delta_1) = \lambda^z{}_x(\delta_1)$$

For all $d_1 \in K$. Hence, we have $\lambda^y{}_x = \lambda^x{}_x = \lambda_x$ for all $x, y \in B/\{0\}$.

(ii) For any $z \in B/\{0\}$ we have $u_x(\delta, \delta_1)(z) = \lambda^z{}_x(\delta_1)\delta(z) = \lambda_x(\delta_1)\delta(z)$
Hence, $u_x(\delta, \delta_1) = \lambda_x(\delta_1)$. From Lemma 3 we know (ii) is true.

Lemma 7: For any $x, y \in B/\{0\}$, $x \neq y$ and $\delta_1, \delta_2 \in K$, we have.

- $\sigma_x(\delta_1, \delta_2) = 0$, where σ_x: K×K → K is the symmetric bilinear map defined in Lemma 5
- $\sigma_{xy}(\delta_1, \delta_2) = 0$, where σ_{xy}: K×K → K is the bilinear map defined in Lemma 6

Proof: Let $x \in B/\{0\}$. From $[t^x\delta_1, t^x\delta_1 \circ t^x\delta_1] = 0$ for all $\delta_1 \in K$, we have $\sigma_x(\delta_1, \delta_1)(x) = 0$. Since, σ_x is a symmetric linear map we have $\sigma_x(\delta_1, \delta_2)(x) = 0$ for all $\delta_1, \delta_2 \in K$.

For $\delta_1, \delta_2 \in K$, let $y \in B/\{0, x\}$ and $\delta_3 \in K$. Since, $[t^y\delta_3, t^x\delta_1 \circ t^x\delta_2] = [t^y\delta_3 \circ t^x\delta_2, t^x\delta_1] + [t^y\delta_3 \circ t^x\delta_1, t^x\delta_2]$, we have $\sigma_x(\delta_1, \delta_2)(y)\ t^y\delta_3 = \sigma_{yx}(\delta_3, \delta_2)(x)\, t^x\delta_1 + \sigma_{yx}(\delta_3, \delta_1)(x)\, t^x\delta_2$ which implies $\sigma_x(\delta_1, \delta_2)(y) = 0$. So $\sigma_x(\delta_1, \delta_2) \in K_0 = 0$. Hence $\sigma_x(\delta_1, \delta_2) = 0$ for all $\delta_1, \delta_2 \in K$. Similarly, we can prove (ii).

Now we can give the following main theorem of this paper.

Theorem 1: If * is a third power-associative (-1, 1) multiplication on the simple generalized Kac-Moody algebra g, then there exist a linear functional τ: g → ℂ such that

$$u * v = \frac{1}{2}[u,v] + \tau(u)v + \tau(v)u \qquad (18)$$

for all u, v ∈ g.

Conversely, if * is a multiplication on g defined by (18), then * is a third power-associative (-1, 1) multiplication on g

Proof: Define a linear functional τ: g → ℂ by:

$$\tau(\delta + \sum_{x\in B/\{0\}} t^x \delta_x) = f(\delta) + \sum_{x\in B/\{0\}} \lambda^x(\delta_x)$$

It is clear that (18) is true.

From (1) we have (x * y) * y = x * (y * y) for all x, y ∈ B which is equivalent to [x, y]∘y = [x, y∘y]. Using theorem 1 we can determine all (-1, 1) multiplication on simple generalized Kac-Moody algebras.

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