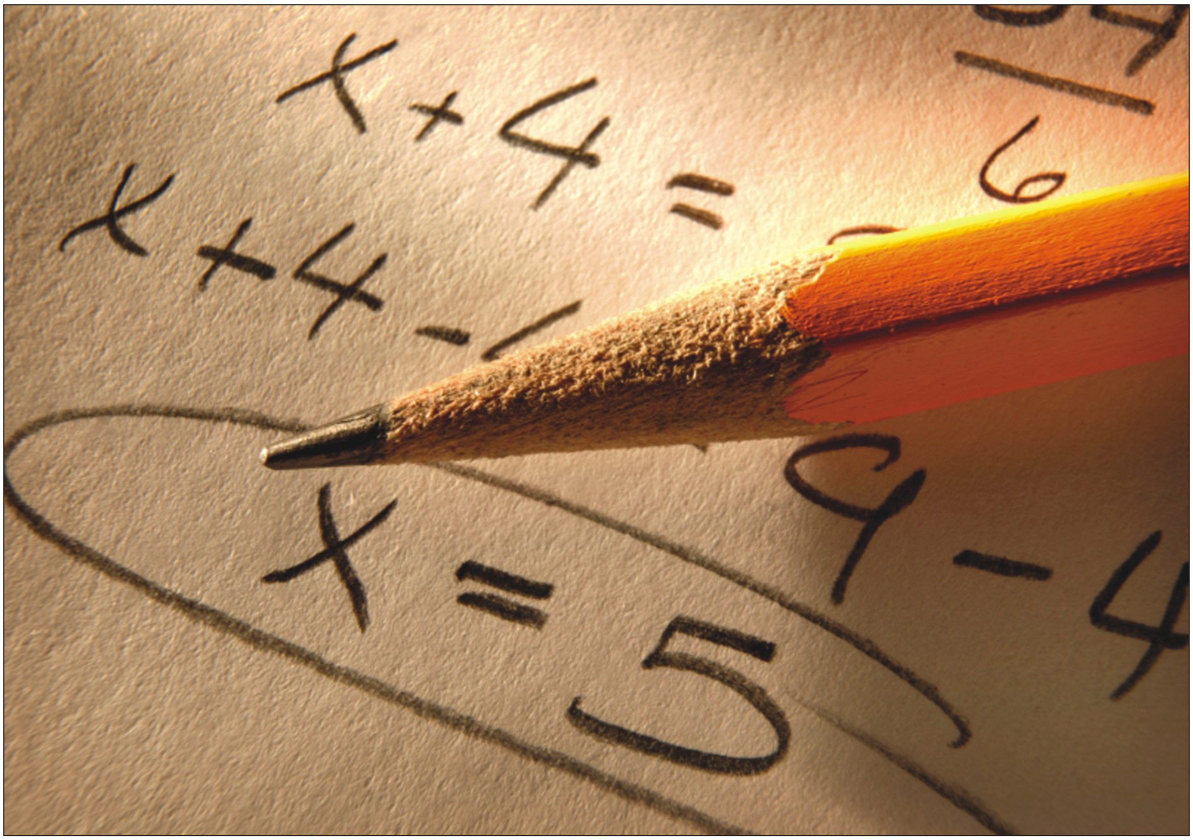

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Research Article
Groupoids in Involution Rings

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Abstract
Describing the structure of certain special types of elements in a ring (in fact in any algebraic structure) is an interesting problem in structure theory. This study was performed to carry out some properties of partial isometries and Moore-penrose invertible elements and the structure of partial isometries and Moore-penrose invertible elements in an involution ring. It is shown that partial isometries in an involution ring is an ordered groupoid and is a sub-groupoid of the groupoid of Moore-penrose invertible elements in R.

Key words: Groupoid, idempotents, involution ring, partial algebra, partial isometry, projections, Moore-penrose inverse

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INTRODUCTION

In the following we briefly recall some definitions needed in the sequel as reported by MacLane[1] for a detailed exposition. A partial binary operation on a set E is a function from a subset D of $E \times E$ to E, the set D is called the domain of the partial binary operation. A partial algebra on a set E is a subset of E on which a partial binary operation is defined. We shall denote the partial binary operation on E by juxtaposition and its domain by D_E. An element $u \in E$ is an identity if ug = g whenever $(u, g) \in D$ and hu = h whenever, $(h, u) \in D$. Thus, we have the following definition[1,2]:

Definition 1: A category C is a partial algebra satisfying the following axioms:

- The composite (xy)z is defined if and only if the composite x(yz) is defined. When either is defined they are equal. The common value of the triple composite is denoted by xyz
- If the composite xy and yz are defined then the triple composite xyz is defined; xyz = (xy)z
- For all $x \in C$, there exists identities e, $f \in C$ such that; xe and fx are defined

Clearly, the identities e and f are uniquely determined by x, we write e = d(x) and f = r(x), where, d(x) is the domain identity and r(x) is the range identity. Observe that xy is defined if and only if d(x) = r(y).

Definition 2: A category C is said to be a groupoid if for each $x \in C$ there is an element x^{-1} such that; $x^{-1}x = d(x)$ and $xx^{-1} = r(x)$.

Groupoid generalizes the notion of group, all groups are groupoid with only one vertex. A groupoid can be seen as a group with partial binary operation replacing the binary operation (2010 Mathematics subject classification 06F25).

Definition 3: Let $(G,\cdot)$ be a groupoid and let $\leq$ be a partial order defined on G. Then $(G,\cdot,\leq)$ is an ordered groupoid if the following axioms hold:

- $x \leq y$ implies $x^{-1} \leq y^{-1}$ for all x, $y \in G$
- For all x, y, u, $v \in G$ if $x \leq y$ and $u \leq v$, xu and yv are defined then $xu \leq yv$
- Let $x \in G$ and e be an identity such that $e \leq d(x)$. Then there exists a unique element $(x|e)$, called the restriction of x to e, such that $(x|e) \leq x$ and $d(x|e) = e$
- Let $x \in G$ and let e be an identity such that $e \leq r(x)$. Then there exists a unique element $(e|x)$, called the co-restriction of x to e, such that $(e|x) \leq x$ and $r(e|x) = e$

A ring R is an additive abelian group together with an associative multiplication such that; the multiplication distributes over addition in R.A ring R is called an involution ring. If R is a ring with an operation *called involution such that it satisfies the conditions $(a^*)^* = a$ $(a+b)^* = a^*+b^*$, $(ab)^* = b^*a^*$. An element $\alpha \in R$ is called a symmetric element if $a^* = a$. Let R be an involution ring with unity 1, then by R^{sym}, R^{proj} we denote the set of all symmetric and projections (simultaneously symmetric and idempotent elements), respectively. It is easy to observe that if e is an idempotent then e^* is also idempotent.

Partial isometries in an involution ring: Let R be an involution ring. A partial isometry in an involution ring is an element r such that $r = rr^* r$, where r^* is the image of r under involution and we pronounce it as r star. We denote the set of all partial isometries of an involution ring R by PI(R). A projection in an involution ring is an element e such that $e^2 = e$ and $e^* = e$. Every projection is a partial isometry, since $ee^* e = e$.

Lemma 1: Let R be an involution ring:

- r be a partial isometry in R, then r^*r and rr^* are projections
- e and f be projections in R, then e = ef if and only if e = fe[3,4]

Next, we define a relation $\leq$ on PI(R) by:

$$r \leq t \Leftrightarrow r = rr^* t \text{ and } rr^* = rr^* tt^*$$

Clearly, $\leq$ is a partial order on PI(R). For r, $t \in PI(R)$, we define $r \cdot t$ by:

$$r \cdot t = \begin{cases} rt & \text{if } r^*r = ss^* \\ \text{Undefined} & \text{otherwise} \end{cases}$$

It is easy to see that $\cdot$ is a partial binary operation on PI(R). For r, $t \in PI(R)$ with rt is defined then:

$$rt(rt)^* rt = rt (t^* r^*) rt = r (tt^*) (r^* r) t = r (tt^* t) = rt$$

Thus, $rt \in PI(R)$.

Theorem 1: Let R be an involution ring. Then, the $(PI(R), \leq)$ is an ordered groupoid[4].

Example 1: Consider $M_2(\mathbb{Z})$ the ring of all 2×2 matrices over the ring $(\mathbb{Z})$ $M_2(\mathbb{Z})$ is an involution ring with * as transpose. An element $A \in M_2(\mathbb{Z})$ is said to be a partial isometry if $A = AA^*A$. Let $A \in M_2(\mathbb{Z})$ be a projection then:

$$A^2 = A \text{ and } A^T = \text{Consider the projection}$$

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} a, b, c \in \mathbb{Z}$$

$$A^2 = A \Rightarrow \begin{pmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

solving the matrix equation we obtain projections on $M_2(\mathbb{Z})$ are:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let, $A \in M_2(\mathbb{Z})$ be a unitary matrix, that is, $AA^* = A^*A = I$: Let:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ be an unitary matrix}$$

and:

$$AA^* = I \Rightarrow \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

On solving we obtain all possible unitary matrices in $A \in M_2(\mathbb{Z})$ as follows:

$$\left\{ \begin{array}{l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right.$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since, a matrix is a partial isometry if and only if it is of the form $A = UP$ where, U is a unitary matrix and P is a projection we have partial isometries in $M_2(\mathbb{Z})$ as follows:

$$PI(M_2(\mathbb{Z})) = \left\{ \begin{array}{l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \right\}$$

Now, it is easy to see that PI (M_2 (Z)) is an ordered groupoid.

Moore-penrose invertible elements in an involution ring: Let R be an involution ring. An element $a \in R$ is Moore-penrose invertible, if there exists $b \in R$ such that the following hold:

$$aba = a, \; bab = b, \; (ab)^* = ab, \; (ba)^* = ba \tag{1}$$

Any b satisfies the above condition is called a Moore-penrose inverse of a. Note that there is at most one b such that above conditions hold and such a b is denoted by $a^{\dagger}$ and we call it as a dagger. The set of all Moore-penrose invertible elements of R will be denoted by $R^{\dagger}$ and call it as R dagger. If a is invertible, then $a^{\dagger}$ coincides with the inverse of a.

Moore-penrose inverse is unique when it exists, suppose x and y be Moore-penrose inverses of a, then:

$$xax = x, \; axa = a, \; (xa)^* = xa; \; (ax)^* = ax$$

$$yay = y, \; aya = a, \; (ya)^* = ya; \; (ay)^* = ay$$

Now:

$$ax = (aya)x = (ay)^* (ax)^* = y^* (a^* x^* a^*) = y^* a^* = (ay)^* = ay$$

Similarly, we can show that xa = ya, Thus:

$$x = xax = yax = yay = y^5$$

Lemma 2: An element r in an involution ring R is a partial isometry if and only if it admits Moore-penrose inverse and $r^\dagger = r^*$.

Proof: Suppose $r \in R$ is a partial isometry then:

$$r = rr^* r \text{ and } r^* = r^* rr^*$$

so, $r^* r$ and rr^* are projections hence, r^* satisfies all the conditions of Moore-penrose inverse given in the Eq. 1 and $r^\dagger = r^*$. Conversely assume that r is Moore-penrose invertible and $r^\dagger = r^*$, then:

$$r = rr^* r \text{ and } r^* = r^* rr^*$$

and $r^* r, rr^*$ are symmetric. Hence, r is a partial isometry.

Lemma 3: Let, R be an involution ring and $r \in R$ is Moore-penrose invertible then $rr^\dagger$ and $r^\dagger r$ are projections.

Proof: Given that r is Moore-penrose invertible then:

$$r = rr^\dagger r\, r^\dagger = r^\dagger rr^\dagger$$

$$rr^\dagger \text{ and } r^\dagger r \in R^{sym}$$

Now, it is enough to show that $rr^\dagger$ and $r^\dagger r$ are idempotents. For:

$$(rr^\dagger)^2 = (rr^\dagger)(rr^\dagger) = (rr^\dagger r)r^\dagger = rr^\dagger$$

$$(r^\dagger r)^2 = (r^\dagger r)(r^\dagger r) = (r^\dagger r)r^\dagger = r^\dagger r$$

Let, $R^\dagger$ denote the set of all elements in R which have Moore-penrose inverses. Define * on $R^\dagger$ by, for $r, t \in (R)^\dagger$:

$$r * t = \begin{cases} rt & \text{if } r^\dagger r = tt^\dagger \\ \text{Undefined} & \text{otherwise} \end{cases} \quad (2)$$

Clearly * is a partial binary operation on $R^\dagger$, for $r, t \in R^\dagger$ such that the product r*t is defined, then r*t = rt and $r^\dagger r = tt^\dagger$. So, r*t is Moore-penrose invertible and its Moore-penrose inverse is given by:

$$(r^*t)^\dagger = (rt)^\dagger = t^\dagger r^\dagger$$

$$(rt)t^\dagger r^\dagger (rt) = r(tt^\dagger)(r^\dagger r)\, t = (rr^\dagger r)\, t = rt$$

$$(t^\dagger r^\dagger)(rt)(t^\dagger r^\dagger) = t^\dagger (r^\dagger r)(tt^\dagger) r^\dagger = t^\dagger (r^\dagger rr^\dagger) = t^\dagger r^\dagger$$

$$(rt)(rt)^\dagger = rtt^\dagger r^\dagger = r(r^\dagger r)r^\dagger = rr^\dagger \in R^{sym}$$

$$(rt)^\dagger (rt)(rt)^\dagger = t^\dagger r^\dagger (rt) = t^\dagger t \in R^{sym}$$

Theorem 2: Let, R be an involution ring. Then $(R^\dagger, *)$ is a groupoid.

Proof: Suppose that (r*t)*u is defined, then:

$$r^\dagger r = tt^\dagger \text{ and } (rt)^\dagger (rt) = uu^\dagger$$

i.e., $t^\dagger r^\dagger rt = uu^\dagger$, but $r^\dagger r = tt^\dagger$ and so $t^\dagger t = uu^\dagger$. Hence, (t*u) is defined.

Now:

$$tu(tu)^\dagger = t(uu^\dagger)t^\dagger = tt^\dagger = r^\dagger r$$

Thus, r*(t*u) is defined and:

$$(r^*t)^*u = r^*(t^*u)$$

Similarly, the converse follows.

Suppose, that r*t and t*u are defined, then:

$$r^\dagger r = tt^\dagger \text{ and } t^\dagger t = uu^\dagger$$

so:

$$tu(tu)^\dagger = t(uu^\dagger)t^\dagger = t(t^\dagger t)t^\dagger = tt^\dagger = rr^\dagger$$

thus, s*(t*u) is defined.

Let, e be a projection and suppose that r*e is defined, then:

$$r^\dagger r = e$$

and:

$$r^*e = rr^\dagger r = r$$

Similarly, if e*s is defined then e*s = s.

Let, e be an identity and the product $e^*(e^\dagger{}^*e)$ is defined, then $e^*(e^\dagger e) = e$, thus $e = e^\dagger e$ and $e^\dagger e$ is a projection.

Observe that $r = (rr^\dagger) r = r(r^\dagger r)$, so it follows that $R^\dagger$ is a groupoid where $r^{-1} = r^\dagger$.

Theorem 3: The ordered groupoid of partial isometries PI(R) in an involution ring R is a sub-groupoid of the groupoid of Moore-penrose invertible elements in R.

Proof: From Lemma 2 partial isometries are Moore-penrose invertible, hence:

$$PI(R) \subset R^{\dagger}$$

To prove that (PI(R),·, $\leq$) is a sub-groupoid of ($R^{\dagger}$, *) it is enough to show that · is the restriction of * to the set of partial isometries. For, let r,t∈PI(R) then $r^{\dagger} = r^*$ and $t^{\dagger} = t^*$:

$$r * t = \begin{cases} rt & \text{if } r^* r = tt^* \\ \text{Undefined} & \text{otherwise} \end{cases}$$

$$r*t = r \cdot t \text{ for all } r,t \in PI\ (R)$$

Thus, ordered groupoid of partial isometries is a sub-groupoid of groupoid of Moore-penrose invertible elements.

CONCLUSION

This is an attempt to extend the groupoid techniques in the study of structure of semigroups to the study of the structure of rings. Here, it is shown that Moore-penrose invertible elements in an involution ring is a groupoid and the ordered groupoid of partial isometries is sub-groupoid of thi groupoid of Moore-penrose invertible elements. Also, this study provide an example of ordered groupoid of partial ismetries in the involution ring M_2 (Z) of all 2×2 matrices of integer entries.

SIGNIFICANCE STATEMENT

This study discovered that set of all Moore-penrose elements in an involution ring forms a groupoid and the ordered groupoid of prtial isometries is sub-groupoid of this gropouid which is beneficial for the study of structure of involution rings.

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