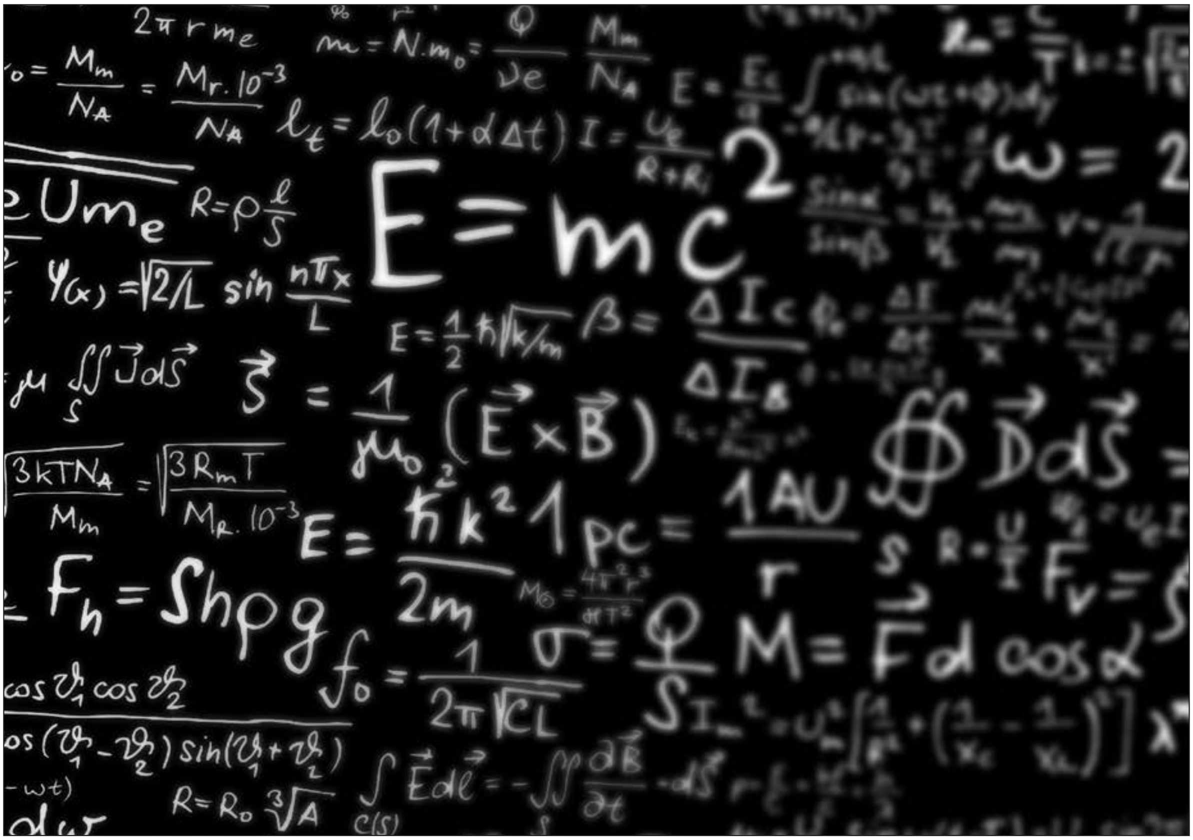

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 1 (3): 165-169, 2008
ISSN 1994-5418

Efficient Approximation for the von Mises Concentration Parameter

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Abstract: This study discusses some of the approximation which can be used to obtain the maximum likelihood estimate of the concentration parameter of the von Mises distribution. This study shows that the efficient approximation may also be obtained by solving the equation for the ratio of the modified Bessel function of first kind of order one and first kind of order zero. The closed-form solution for parameter concentration is also given. We have found out that the new proposed method performs well especially for large κ.

Key words: Concentration parameters, Von Mises distribution, Bessel function, directional data

INTRODUCTION

The von Mises distribution is denoted by $M(\mu_0, \kappa)$ where, μ_0 $(0 \le \mu_0 < 2\pi)$ is the mean direction and k is the concentration parameter. The probability density function is given by $g(\theta; \mu_0, \kappa) = \{2\pi I_0(\kappa)\}^{-1} \exp\{\kappa \cos(\theta - \mu_0)\}$, where I_0 denotes the modified Bessel function of the first kind with order zero as described by Mardia and Jupp (2000). Some of the applications of the von Mises distribution is in the circular linear functional model which was firstly introduced by Hussin (2001) and the circular regression model as proposed by Lund (1999) and Down and Mardia (2002).

Suppose $\theta_1, \ldots, \theta_n$ are a random sample from $M(\mu_0, \kappa)$, then the maximum likelihood estimator of κ is $\hat{\kappa}$ and is given by the solution of $A(\hat{\kappa}) = \bar{R} = (\bar{C}^2 + \bar{S}^2)$, where, $\bar{C} = n^{-1} \sum \cos\theta_i$ and $\bar{S} = n^{-1} \sum \sin\theta_i$. Further $A(x) = I_1(x)/I_0(x)$, where, I_1 is the modified Bessel function of the first kind with order one. This study discuss some functions which approximate A^{-1} that can be obtained accurately without using tables, algorithm, interpolation formula or sophisticated computer program.

There are several approximations available for A^{-1} (x) for all *x* in (0,1). For instance Amos (1974) proved that:

$$\frac{x}{\frac{1}{2} + \left(x^2 + \frac{9}{4}\right)^2} < A(x) < \frac{x}{\frac{1}{2} + \left(x^2 + \frac{1}{4}\right)^{\frac{1}{2}}}, \text{ for } x \ge 0$$

and hence A^{-1} (x) is approximately given by:

$$f(x) = \frac{x}{1 - x^2} \left[\frac{1}{2} + \left\{ 1.46\left(1 - x^2\right) + \frac{1}{4} \right\}^{\frac{1}{2}} \right]$$

Mardia and Zemroch (1975) give a computer algorithm for calculating A^{-1} (x) together with the tables which are obtained iteratively. Meanwhile, by using the power series for the Bessel function I_0 (x) and I_1(x), Dobson (1978) has given the approximation of A^{-1} (x) as follows:

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$$f(x) = \begin{cases} 2x + x^3 + \dfrac{5x^5}{6}, & x < 0.65 \\ \dfrac{9 - 8x + 3x^2}{8(1-x)}, & x \geq 0.65 \end{cases}$$

It is shown that this approximation gives less maximum relative error compared to Amos (1974). Further, an improved approximation for $A^{-1}(x)$ was given by Best and Fisher (1981) which is

$$f(x) = \begin{cases} 2x + x^3 + \dfrac{5x^5}{6}, & x < 0.53 \\ -0.4 + 1.39x + \dfrac{0.43}{1-x}, & 0.53 \leq x < 0.85 \\ \dfrac{1}{x^3 - 4x^2 + 3x}, & x \geq 0.85 \end{cases}$$

Its tabulated values can be found in Fisher (1993).

APPROXIMATION BASED ON THE MODIFIED BESSEL FUNCTION

By definition,

$$A(\hat{\kappa}) = \left(\bar{C}^2 + \bar{S}^2\right)^{\frac{1}{2}} = t \tag{1}$$

and from the power series for the Bessel function $I_0(x)$ and $I_1(x)$, we have

$$A(x) = \frac{I_1(x)}{I_0(x)} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3} \tag{2}$$

Solving (1) and (2), we obtain:

$$(8t - 8)\hat{\kappa}^3 + 4\hat{\kappa}^2 + \hat{\kappa} + 1 = 0$$

or

$$a_3\hat{\kappa}^3 + a_2\hat{\kappa}^2 + a_1\hat{\kappa} + a_0 = 0 \tag{3}$$

where, $a_3 = 8t\text{-}8$, $a_2 = 4$, $a_1 = 1$ and $a_0 = 1$.
By taking transformation of $\hat{\kappa} = \left(y - \dfrac{a_2}{3a_3}\right)$, (Eq. 3) can be written as

$$y^3 + py + q = 0 \tag{4}$$

Where:

$$p = \frac{3a_3a_1 - a_2{}^2}{3a_3{}^2} = \frac{3(t-1) - 2}{24(t-1)^2}$$

and

$$q=\frac{2a_2^3-9a_1a_2a_3+27a_0a_3^2}{27a_3^3}=\frac{4-9(t-1)+54(t-1)^2}{432(t-1)^3}$$

Following Rades and Westergren (1988), the roots for Eq. (4) are of one real root and two complex roots and are given by

$$y_1=u+v,$$

$$y_2=-\left(\frac{u+v}{2}\right)+\left(\frac{u-v}{2}\right)\sqrt{3}i$$

and (5)

$$y_3=-\left(\frac{u+v}{2}\right)-\left(\frac{u-v}{2}\right)\sqrt{3}i.$$

Where:

$$u=\left(\frac{-q}{2}+\sqrt{D}\right)^{1/3},\quad v=\left(\frac{-q}{2}-\sqrt{D}\right)^{1/3}$$

$$D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2$$

Hence, the approximation for $\hat{\kappa}$ is given by

$$\hat{\kappa}=\left(-\frac{q}{2}+\sqrt{D}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}-\sqrt{D}\right)^{\frac{1}{3}}-\frac{1}{6(t-1)} \tag{6}$$

Here we show that this approximation gives more efficient estimation for $\hat{\kappa}$ compare to the approximations given by Dobson (1978) and Best and Fisher (1981).

SIMULATION RESULTS

Computer programs were written using SPLUS language to carry out the simulation study to assess the efficiency of the three different methods of approximating the concentration parameter κ. Circular samples of length $n = 100$ were generated from von Mises distribution with mean 0 and κ = 2, 4, 6, 8, 10, 12 14 and 16.

Let s be the number of simulations and the following computation were obtained from the simulation study.

- Mean $\bar{\hat{\kappa}}=\frac{1}{s}\sum\hat{\kappa}_j$
- Estimated Bias = $\bar{\hat{\kappa}}-\kappa$
- Absolute Relative Estimated Bias (%) $=\left(\frac{|\bar{\hat{\kappa}}-\kappa|}{\kappa}\right)\times 100\%$
- Estimated Standard Error = $\sqrt{\frac{1}{s-1}\sum\left(\hat{\kappa}_j-\bar{\hat{\kappa}}\right)^2}$
- Estimated Root Mean Square Error (RMSE) = $\sqrt{\frac{1}{s}\sum\left(\hat{\kappa}_j-\kappa\right)^2}$
- Relative Efficiency = $\frac{\text{Estimated RMSE of Dobson or New}}{\text{Estimated RMSE of Fisher}}$

Table 1: Simulation results for various true value of parameter concentration

	True value of concentration parameters	$\kappa=2$	$\kappa=4$	$\kappa=6$	$\kappa=8$	$\kappa=10$	$\kappa=12$	$\kappa=14$	$\kappa=16$
Mean	$\hat{\kappa}^{\text{Dobson}}$	2.0571	4.0795	6.1779	8.2421	10.3176	12.3831	14.4343	16.5612
	$\hat{\kappa}^{\text{Best and Fisher}}$	2.0348	4.0884	6.1516	8.2540	10.2998	12.3751	14.4193	16.5132
	$\hat{\kappa}^{\text{New}}$	2.0152	4.0634	6.1549	8.1838	10.2836	12.3613	14.4093	16.4854
Est Bias	$\hat{\kappa}^{\text{Dobson}}$	0.0571	0.0795	0.1779	0.2421	0.3176	0.3831	0.4343	0.5612
	$\hat{\kappa}^{\text{Best and Fisher}}$	0.0348	0.0884	0.1516	0.2540	0.2998	0.3751	0.4193	0.5132
	$\hat{\kappa}^{\text{New}}$	0.0152	0.0634	0.1549	0.1838	0.2836	0.3613	0.4093	0.4854
Absolute	$\hat{\kappa}^{\text{Dobson}}$	2.8569	1.9894	2.9665	3.0269	3.1768	3.1931	3.1024	3.5079
Relative	$\hat{\kappa}^{\text{Best and Fisher}}$	1.7421	2.2101	2.5280	3.1750	2.9988	3.1260	2.9953	3.2078
Bias (%)	$\hat{\kappa}^{\text{New}}$	0.7629	1.5853	2.5826	2.2984	2.8361	3.0111	2.9242	3.0340
Est SE	$\hat{\kappa}^{\text{Dobson}}$	0.2175	0.5467	0.8621	1.1337	1.4594	1.7827	2.0498	2.3967
	$\hat{\kappa}^{\text{Best and Fisher}}$	0.2589	0.5466	0.8528	1.1491	1.4699	1.7586	2.0598	2.3538
	$\hat{\kappa}^{\text{New}}$	0.2305	0.5489	0.8502	1.1313	1.4621	1.7186	2.0231	2.3467
Est RMSE	$\hat{\kappa}^{\text{Dobson}}$	0.0505	0.3052	0.7748	1.3437	2.2304	3.3244	4.3897	6.0580
	$\hat{\kappa}^{\text{Best and Fisher}}$	0.0682	0.3065	0.7502	1.3846	2.2502	3.2331	4.4178	5.8028
	$\hat{\kappa}^{\text{New}}$	0.0533	0.3053	0.7467	1.3135	2.2180	3.0838	4.2598	5.7416
Efficiency	$\hat{\kappa}^{\text{Dobson}}$	0.7408	0.9957	1.0326	0.9704	0.9912	1.0282	0.9936	1.0439
	$\hat{\kappa}^{\text{Best and Fisher}}$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$\hat{\kappa}^{\text{New}}$	0.7820	0.9960	0.9953	0.9486	0.9856	0.9538	0.9642	0.9894

The values of mean, estimated bias, absolute relative bias, estimated standard error and RMSE are computed for the Dobson's method, Best and Fisher's method and the new proposed method. The results are given in rows 1-15 of Table 1. We also compare the efficiency of Dobson and the new proposed method relative to the Best and Fisher method and the values are given in row 16-18 of (Table 1).

It appears from rows 1 to 3 that the mean estimate obtained from the new proposed method is very close to the true k value compared to the estimates obtained by Dobson and Best and Fisher approximation method. It can be seen clearer from rows 4 to 6 as the biases of estimates for new proposed method are closer to zero. Note also that biases are an increasing function of true κ. Further, it is also clear that, for $\kappa \geq 6$, the absolute relative estimated bias, estimated standard error and estimated root mean square error are the smallest for the new proposed method and followed by the Best and Fisher's method as given in row 7-15. However, for smaller κ, Dobson's method has the smallest estimated standard error and estimated root mean square error. To make a direct comparison between these approximation methods, the relative efficiency of Dobson's method and new proposed method relative to Best and Fisher's method were computed and results are given in row 16-18. Result in row 18 shows that new proposed method has the relative efficiency smaller than one for all values of true κ. These indicate that new proposed method is relatively better than Best and Fisher approximation method especially for small k. New proposed method is also found to be relatively better than the Dobson's method except for small κ.

CONCLUSION

The objective of present study is to evaluate the performance of three different approximating methods of concentration parameter κ, namely, the Dobson's method, Best and Fisher's method and new proposed method through simulation study. Generally, it appears that, for large κ, the new proposed method have a better performance than the Dobson's and Best and Fisher's methods. For smaller κ, new proposed method has the least biases and absolute relative biases but Dobson's method is more efficient with smaller values of estimated standard error and RMSE. Hence, it is shown that new proposed method is superior for large κ. However, the new proposed method and the Dobson's method could be used for small κ.

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