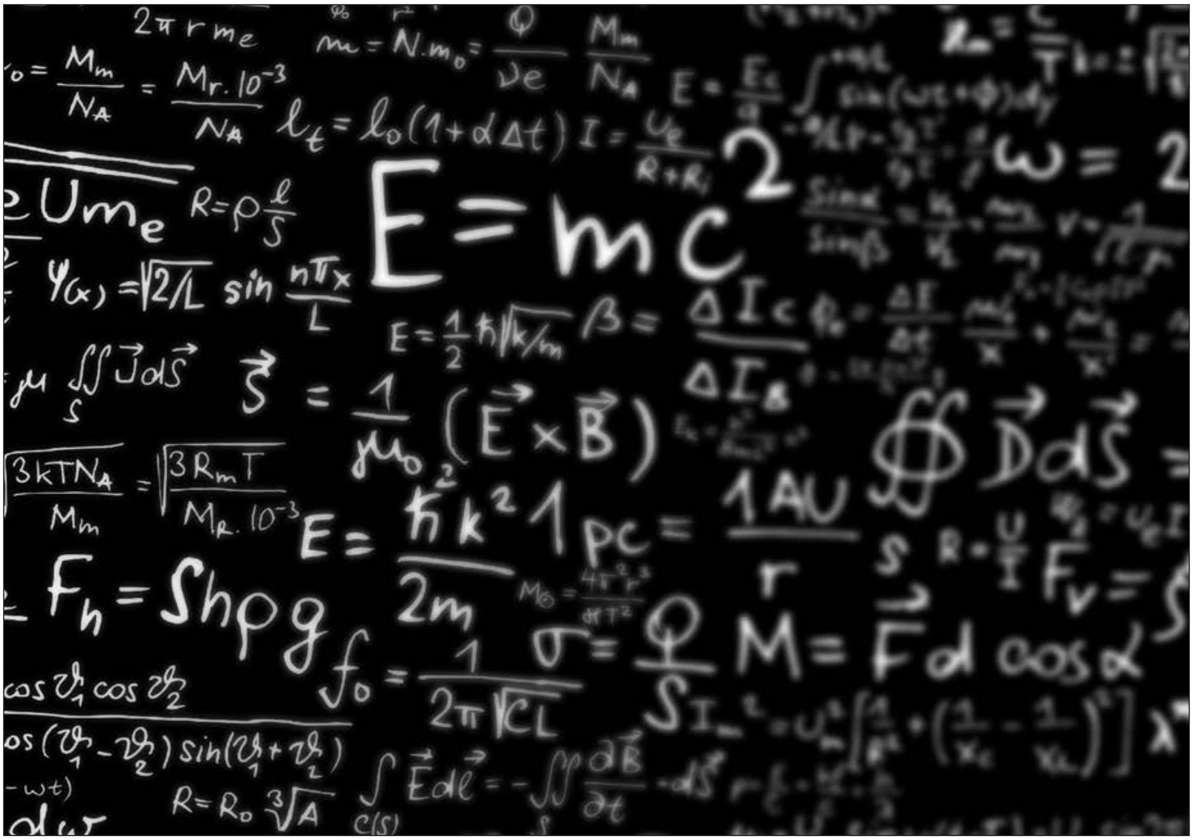

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The Exact Packing Dimension of a Set of Zero Heat Capacity

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Abstract: We consider the packing measure properties of subsets of R^{n+1} of zero heat capacity relative to the heat equation:

$$\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}=\frac{\partial u}{\partial t}$$

The exact packing dimension is determined for subset of R^n, $n\geq 3$ for which u(x, t) is unbounded for $(x, t)\in R^{n+1}$.

Key words: Multifractal spectrum, packing measure and dimension, Brownian motion, heat equation, heat capacity

INTRODUCTION

Taylor and Watson (1985) used a Hausdorff measure classification to determine which subsets of R^{n+1} are of zero heat capacity with respect to the heat equation:

$$\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}=\frac{\partial u}{\partial t} \tag{1}$$

where, R^n is the n-dimensional Euclidean space.

Watson (1978) deduced that for a set in R^n with zero Lebesgue measure, there is a fundamental solution of (1) on $R^n\times(0, \infty)$ which tends to infinity as any point of the given set is approached and that for such Borel subset $E\subset R^n = (R^n\times(t_0))$ of a characteristics hyperplane, the heat capacity relative to the strip $R^n\times(b, c)$ is precisely the n-dimensional Lebesgue measure of E on the hyperplane.

In this research we exploit the well known fact that the transition density of Brownian motion in R^n is just the heat kernel

$$u(x,t)=\begin{cases}(4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{\|x\|^2}{4t}\right) & \text{if } t>0\\ 0 & \text{if } t\leq 0\end{cases} \tag{2}$$

which satisfies the heat Eq. 1. We show, in this study that packing measure zero is equivalent to heat capacity zero on the hyperplane and we also determine the exact packing dimension of subsets $x\in R^n$ for which u(x, t) is unbounded for all t.

Preliminaries

We will use Euclidean norm sign to denote both the distance $\|x-y\|$ between two point in R^n and the Lebesgue meansure $\|E\|$ of a subset of $E\subset R^n$.

We use c_1 c_2... to denote finite positive constants whose precise values are unimportant.

Now let us recall the general setting. Consider a situation where heat is distributed from a heat source over n+1 dimensional Euclidean space R^{n+1}. This heat source gives rise to a function $V : R^{n+1} \rightarrow R$ which assigns a heat potential V(x, t) to each point of $(x, t) \in R^{n+1}$ through a generating kernel u(x, t) so that if we consider a postive Borel measure π on R^{n+1} as the distribution of heat source, then the integral:

$$V(x,t) = \int_{R^{n+1}} u(x-y, t-s)\pi(y,s) \tag{3}$$

is called a heat potential if it is finite on a dense subset of R^{n+1}

We now define the heat capacity of $E \subset R^{n+1}$

Definition 1

For a compact set k, the heat capacity C(k) is defined by:

$C(k) = \sup\{\pi(k) : V(x, t) \leq 1 \text{ on } R^{n+1}, \pi \text{ is supported by } k, \pi \geq 0\}$.

Then the capacity of an arbitrary Borel set E is:

$$C(E) = \sup\{C(k): k \text{ is compact, } K \subseteq E\}$$

The heat capacity of E is zero if and only if V(x, t) is unbounded for every π such that $\pi(E)>0$ whenever $(x, t) \in E$

Equivalently a set E is called polar if and only if there is a potential V on R^{n+1} such that $V(x, t) = \infty$,whenever $(x, t) \in E$

The set E of heat capacity zero are precisely the polar sets.

Specifically, in the present study, we are mainly interested in the packing dimension of the subset $S \subset R^n$ for which $E \subset R^{n+1}$ has zero heat capacity.

So, in what follows, will concern ourselves with the case where the distribution of heat sources generating the potential V(x, t) is invariant under translation in the direction of the t axis so that π can be writen as the direct product of a completely additive function m of Borel sets in n-dimensions and Lebsegue measure on the t axis.

Thus we write:

$$V(x,t) = \int_{R^n} \left\{\int_0^\infty u(y-x,s)ds\right\} m(dy) \tag{4}$$

Where:

$\int_0^\infty u(y-x,s)ds$

is the well known potential kernel of Brownian motion with respect to Lebesgue measure and:

$$\int_0^\infty u(y-x,s)ds = \frac{c_1}{\|y-x\|^{n-2}} \tag{5}$$

where, c_1 is a constant.

The Packing Dimension

One of the several distinct techniques in investigating the size of subsets of zero Lebesque in R^n is the notion of packing dimension due to Taylor and Triort (1985).

Packing dimension is defined via the packing measure as follows:

Start with a class Φ of monotone functions h : $(0, \delta)\rightarrow(0, 1)$ which is non-decreasing, right continuous and satisfies $h(0+) = 0$ and for which there is a constant:

$$c_1 > 0 \text{ such that } h(2s) \le c_1 h(s) \text{ for } 0 < s < \frac{\delta}{2} \tag{6}$$

We obtain the packing measure by a two stage definition. First, we define a pre-measure:

$$h-\overline{p}(E) = \lim_{\sigma\downarrow 0}\sup\left\{\sum_{i=1}^{\infty} h(2r_i) : B_{r_i}(x_i) \text{ disjoint}, x_i \in E, r_i < \sigma\right\} \tag{7}$$

where, $B_{ri}(x_i)$ denotes the open ball centred at x_i, radius r_i Eq. 7 is not an outer measure because it is not countably subadditive:

However it leads to an outer measure by defining:

$$h-p(E) = \inf\left\{\sum_{i=1}^{\infty} h-\overline{p}(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\right\} \tag{8}$$

which can be thought of as a generalization of the Lebesgue measure using maximal packing of E by balls, so that if $h(s) = s^n$ then h-p() on R^n is n-dimensional Lebesgue measure.

Thus, to measure the Borel subset of $E\subseteq R^n$ we need :

$$\lim_{s\to 0+}\frac{s^n}{h(s)} = 0$$

so that if $h(s) = s^\alpha$, $\alpha > 0$, there is a unique value α for which the packing measure h-p (E) drops from infinity to zero
that is, if $E\subseteq R^n$ is bounded, there is a number say $\beta\in[0, n]$ such that :

$$s^\alpha - p(E) = \begin{cases} \infty & \forall\ \alpha<\beta\ \alpha>0 \\ 0 & \forall\ \alpha>\beta\ \alpha>0 \end{cases} \tag{9}$$

so that $s^\alpha - p(R^n) = 0 \qquad \forall \qquad \alpha > n$

This in turn means that E is less occupied than if it were α-ε dimensional for $\varepsilon>0$.

We define the index Dim E as

$$\begin{aligned} \text{Dim E} &= \inf\{\alpha > 0 : s^\alpha - p(E) = 0\} \\ &= \sup\{\alpha > 0 : s^\alpha - p(E) = \infty\}, \text{ which denotes} \end{aligned} \tag{10}$$

the packing dimension of E.

This index gives the notion of size to sets of Lebesgue measure zero and takes value n on each subset of R^n of positive n-dimensional Lebesgue measure and for all subsets of R^n,we have:

$$\text{Dim } E \le n$$

We refer to Taylor (1985) for a convenient reference on the use of packing measure for the analysis of random sets arising from the sample paths of Stocastic processes.

Another useful tool to study the size of a Polar set is the notion of multifractal analysis of occupation measure of a Brownian motion. Let $B_r(x)$ denote the ball in R^n of radius r centred at x and $\{X(t)\}_{t\ge 0}$ denote Brownian motion in R^n, $n\ge 3$, then the occupation measure of X(t) on B_r (x) is $\Psi(B_r(x))$,

$$\text{where, } \psi(E) = \left|\{t \in [0,1] : X(t) \in E\}\right| \tag{11}$$

is the portion of the time interval [0, 1] which the process spends in the Borel set E.

Taylor and Triort (1985) relates the lower density of Ψ at x, denoted by :

$\liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{h(2r)}$, to the packing measure as follows:

Lemma 1

Let Ψ be a Borel measure on R^n and $h\epsilon\Phi$. Then for any Borel set $E\subseteq R^n$

$$h - p(E) \ge c_1\, \psi(E) \inf_{x\in E} \left(D_h(x)\right)^{-1}$$

where c_1 is a constant and

$$D_h(x) = \liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{h(2r)}$$

is the lower h-density of Ψ at x

This is a natural random measure associated with the range of a Brownian motion.

Let φ: $R^n \to R$ be a measureable function. Let D be a real-valued function defined on R^n

Then the spectrum with respect to φ and D is defined by $D\{x\epsilon R^n : \varphi(x) = \alpha\}$

Multifractal analysis consists of studying the size of the level set

$$E_\alpha = \{x : \varphi(x) = \alpha\}$$

In what follows, we consider spectra involving logarithmic order of magnitude. For a random measure Ψ associated to a Brownian motion, let $\varphi(x)$ be the lower density of Ψ at x i.e.:

$$\varphi(x) = \liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{h(2r)}$$

Then we compute the logarithmic multifractal spectrum of Ψ, which is defined as follows

Definition 2

For occupation measure Ψ associated to a Brownian motion and for every $\alpha \ge 0$,

$$E_\alpha = \{x : \varphi(x) = \alpha\}$$

The mapping $D:\alpha \geq 0 \mapsto \mathrm{Dim}(E_\alpha)$ is the logarithimic spectrum of Ψ where Dim E denotes the packing dimension of E.

This provides information about the underlying geometric structure of the supports of Ψ, which are sets of Lebesgue measure zero.

Our first result is analogous to the one due to Watson (1978) for Lebesgue measure and states:

Theorem 1

Let E be a Borel set in R^n, $n \geq 3$ and suppose that s^n - p(E) = 0,
then C(E) = 0

Proof:

Let $(x_0, t_0) \in R^{n+1}$ and let $B_\sigma(x_0) = \{y \in R^n : \|x_0 - y\| < \sigma\}$., $\sigma > 0$

The potential $V(x_0, t_0)$ can be written as the sum of integrals over $B_\sigma(\pi_0) \times R$ and its complement.

Thus

$$V(x_0, t_0) \geq \int_{B_r(\pi_0) \cap E} \frac{c_3}{\|x_0 - y\|^n} dm$$
$$\geq \int_{B_r(\pi_0) \cap E} \frac{c}{\sigma^n} dm = \frac{cm(B_\sigma(x_0) \cap E)}{\sigma^n}$$

Since this holds for all x_0, σ, t_0 it follows that $V(x, t) = \infty$ if

$$\liminf_{\sigma \to 0} \frac{m(B_\sigma(x_i) \cap E)}{\sigma^n} = \infty$$

A stansard application of lemma 1 then shows that $s^n - p(E) = 0$ and hence C(E) = 0

Corollary 1

If $E \subset R^n$ and has packing measure zero then there exists a positive.
temperature V on $R^n \times (0, \infty)$ such that

$$\lim_{(x,t) \to (y, 0+)} V(x,t) = \infty \ \forall \ y \in E$$

Thus $E = (E \times \{0\})$ has zero capacity, in R^{n+1}

Corollary 2

Let $E \subset R^n$, $B \subset R$, λ = Dim E and let X(t) be a standard Brownian motion in R^n then if $\lambda < n$:

$$P\{X(t) \in E, \text{ for some } t \in B\} = 0$$

Brownian Motion and the Distribution of Heat Source

Let $R_T = \{x \in R^n : x = X(t), 0 \leq t \leq T\}$ denote the range of n-dimensional Brownian motion X(t) from $0 \leq t \leq T$ to R^n.

It is well known that for $n \geq 2$ the Brownian motion process hits a point $x \in R^n$ with probability zero which implies that the Lebesgue measure of R_T is zero for $n \geq 2$.

This has an equally inportant interpretation that Brownian path X(t) avoids a set E with probability one i.e., $P(R_T \cap E = \phi) = 1$, if C(E)=0 (Takeuchi, 1964), so that on the set $E \subseteq R^n$, V(x, t) fails to be bounded for all t.

In what follows we consider the occupation measure:

$\Psi(E) = |\{t \in [0, 1]: X(t) \in E\}|$ as the distribution of heat source so that for any $x \notin \langle x_t : 0 \leq t \leq 1\rangle$ and r, $\Psi(B_r(x)) = 0$

Note that for 0<t<1, $\Psi(B_r(x)) \leq T_1(r) + T_2(r)$,

Where:

$$T_1(r) = \int_0^\infty I_{B_r(0)} X(s) ds \tag{12}$$

is the total time spent by X(t) in $B_r(0)$ and $T_2(r)$ is the corresponding sojourn time for an independent copy of the dual X^1 of X obtained by time reversal.

In order to obtain further information about the size of E such that $E \cap R_T = \phi$, it is of interest to find a gauge function h such that

$$\liminf_{r\downarrow 0} \frac{\psi(B_r(x)}{h(r)} = 0.$$

This allows us to consider the set

$$E_0 = \left\{x \in R^n : \liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{h(r)} = 0\right\}.$$

since for any $x \notin \{X(t) : 0 \leq t \leq 1\}$ and r small enough $\Psi(B_r(x)) = 0$

Dembo *et al*. (2000a, b) studied the random set

$$\text{set } E_a = \left\{x \in R^n : \liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{r^2 |\log r|} = a\right\}$$

They proved the following

Theorem 1 (Dembo *et al.*, 2000a)

Let X(t), a Brownian motion in R^n, $n \geq 3$ and $a \in (O, c_n)$

$$\text{Dim}\left\{x \in R^n : \liminf \frac{\psi(B_r(x))}{r^2} \geq a\right\} \leq 2 - I_n(a) \qquad \text{a.s.}$$

$$\text{Where, } I_n(a) = \frac{a}{4}\left(\max\left\{0, n-2-\frac{2}{a}\right\}\right)^2 \text{ and } c_n = \inf\{a : I_n(a) = 2\} = \frac{2}{n - 2\sqrt{n-1}}$$

Theorem: 2 (Dembo *et al.*, 2000b)

Let X(t) be a Brownian motion in R^n, $n \geq 3$ then for all n>1

$$\text{Dim}\left\{x \in R^n : \liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{r^2 |\log r|} = a\right\} = 2 \qquad \text{a.s.}$$

With a little modification, their methods extend, only with obvious changes to the following

Theorem 3

For X(t) a Brownian motion in R^n, $n \geq 3$

$$\text{Dim } E_0 = \left\{x \in R^n : \liminf_{r \to 0} \frac{\psi(B_r(x))}{h(r)} = 0\right\} < 2 \qquad \text{a.s}$$

We focus on constructing a measure function h for which

$$\liminf_{r\to 0} \frac{\psi(B_r(x))}{h(r)} = 0$$

To this end we state the following

Theorem 4

If X(t) is a Brownian motion in R^n, $n\geq 3$, T_1 (r) and T_2 (r) are independent copies of

$$T(r)=\int_0^\infty I_{B_r(0)}X(s)ds$$

Suppose

$$h(r)=r^2\left(\log\frac{1}{r}\right)^{\alpha}\left(\log\log\frac{1}{r}\right)^{-1},\alpha>1$$

Then, with probability 1:

$$\liminf_{r\to 0}\frac{T_1(r)+T_2(r)}{h(r)}=0$$

To prove the above theorem, we will need a few preliminary facts, in particular, the following lemmas

Lemma 2 (Taylor, 1972)

For a Brownian motion X(t) in n-space, there is a λ_0 such that $\forall\ \ \varepsilon>0$

$$\exp\left\{-\frac{1}{2}(1+\varepsilon)\lambda^2\right\}<P\left\{|X(t)|>\lambda t^{\frac{1}{2}}\right\}$$
$$<\exp\left\{-\frac{1}{2}(1-\varepsilon)\lambda^2\right\}$$

Whenever $\lambda>\lambda_0=\lambda_0\ (\varepsilon,\ n)$

Lemma 3 (Perkins and Taylor, 1987)

If $\Gamma_E=\inf\{t>0:X(t)\in E\}$

$X(0)=x\in R^n$

With $|x|=\rho$ and $\Gamma_1=B_{r_1}(0)$ is a ball with centre of the origin and radius r_1

Then for a standard Brownian motion process in n-space, $n\geq 3$:

$$P_x\left\{\Gamma_{\Gamma_1}<\infty\right\}=\left(\frac{r_1}{\rho}\right)^{n-2}$$

Lemma 4

For a Brownian path X(t) in R^n, $n\geq 3$ and a fixed λ, if

$$h(r)=r^2\left(\log\frac{1}{r}\right)^{\alpha}\left(\log\log\frac{1}{r}\right)^{-1},$$

Define $D_k=\{T(a_{k+1})>\lambda h(a_k)\},a_k=\rho^{-k},\rho>1$

Then $P(D_k)=0$ a.s for $\alpha>1$

Proof

From the obvious relationship

$$\{\omega : T(r,\ \omega)\geq s\}\subseteq\{w:\ X(s,\ \omega)<r\}$$

We have

$$\{T(a_{k+1}) > \lambda h(a_k)\} \subseteq \{X(\lambda h(a_k)) < a_{k+1}\}$$

We replace this event by a larger event.

$$|X(\lambda h(a_k)) - X(\lambda h(a_{k+1}))| < a_{k+1}.$$

X(t) does not leave $B_{\alpha_{k+1}}$ (0) before the time t>λ h(α_k)

Let $F_k = A_k \cap B_k \cap C_k$,

where, $A_k = \{X(\lambda\ h(a_{k+1})\ ) > a_{k+1}\ \}$.

$B_k = |X\{\lambda(h(a_k)) - X(\lambda\ h(a_{k+1})\ \}| < (1-\varepsilon)a_{k+1}$

C_k = { X(t) does not leave $B_{\alpha_{k+1}}$ (0) Before time t > λ λ h(α_k)}

By the choice of a_k we take ρ large enough so that $p(B_k \backslash A_k) \le 1\text{-}\varepsilon = C_1$.

By Lemma 3 $P(C_k \backslash B_k \cap A_k) \le C_2$

so that

$$\begin{aligned} P(F_k) &= P(A_k \cap B_k \cap C_k) \\ &= P(B_k \setminus A_k)P(C_k \setminus B_k \cap A_k)P(A_k) \\ &\le c_3 P(A_k) \end{aligned}$$

$$\text{But } P(A_k) = P\left(X\left(\lambda a_{k+1}^2(\log\frac{1}{a_{k+1}}\ )\right)^{\alpha}\left(\log\log\frac{1}{a_{k+1}}\right)^{-1} > a_{k+1}\ \right), \text{ by scaling}$$

$$= P\left(X(a_{k+1}) > a_{k+1}\left[\ \lambda a_{k+1}(\log\frac{1}{a_{k+1}}\ )^{\alpha}\left(\log\log\frac{1}{a_{k+1}}\right)^{-1}\ \right]^{-\frac{1}{2}}\ \right)$$

$$= P\left(X(a_{k+1}) > \lambda^{-\frac{1}{2}} a_{k+1}^{\frac{1}{2}}(\log\frac{1}{a_{k+1}}\ )^{-\frac{\alpha}{2}}\left(\log\log\frac{1}{a_{k+1}}\right)^{+\frac{1}{2}}\ \right)$$

Thus by Lemma 2

we have:

$$\begin{aligned} P(D_k) &\le \exp\left(-\frac{1}{2}(1-\varepsilon)\lambda^{-1}(\log\frac{1}{a_{k+1}}\ )^{-\alpha}\left(\log\log\frac{1}{a_{k+1}}\right)\ \right) \\ &= \exp\left(\log(\log\frac{1}{a_{k+1}})^{-(1-\varepsilon)}\frac{1}{2}\lambda^{-1}(\log\frac{1}{a_{k+1}}\ )^{-\alpha}\right) \\ &= \frac{1}{2}\lambda^{-1}(\log\frac{1}{a_{k+1}})^{-(1-\varepsilon)}(\log\frac{1}{a_{k+1}}\ )^{-\alpha} \\ &= \frac{1}{2}\lambda^{-1}(\log\frac{1}{a_{k+1}})^{-(1-\varepsilon)\ -\alpha} \\ &= \frac{1}{2\lambda}(\log\frac{1}{a_{k+1}})^{-(1-\varepsilon)\ -\alpha)} \end{aligned}$$

For $\alpha > 1$,

$$\sum_{k=1}^{\infty} P(D_k) \le \frac{1}{2\lambda}\sum\frac{1}{(\log\frac{1}{a_{k+1}})^{(1-\varepsilon)+\alpha}} < \infty$$

By Borel Cantelli Lemma

$P(D_k) = 0$

We now prove Theorem 4

For a fixed

$$\lambda_1, 0 < \sigma < 1, \bar{\delta} = \max\{\sigma, 1-\sigma\}$$

Define

$$E_k = \{T_1(a_{k+1}) + T_2(a_{k+1}) > \lambda h(a_k)\}\ a_k = \rho^{-k}, \rho > 1$$

Then $E_k \subseteq \{T_1(a_{k+1}) > \sigma\lambda h(a_k)\} \cup \{T_2(a_{k+1}) > (1-\sigma)\lambda h(a_k)\}$

so that $P(E_K) \leq cP(D_k)$, where

$$D_k = \left\{T(a_{k+1}) > \bar{\sigma}\lambda h(a_k)\right\} \text{ for a constant } c$$

But by Lemma 4

$$P(D_k) = 0 \text{ so that } P(E_k) = 0$$

and thus E_k happens finitely often a.s for each λ and hence

$$P\left(\liminf_{r\downarrow 0} \frac{T_1(r) + T_2(r)}{h(r)} = 0\right) = 1$$

since λ is any fixed number.

By the Blumental zero-one law, we have $P\left(\liminf_{r\downarrow 0} \frac{T_1(r) + T_2(r)}{h(r)} = 0\right) = 1$

so that $P\left(\liminf_{r\downarrow 0} \frac{\psi_1(B_r X(t_0))}{h(r)} = 0\right) = 1$ and hence

$$\liminf_{r\downarrow 0} \frac{\psi(B_r(x))}{h(r)} = 0 \qquad \text{a.s}$$

Theorem 5

Suppose X(t) is Brownian motion in R^n, $n \geq 3$ and

$$R_1 = \{x \in R^n : x = X(t), 0 \leq t \leq 1\}$$

Then for any compact set $E C R^n$

$$p\{R_T \cap E \neq \phi\} = \begin{cases} 1 \text{ if } \operatorname{Dim} E > 2 \\ 0 \text{ if } \operatorname{Dim} < 2 \end{cases}$$

Proof

We first show that Dim E<2 implies $R_1 \cap E = \phi$ a.s

This follows from the fact that if $x \notin R_T$ then $\Psi(B_r(x)) = 0$ and by theorem 3 that

Dim E < 2 implies

$R_1 \cap E = \phi$ a.s.

It remains to show that if Dim E > 2 then $R_T \cap E \neq \phi$ a.s.

Pruit and Taylor (1996) have proved that if $\lambda > 2$

then Dim $R_1 \geq \lambda$ a.s.

This implies that whenever Dim(E)>2, then $R_1 \cap E \neq \phi$ a.s.

CONCLUSION

We have characterized the geometric structure of the set of points $x \in R^n$ for which the solution u(x, t) of the heat equation in R^n is unbounded for all t, by giving the size of such set of points (more precisely, the packing dimension of such set of points). Such points have, until very recently, been considered of little interest since the set of such points is of Lebesque measure zero. We have shown that the set has packing dimension 2 and thus is big. Moreover, we have shown that a set of zero heat capacity has zero packing measure in n-dimensional space.

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