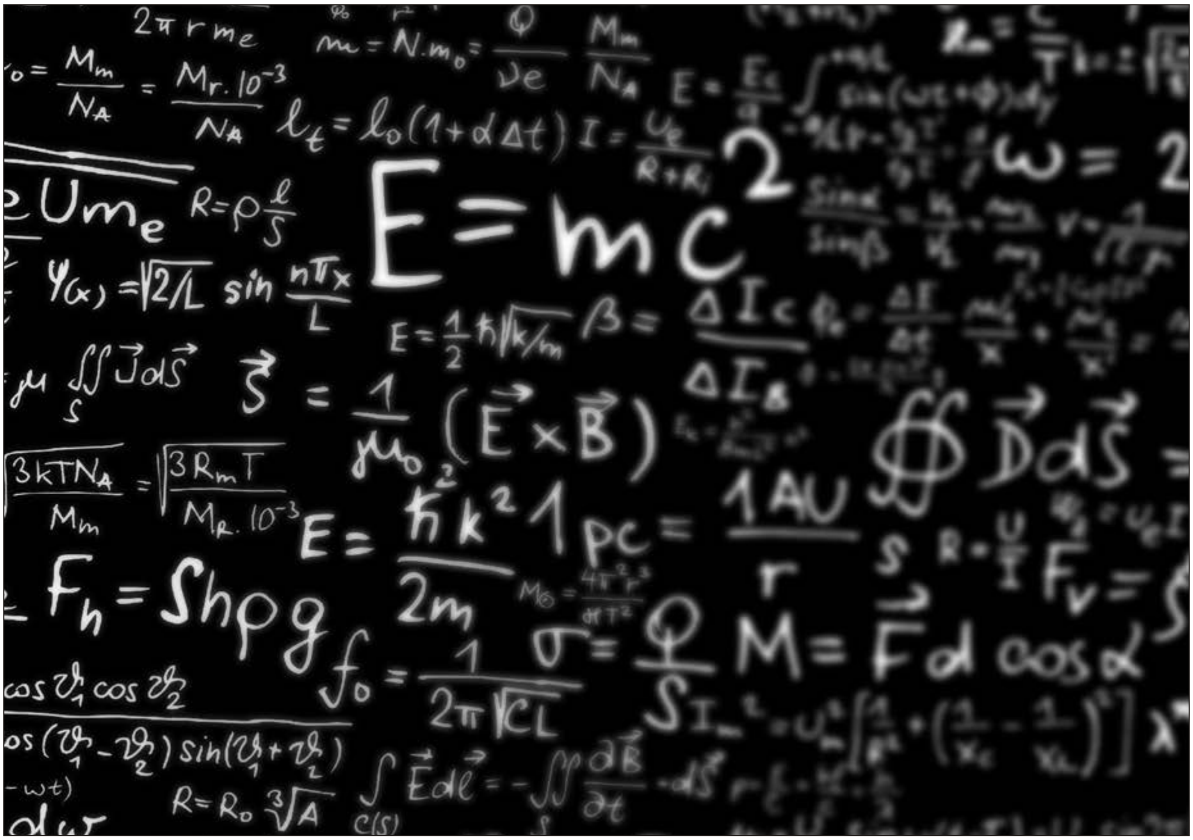

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# Recurrence Relations for Single and Product Moments of k-th Record Values from Linear-Exponential Distribution and a Characterization

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**Abstract:** In this study we give some recurrence relations satisfied by single and product moments of k-th upper record values from the linear-exponential distribution. Using a recurrence relation for single moments we obtain a characterization of linear-exponential distribution.

**Key words:** Order statistics, single moments, product moments, k-th upper record values, recurrence relations, linear-exponential distribution, characterization

## INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x) = P[X \leq x]$ and pdf $f(x)$. For a fixed $k \geq 1$ we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of k-th upper record times of $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1,$$
$$U_{n+1}^{(k)} = \min\left\{j > U_n^{(k)} \ : \ X_{j\,:\,j+k-1} > X_{U_n^{(k)}\,:\,U_n^{(k)}+k-1}\right\}$$

For $k = 1$ and $n = 1,2,\ldots$, we write $U_n^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of k-th upper record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_n}, n \geq 1$, which are record values of $\{X_n, n \geq 1\}$ (Ahsanullah, 1995). Moreover, $Y_1^{(k)} = \min(X_1, X_2, \ldots X_k) = X_{1:k}$.

Record values and associated statistics are of great importance in several real-life problems involving weather, economic and sports data. The statistical study of record values started with Chandler (1952) and has now spread in various directions. Properties of record values have been extensively studied in the literature. Various developments on records and related topics have been reviewed by Glick (1978), Nevzorov (1987), Resnick (1987), Arnold and Balakrishnan (1989) and Arnold *et al*. (1992, 1998).

In the present study, we have established some recurrence relations for single and product moments of the k-th upper record values from the linear-exponential distribution. A characterization of linear-exponential distribution has also been obtained on using a recurrence relation for single moments. Similar results for Weibull distribution have been derived by Pawlas and Szynal (2000).

We shall denote:

$$\mu_{(n):\,k}^{(r)} = E((Y_n^{(k)})^r), \quad r, n = 1, 2, \ldots,$$
$$\mu_{(m,n):\,k}^{(r,s)} = E((Y_m^{(k)})^r (Y_n^{(k)})^s), \quad 1 \leq m \leq n-1 \ \text{ and } \ r, s = 1, 2, \ldots,$$
$$\mu_{(m,n):\,k}^{(r,o)} = E((Y_m^{(k)})^r (Y_n^{(k)})^o) = E((Y_m^{(k)})^r) = \mu_{(m):\,k}^{(r)},$$
$$1 \leq m \leq n-1 \ \text{ and } \ r = 1, 2, \ldots,$$
$$\mu_{(m,n):\,k}^{(o,s)} = E((Y_m^{(k)})^o (Y_n^{(k)})^s) = E((Y_n^{(k)})^s) = \mu_{(n):\,k}^{(s)},$$
$$1 \leq m \leq n-1 \ \text{ and } \ s = 1, 2, \ldots$$

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## RELATIONS FOR SINGLE AND PRODUCT MOMENTS

A random variable X is said to have a linear-exponential distribution if the pdf is of the form:

$$f(x) = (\lambda + \nu x)e^{-(\lambda x + \nu x^2/2)}, \quad 0 \le x < \infty, \quad \lambda, \nu > 0 \tag{1}$$

and the cdf is of the form:

$$F(x) = 1 - e^{-(\lambda x + \nu x^2/2)}, \quad 0 \le x < \infty \tag{2}$$

It can be seen that

$$f(x) = (\lambda + \nu x)[1 - F(x)] \tag{3}$$

The relation in Eq. 3 will be employed in this study to derive recurrence relations for the moments of k-th record values from the linear-exponential distribution and to give a characterization of the linear-exponential distribution.

Let $\{Y_n^{(k)}, n \ge 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$, be a sequence of k-th upper record values from Eq. 1. Then the pdf of $Y_n^{(k)}$ ($n \ge 1$) is given by:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x) \tag{4}$$

and the joint density function of $Y_m^{(k)}$ and $Y_n^{(k)}$, $1 \le m < n$, $n \ge 2$, is given by:

$$\begin{aligned} f_{Y_m^{(k)}, Y_n^{(k)}}(x,y) = {} & \frac{k^n}{(m-1)!(n-m-1)!}[\log(1-F(x)) - \log(1-F(y))]^{n-m-1} \\ & \times[-\log(1-F(x))]^{m-1}\frac{f(x)}{1-F(x)}[1-F(y)]^{k-1}f(y), \quad x < y \end{aligned} \tag{5}$$

(Dziubdziela and Kopocinski, 1976; Grudzien, 1982).

**Theorem 1**

Fix a positive integer $k \ge 1$. For $n \ge 1$ and $r = 0, 1, 2, \ldots$,

$$\mu_{(n):k}^{(r)} = \frac{k\lambda}{(r+1)}\mu_{(n):k}^{(r+1)} - \frac{k\lambda}{(r+1)}\mu_{(n-1):k}^{(r+1)} + \frac{k\nu}{(r+2)}\mu_{(n):k}^{(r+2)} - \frac{k\nu}{(r+2)}\mu_{(n-1):k}^{(r+2)} \tag{6}$$

**Proof**

For $n \ge 1$ and $r = 0, 1, 2, \ldots$, we have from Eq. 4 and 3

$$\begin{aligned} \mu_{(n):k}^{(r)} = {} & \frac{k^n\lambda}{(n-1)!}\int_0^\infty x^r[-\log(1-F(x))]^{n-1}[1-F(x)]^k dx \\ & + \frac{\nu k^n}{(n-1)!}\int_0^\infty x^{r+1}[-\log(1-F(x))]^{n-1}[1-F(x)]^k dx \end{aligned}$$

Integrating by parts, treating $[\{-\log(1-F(x))\}^{n-1}(1-F(x))^k]$ for differentiation and rest of the integrand for integration, we get Eq. 6.

**Remark 1**

Putting $k = 1$ in the Eq. 6, we deduce the recurrence relation for the single moments of upper 1-record values from the linear-exponential distribution, established by Saran and Pushkarna (2000).

**Theorem 2**

For $1 \le m \le n-2$ and $r,s = 0,1,2,...,$

$$\mu_{(m,n):k}^{(r,s)} = \frac{k\lambda}{(s+1)}\mu_{(m,n):k}^{(r,s+1)} - \frac{k\lambda}{(s+1)}\mu_{(m,n-1):k}^{(r,s+1)} + \frac{k\nu}{(s+2)}\mu_{(m,n):k}^{(r,s+2)} - \frac{k\nu}{(s+2)}\mu_{(m,n-1):k}^{(r,s+2)} \tag{7}$$

and for $m \ge 1$, $r,s = 0,1,2,...,$

$$\mu_{(m,m+1):k}^{(r,s)} = -\frac{k\lambda}{(s+1)}\mu_{(m):k}^{(r+s+1)} + \frac{k\lambda}{(s+1)}\mu_{(m,m+1):k}^{(r,s+1)} - \frac{k\nu}{(s+2)}\mu_{(m):k}^{(r+s+2)} + \frac{k\nu}{(s+2)}\mu_{(m,m+1):k}^{(r,s+2)} \tag{8}$$

**Proof**

From Eq. 5 for $1 \le m \le n-1$, $r,s = 0,1,2,...$ and on using Eq. 3, we get:

$$\begin{aligned}\mu_{(m,n):k}^{(r,s)} &= E[(Y_m^{(k)})^r (Y_n^{(k)})^s] \\ &= \frac{k^n}{(m-1)!(n-m-1)!}\int_x x^r[-\log(1-F(x))]^{m-1}\frac{f(x)}{1-F(x)}I(x)dx\end{aligned} \tag{9}$$

Where:

$$\begin{aligned}I(x) = \lambda\int_y y^s[\log(1-F(x)) - \log(1-F(y))]^{n-m-1}[1-F(y)]^k dy \\ + \nu\int_y y^{s+1}[\log(1-F(x)) - \log(1-F(y))]^{n-m-1}[1-F(y)]^k dy\end{aligned}$$

Integrating $I(x)$ by parts, we get:

$$\begin{aligned}I(x) = &-\frac{\lambda(n-m-1)}{(s+1)}\int_x^\infty y^{s+1}[\log(1-F(x)) - \log(1-F(y))]^{n-m-2}[1-F(y)]^{k-1}f(y)dy \\ &+\frac{\lambda k}{(s+1)}\int_x^\infty y^{s+1}[\log(1-F(x)) - \log(1-F(y))]^{n-m-1}[1-F(y)]^{k-1}f(y)dy \\ &-\frac{\nu(n-m-1)}{(s+2)}\int_x^\infty y^{s+2}[\log(1-F(x)) - \log(1-F(y))]^{n-m-2}[1-F(y)]^{k-1}f(y)dy \\ &+\frac{\nu k}{(s+2)}\int_x^\infty y^{s+2}[\log(1-F(x)) - \log(1-F(y))]^{n-m-1}[1-F(y)]^{k-1}f(y)dy\end{aligned}$$

Substituting this expression into Eq. 9 and simplifying, it leads to Eq. 7. Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in Eq. 8 can easily be established.

**Remark 2**

Putting $k = 1$ in Eq. 7 and 8, we deduce the recurrence relations for the product moments of upper record values from the linear-exponential distribution, established by Saran and Pushkarna (2000).

## A CHARACTERIZATION OF THE LINEAR-EXPONENTIAL DISTRIBUTION

This section contains a characterization of the linear-exponential distribution. For this we require the following result of Lin (1986).

**Proposition**

Let $n_0$ be any fixed non-negative integer, $-\infty \le a < b \le \infty$ and $g(x) \ge 0$ an absolutely continuous function with $g'(x) \ne 0$ a.e. on (a,b). Then the sequence of functions $\{[g(x)]^n e^{-g(x)}, n \ge n_0\}$ is complete in L(a,b) if and only if g(x) is strictly monotone on (a,b).

Using the above proposition we get a stronger version of Theorem 1.

**Theorem 3**

Fix a positive integer $k \ge 1$ and let r be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by Eq. 1 is that:

$$\mu_{(n):k}^{(r)} = \frac{k\lambda}{(r+1)}\mu_{(n):k}^{(r+1)} - \frac{k\lambda}{(r+1)}\mu_{(n-1):k}^{(r+1)} + \frac{k\nu}{(r+2)}\mu_{(n):k}^{(r+2)} - \frac{k\nu}{(r+2)}\mu_{(n-1):k}^{(r+2)} \tag{10}$$

for n = 1,2,....

**Proof**

The necessary part follows immediately from Eq. 6. On the other hand if the recurrence relation in Eq. 10 is satisfied, then on rearranging the terms in Eq. 10 and using Eq. 4, we have:

$$\begin{aligned} -\frac{k\nu}{(r+2)}\mu_{(n):k}^{(r+2)} = & -\frac{k\lambda}{(r+1)}\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+1}[-\log(1-F(x))]^{n-2}[1-F(x)]^{k-1}f(x)dx \\ & -\frac{k\nu}{(r+2)}\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+2}[-\log(1-F(x))]^{n-2}[1-F(x)]^{k-1}f(x)dx - \mu_{(n):k}^{(r)} + \frac{k\lambda}{(r+1)}\mu_{(n):k}^{(r+1)} \end{aligned}$$

Integrating the first two integrals on the right hand side of the above expression, by parts, we get:

$$\begin{aligned} & -\frac{k\nu}{(r+2)}\frac{k^n}{(n-1)!}\int x^{r+2}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx \\ & = \frac{k^n\lambda}{(n-1)!}\int x^r[-\log(1-F(x))]^{n-1}[1-F(x)]^k dx \\ & -\frac{\lambda k^{n+1}}{(n-1)!(r+1)}\int x^{r+1}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx \\ & +\frac{k^n\nu}{(n-1)!}\int x^{r+1}[-\log(1-F(x))]^{n-1}[1-F(x)]^k dx \\ & -\frac{\nu k^{n+1}}{(n-1)!(r+2)}\int x^{r+2}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx \\ & -\frac{k^n}{(n-1)!}\int x^r[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx \\ & +\frac{\lambda k^{n+1}}{(n-1)!(r+1)}\int x^{r+1}[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1}f(x)dx \end{aligned}$$

which reduces to

$$\begin{aligned} & \frac{k^n}{(n-1)!}\int x^r[-\log(1-F(x))]^{n-1}[1-F(x)]^{k-1} \\ & \quad \times\{-\lambda(1-F(x)) - \nu x(1-F(x)) + f(x)\}dx = 0 \end{aligned}$$

It now follows from the above proposition with $g(x) = -\log(1-F(x))$ that:

$$(\lambda + \nu x)(1 - F(x)) = f(x)$$

which proves that f(x) has the form as in Eq. 3.

Now we shall show how Theorem 3 can be used in a characterization of the linear-exponential distribution in terms of moments of minimal order statistics. Putting n = 1 in Eq. 10, we get:

$$\mu_{(1):\,k}^{(r)} = \frac{k\lambda}{(r+1)}\mu_{(1):\,k}^{(r+1)} + \frac{k\nu}{(r+2)}\mu_{(1):\,k}^{(r+2)},$$

for any fixed integer $k \geq 1$. This result leads to the following theorem.

**Theorem 4**

Let r be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given in Eq. 1 is that:

$$\mu_{(1):\,k}^{(r)} = \frac{k\lambda}{(r+1)}\mu_{(1):\,k}^{(r+1)} + \frac{k\nu}{(r+2)}\mu_{(1):\,k}^{(r+2)}, \quad \text{for } k = 1, 2, \ldots \tag{11}$$

**Proof**

The necessary part follows from Eq. 6. On the other hand if Eq. 11 is satisfied then

$$\int_0^\infty x^r[1-F(x)]^{k-1}f(x)dx = \frac{k\lambda}{(r+1)}\int_0^\infty x^{r+1}[1-F(x)]^{k-1}f(x)dx + \frac{k\nu}{(r+2)}\int_0^\infty x^{r+2}[1-F(x)]^{k-1}f(x)dx$$

Integrating the integrals on the right-hand side of the above expression by parts, we get:

$$\int_0^\infty x^r[1-F(x)]^{k-1}f(x)dx = \lambda\int_0^\infty x^r[1-F(x)]^k dx + \nu\int_0^\infty x^{r+1}[1-F(x)]^k dx,$$

which further reduces to

$$\int_0^\infty x^r[1-F(x)]^{k-1}\{f(x) - (1-F(x))\lambda - \nu x(1-F(x))\}dx = 0, \quad k = 1, 2, \ldots \tag{12}$$

Now applying a generalization of the Muntz-Sz'asz Theorem (Hwang and Lin, 1984) to Eq. 12, we get:

$$(\lambda + \nu x)(1 - F(x)) = f(x)$$

which proves that

$$F(x) = 1 - e^{-(\lambda x + \nu x^2/2)}, \quad 0 \leq x < \infty, \ \lambda, \nu > 0$$

## CONCLUSION

In this study some recurrence relations for single and product moments of k-th upper record values from the linear -exponential distribution have been established, which generalize the corresponding results for upper 1-record values from the linear-exponential distribution due to Saran and Pushkarna (2000). Further, these recurrence relations have been utilized to obtain a characterization of the linear-exponential distribution by using a result of Lin (1986). Similar results for Weibull distribution have been obtained by Pawlas and Szynal (2000).

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