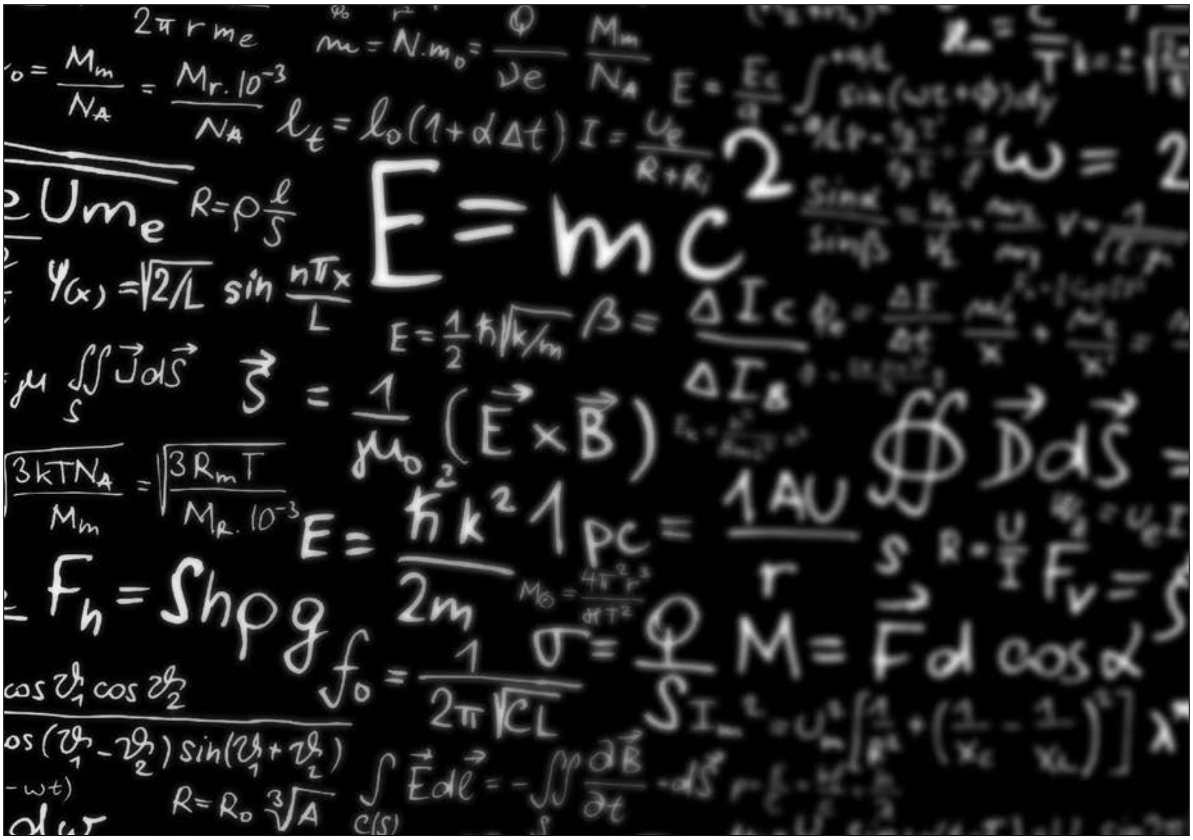

Asian Journal of Mathematics & Statistics

ISSN 1994-5418

Asian Journal of Mathematics and Statistics 2 (3): 41-54, 2009
ISSN 1994-5418

Modelling the Dependence of Parametric Bivariate Extreme Value Copulas

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Abstract: In this study, we consider the situation where contraints are made on the domains of two random variables whose joint copula is an extreme value model. We introduce a new measure which characterize these conditional dependence. We proved that every bivariate extreme value copulas is totally characterized by a conditional dependence function. Every two-dimensional distribution is also shown to be max-infinite divisible under a restriction on the new measure. The average and median values of the measure have been computed for the main bivariate families of parametric extreme value copulas.

Key words: Copulas, pickands dependence function, discordance function, tail-dependence, median discordance

INTRODUCTION

Modelling the dependence between several random variables lies at the heart of the subject of Extreme Value Theory (EVT). In this theory, many structures have been developed to describe and to measure this dependence (Pickands, 1981; Coles, 2001). Some of these structures remain invariant under strictly increasing transformations of lower dimensional marginal variables. It is the case of copulas, multivariate distributions whose univariate margins are uniformly distributed on the unit interval I = [0, 1] and which establish, via Sklar theorem, a close connection between every multivariate distribution function and its univariate margins (Sklar, 1996; Joe, 1997).

In Bivariate Extreme Value (BEV) study, the tail dependence parameters estimate numerically the importance of asymptotic dependence between two random variables. Let H be the distribution function of a random pair (X_1, X_2) with univariate margins $\{H_i, i = 1, 2\}$. The tail dependence parameter λ of H, given by:

$$\lambda = \lim_{x \longrightarrow x_H^*} \frac{\overline{H}(x,x)}{1 - H_i(x)} = \lim_{t \longrightarrow 1^-} P(H_1(X_1) > t / H_2(X_2) > t) \tag{1}$$

Quantifies the probability to observe a large X_1 assuming that X_2 is also large, where H and x_H^* denote, respectively the survivor function and the right endpoint of H.

In many latest studies published on the topic of EVT and applications (Beirlant *et al.*, 2005; Michel, 2006), it has been shown that no unique parametric structure can summary the family of multivariate distributions like in univariate situation. Nevertheless, if the univariate

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margins are given, the dependence of the joint distribution can be characterized by one of equivalent measures like Pickands dependence function (Pickands, 1981), exponent measure or stable tail dependence function (Michel, 2006; Degen, 2006). However, these measures do not take into account the conditions that would be made on lower dimensional margins.

The main contribution of this study is to construct a new measure and function which describe the dependence of the family of BEV copulas under given conditions on the marginal variables. This measure enables us to compute the everage and medium values of the conditional dependence and the new dependence function allows the characterization of the main BEV parametric copulas.

MATERIALS AND METHODS

In this study, we consider the following problem: Let consider a situation where the bivariate distribution function H associated to a copula C is a BEV model and we are interested to model a structure which describes the dependence of H under given conditions on X_1 and X_2. Therefore, it is desirable to model a bivariate function which gives at any realization (x_1, x_2) of (X_1, X_2) the probability of the discordances like $\{X_i>x_i/X_j\le x_j\}$ or $\{X_i\le x_i/X_j>x_j\}$ where x_i and x_j are given realizations of X_i and X_j respectively with $1\le i\ne j\le 2$. For this purpose the total positivity and max-infinite divisibility of a multivariate distribution function and the Pickands characterization of the dependence of multivariate extreme value structures would be useful.

A bivariate copula is defined as follows (Sklar, 1996; Nelsen, 1999).

Definition 1

A two-dimensional copula (or 2-copula) is a bivariate function C: $I^2 \to I$ such that:

- $C(u_1, 0) = C(0, u_2) = 0$ for all $(u_1, u_2) \in I^2$; C is said grounded
- $C(u_1, 1) = u_1$ and $C(1, u_2) = u_2$, for all $(u_1, u_2) \in I^2$ the univariate margins of C are uniformly distributed on I
- C is 2-increasing: $C(u'_1, u'_2)-C(u'_1, u_2)-C(u_1, u'_2)+C(u_1, u_2)\ge 0$ for all (u_1, u_2), $\|\in {}^2$, $(u'_1, u'_2) \in I^2$ such as and $u_1\le u'_1$ and $u_2\le u'_2$

The following result gives the connection between this formal definition of a copula and its application in statistical modeling (Sklar, 1996; Nelsen, 1997).

Theorem 1: (Sklar Theorem)

Let H be a m-dimensional distribution with margins $H_1,...,H_m$. Then, there exists a copula C such that for all $x = (x_1,...,x_m) \in \bar{R}^m$, where $\bar{R} = R \cup \{\pm\infty\}$.

$$H(x_1,...,x_m) = C[H_1(x_1),...,H_m(x_m)] \tag{2}$$

If $H_1,...,H_m$ are all continuous, then C is unique; otherwise C is uniquely determined on $RanH_1\times...\times RanH_m$. Conversely, for a copula C and continuous margins $H_1,...,H_m$ the function H defined by Eq. 1 is m-dimensional distribution with margins $H_1,...,H_m$.

In bivariate case, the copula C defined by Eq. 2 is a BEV copula if there exists normalizing sequences $\{\sigma_{i,n}>0\}$ and $\{\mu_{i,n}\in R\}$, i = 1, 2; such that, for $(u_1, u_2 \in I^2)$.

$$P\left(\frac{M_{1,n}-\mu_{1,n}}{\sigma_{1,n}} \le H_1^{-1}(u_1); \frac{M_{2,n}-\mu_{2,n}}{\sigma_{2,n}} \le H_2^{-1}(u_2)\right) \xrightarrow[n\to+\infty]{} C(u_1,u_2) \tag{3}$$

where, $H_i^{-1}(p) = \inf\{x/H_i(x) \geq p\}$ is the quantile function of the marginal variable X_i and $(M_{1n};M_{2n}) = \left(\max_{1\leq j\leq n}(X_{1,j});\ \max_{1\leq j\leq n}(X_{2,j})\right)$ being the vector of componentwise maxima of the n independent and identically distributed vectors $(X_{1,1}:\ X_{2,1}),\ldots,(X_{1,n},\ X_{2,n})$ with commun distribution function F.

If Eq. 3 holds, F is said to belong to the Max-Domain of Attraction of H (MDA(H)) and C verifies, for all $(u_1, u_2)\in I^2$:

$$\begin{cases} C^t(u_1,u_2) = C(u_1^t,u_2^t) \text{ for all } t > 0 \\ C(u_1,u_2) = \lim\limits_{n\longrightarrow+\infty} F^n\left[F_1^{-1}(\tilde{u}_1),F_2^{-1}(\tilde{u}_2)\right] \end{cases} \tag{4}$$

where, $\tilde{u}_i$ is defined as:

$$\tilde{u}_i = \mu_i - \frac{\sigma_i}{\xi_i}\left[1-(-\ln u_i)^{-\xi_i}\right];\{u_i \in I, i = 1,2\} \tag{5}$$

$\{\mu_i \in R\}$; $\{\xi_i \in R\}$ and $\{\sigma_i>0\}$ being respectively the location, shape and scale parameters of the variable X_i.

Moreover, every BEV copula has the following canonical representation:

$$C(u_1,u_2) = \exp\left\{-(\tilde{u}_1+\tilde{u}_2)A_C\left(\frac{\tilde{u}_1}{\tilde{u}_1+\tilde{u}_2}\right)\right\},(u_1,u_2)\in I^2 \tag{6}$$

The function A_C is usually refered to Pickands dependence function of C. It satisfies:

$$A(t) = \int_0^1 \max((1-t)q, t(1-q))dH(q) \text{ and } \frac{1}{2} \leq A([0,1]) \leq 1$$

where, H is a positive finite measure on I (Pickands,1981).

Furthermore, every margin $\{H_i, i = 1, 2\}$ of H is one of the three univariate extreme value distributions Φ, Λ and Ψ defined as:

$$\begin{cases} \Phi_{\frac{1}{\xi_i}}(x_i) = \exp\left\{-(x_i)^{\frac{-1}{\xi_i}}\right\};\xi_i > 0, x_i > 0 \text{ with } \sigma_{i,n} = n^{\frac{1}{\xi_i}};\mu_{i,n} = 0 \\ \Lambda(x_i) = \exp\{-\exp(-x_i)\};\xi_i = 0, x_i \in R \text{ with } \sigma_{i,n} = 1;\mu_{i,n} = \log n \\ \Psi_{\frac{1}{\xi_i}}(x_i) = \exp\left\{-(x_i)^{\frac{1}{\xi_i}}\right\};\xi_i < 0, x_i \leq 0 \text{ with } \sigma_{i,n} = n^{\frac{-1}{\xi_i}};\mu_{i,n} = 0 \end{cases} \tag{7}$$

Note that the notions of he Max-infinite divisibility and the Total Positivity of multivariate distributions are inherently associated with EVT (Joe, 1997).

Definition 2

A multivariate distribution function H is Max-infinite divisible (max-id) if, for all $\lambda>0$, H^λ is also a distribution function.

- A non-negative function f defined on a subset D of R^2 is Totally Positive of order 2 (TP_2) on D if, for all $(x_1, x_2) \in D$; $(x'_1, x'_2) \in D$; $f(x_1, x_2)\, f(x'_1, x'_2) \geq f(x'_1, x_2)\, f(x_1, x'_2)$ provided that $x_1<x'_1$ and $x_2<x'_2$ in D

RESULTS

Discordance Measure of Multivariate Copulas

Let $X = \{(X_1,...,X_n), n\geq 2\}$ define a random vector and k be a natural number such that $1\leq k<n$. For any proper subset of k elements N_k of $N = \{1,...,n\}$, we consider the following partition $\tilde{X}$ of X such as:

$$\tilde{X} = (\tilde{X}_{N_k}, \tilde{X}_{\bar{N}_k})$$

where, $\tilde{X}_{N_k} = (X_{N_k,1};...;X_{N_k,k})$ is the k-dimensional marginal vector of X whose component indexes are ordered in the subset N_k and $\tilde{X}_{\bar{N}_k} = (X_{\bar{N}_k,1};...;X_{\bar{N}_k,n-k})$ is the (n-k)-dimensional marginal vector of X whose component indexes are ordered in $\bar{N}_k = C_{N_n}^{N_k}$, the complementary of N_k in N.

Definition 3

Let N_k be a proper subset of N. For all realization $x = (x_1,...,x_n)$ of X, we define the N_k-upper discordance degree of X (or the discordance degree of X in the direction of N_k) as the conditional probability:

$$\delta_{N_k}^{+}(x) = P\left(\tilde{X}_{N_k} > \tilde{x}_{N_k} / \tilde{X}_{\bar{N}_k} \leq \tilde{x}_{\bar{N}_k}\right)$$

where, $\tilde{x}_{N_k}$ and $\tilde{x}_{\bar{N}_k}$ are realizations of vectors $\tilde{X}_{N_k}$ and $\tilde{X}_{\bar{N}_k}$, respectively.

Similarly, the lower discordance degree of X is given by:

$$\delta_{\bar{N}_k}^{-}(x) = P\left(\tilde{X}_{N_k} \leq \tilde{x}_{N_k} / \tilde{X}_{\bar{N}_k} > \tilde{x}_{\bar{N}_k}\right)$$

If C is the copula associated to the distribution function of X by Eq. 2, then $\delta_{N_k}^{+}$ is the upper discordance degree of C and denoted $\delta_{N_k,C}^{+}$.

For example, for any random pairwise there exists four discordance degrees $\{\delta_{i,j}^{\pm};1\leq i\neq j\leq 2\}$. If in addition the distribution function H of is continuous with margins we check easily that, for all $(x_1, x_2)\in R^2$.

$$\begin{cases} \delta_{1,1}^{+}(x_1,x_2) = P(X_1 > x_1 / X_2 \leq x_2) = 1 - H(x_1,x_2)H_2^{-1}(x_2) \\ \delta_{1,1}^{-}(x_1,x_2) = P(X_1 \leq x / X_2 > x_2) = 1 - \bar{H}(x_1,x_2)\bar{H}_2^{-1}(x_2) \\ \delta_{1,2}^{+}(x_1,x_2) = P(X_2 > x_2 / X_1 \leq x_1) = 1 - H(x_1,x_2)H_1^{-1}(x_1) \\ \delta_{1,2}^{-}(x_1,x_2) = P(X_2 \leq x_2 / X_1 > x_1) = 1 - \bar{H}(x_1,x_2)\bar{H}_1^{-1}(x_1) \end{cases} \tag{8}$$

where, $\{H^{-1}_i>0\}$ and $\bar{H}$ are the inverse and the survivor function, respectively of the univariate margin with $\bar{H}(x_1,x_2) = 1 - H_1(x_1) - H_2(x_2) + H(x_1,x_2)$.

Let $M_{e,H}$ denote the median point of a continuous distribution function H such as:

$$M_{e,H} = \left(H_1^{-1}\left(\frac{1}{2}\right),...,H_n^{-1}\left(\frac{1}{2}\right)\right)$$

Definition 4

Let C be the copula of H. We define the N_k-upper median discordance of C, denoted by $\tilde{\delta}^+_{N_k,C}$ as the value taken by the measure at the median point. Then,

$$\tilde{\delta}^+_{N_k,C} = \delta^+_{N_k,C}(M_{e,X}) = 1 - C\left(\frac{1}{2},\dots,\frac{1}{2}\right)\left[C_{H_{N_k}}\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right]^{-1}$$

where, $C_{H_{N_k}}$ is the k-variate copula associated to the distribution fonction H_{N_k} of the marginal variable X_{N_k}. The N_k-lower median discordance of C is defined similarly.

Note that each of the four discordance degrees is linked to the others by functional transformations. Therefore, we can restrict the further characterizations to the first degree $\delta^+_{1,1}$ termed the discordance degree of C and denoted δ_C.

For example, let $\{C_{\theta_1,\theta_2};\theta_i > 0\}$ be the trivariate copula of parametric, asymmetric and logistic model expressed by:

$$C_{\theta_1,\theta_2}(u_1,u_2) = \exp\left\{-\left[\sum \tilde{u}_i^{-\theta_1} - \left(\tilde{u}_1^{-\theta_1\theta_2} + 2^{\theta_2}u_2^{-\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}} - \left(u_3^{-\theta_1\theta_2} + 2^{\theta_2}u_2^{-\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}}\right]\right\}$$

where, $\tilde{u}_i$ satisfies Eq. 5. The corresponding discordance degree is given by:

$$\delta_C = \delta_C(\theta_1,\theta_2) = 1 - 2^{\frac{-1}{\theta_2}-\left[3-2\left(1+2^{\theta_2}\right)^{\frac{1}{\theta_1}}\right]}$$

and the simple choice: $\theta_1 = \theta_2 = 1/2$ yields $\delta_C \approx 0.88$.

The following result gives the Max-infinite divibility of a bivariate distribution function under conditions made on its discordance degree.

Theorem 2

Let H the distribution function of a random pairwise (X_1, X_2) with copula C. If the discordance degree $\delta_C(x_1, x_2)$ of C is non-decreasing in x_2, then H is max-id.

Discordance Function of BEV Copulas

The following result characterizes the discordance degree of a BEV copula by a conditional dependence function.

Theorem 3

Let C a BEV copula with discordance degree δ_C. Then, there exists a function B_C defined from R^2 to $\left[\frac{-1}{2},1\right]$ such that, for all $(x_1, x_2) \in R^2$.

$$\delta_C(x_1,x_2) = 1 - \exp\left\{-\left(y_1(x_1) + y_2(x_2)\right)B_C\left(\frac{y_2(x_2)}{y_1(x_1)}\right)\right\} \tag{9}$$

where, $y_i(x_i) = \left[1 + \xi_i\left(\frac{x_i - \mu_i}{\sigma_i}\right)\right]_+^{\frac{-1}{\xi_i}}$ with $\{\mu_i \in R\}$, $\{\xi_i \in R\}$ and $\{\sigma_i > 0\}$ given by Eq. 5.

Definition 5

B_C is called the discordance function of the BEV copula C.
Consider for example the symmetric and mixed model of Tajvidi (1996) expressed as:

$$C_{\theta_1,\theta_2}(u_1,u_2) = u_1 u_2 \exp\left\{-\theta_2\left(\left(u_1^{-\theta_1} + u_2^{-\theta_1}\right)^{\frac{-1}{\theta_1}}\right)\right\}; \theta_1 \geq 0; 0 < \theta_2 \leq 1$$

The corresponding discordance function is given by:

$$B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left[1-\theta_2\left[1+t^{-\theta_1}\right]^{\frac{-1}{\theta_1}}\right]; t > 0$$

The following result shows that all BEV copula is complētly characterized by its discordance function.

Theorem 4

Let C be the copula of a continuous BEV distribution H with discordance function B_C. Then, verifies the following properties:

- $C(u_1,u_2) = u_2 \exp\left\{\left[\ln(u_1 u_2) B_C\left(\frac{\ln u_2}{\ln u_1}\right)\right]\right\}; 0 < u_1 \leq 1; u_2 \in I$
- If $\tilde{H}$ is a bivariate distribution such as $\tilde{H} \in MDA(H)$, then $C_{\tilde{H}} = C$

Average Value of the Median Discordance of BEV Copulas

The characterization of the tail dependence by Eq. 1 means also that the tail dependence parameter is a pure copula property, i.e., is independent of margins. Indeed, if C is the copula associated the distribution H, the tail dependence parameter is equivalently defined by:

$$\lambda = \lim_{u \longrightarrow 1^-} \frac{1-2u+C(u,u)}{1-u} = 2 - \lim_{u \longrightarrow 1^-} \frac{\log C(u,u)}{\log u} \tag{10}$$

The couple (X_1, X_2) is said upper tail dependent if $\lambda>0$ and upper tail independent if $\lambda = 0$. Particularly, if the copula C is a BEV model, its diagonal section C(u, u) is differentiable for all $u \in I$ so that it yields the very significant equality $\lambda = 1-2B_C(1)$; B_C being the discordance function of C. Then, the median discordance of C is given by:

$$\tilde{\delta}_{C_\theta} = 1 - 2C\left(\frac{1}{2};\frac{1}{2}\right)$$

and then $\tilde{\delta}_{C_\theta} = 1 - 2^{-2B(1)}$.

Furthermore, assume that the parameter θ of a parametric BEV copula C_θ belongs to an opened domain $]\theta_{min}, \theta_{max}[$ in R. Therefore, estimation on $\tilde{\delta}_{C_\theta}$ may now be reduced to estimation of the parameter θ. Let $\{\tilde{\delta}^*_{i,C_\theta}; i=1,2\}$ denote the two limiting values of $\tilde{\delta}_{C_\theta}$ when θ tends to θ_{min} or θ_{max}. It follows that, for a n-parametric BEV copula there exists 2^n limiting values of the parameters so that we can compute their average value.

Definition 6

Given a n-parametric BEV copula $\{C_\theta, \theta = (\theta_1,...,\theta_n)\}$. We define the average value of the median discordance of C as the real number:

$$\bar{\delta}_{C_\theta} = \sum_{i=1}^{2^n} \frac{\tilde{\delta}^*_{i,C_\theta}}{2^n}$$

where, $\{\tilde{\delta}^*_{i,C_\theta}; i = 1,...,n\}$ are the limiting values of $\tilde{\delta}_{i,C_\theta}$.

For the above mixed model of Tajvidi, we have $\theta_1 \le 0$; $0<\theta_2\le 1$. Therefore, there exists four limiting values: $\tilde{\delta}_{0,0}$; $\tilde{\delta}_{0,1}$; $\tilde{\delta}_{+\infty,0}$; $\tilde{\delta}_{+\infty,1}$; then $\bar{\delta}_{C_\theta} = 0.75$. Table 1-3 give the averages of the median discordance of the main parametric model of BEV copulas.

Application to the Parametric Models of BEV Copulas

The BEV copulas arise from symmetric or asymmetric generalization of two known differentiable parametric models: the logistic model and the mixted model (Joe, 1997; Degen, 2006; Müller, 2006).

Table 1: Discordance function of logistic BEV copulas and extensions (1)

Copula $C_\theta(u_1, u_2)$, $(u_1, u_2) \in I^2$, discordance $B_\theta(t)$, t>0	Median, average
Logistic model of Gumbel (Joe, 1997), $\theta \ge 1$:	
$C_\theta(u_1,u_2) = \exp\left\{-\left(\tilde{u}_1^\theta + \tilde{u}_2^\theta\right)^{\frac{1}{\theta}}\right\}$; $B_\theta(t) = \frac{1}{1+t}\left[\left(1+t^{-\theta}\right)^{\frac{1}{\theta}} - 1\right]$	$\tilde{\delta}_C = 1 - 2^{1-2^{\frac{1}{\theta}}-1}$ $\bar{\delta}_C = 0.25$
Negative and logistic model of Galambos (Joe, 1997), $\theta \ge 0$:	
$C_\theta(u_1,u_2) = u_1u_2 \exp\left\{\left(\tilde{u}_1^{-\theta} + \tilde{u}_2^{-\theta}\right)^{\frac{-1}{\theta}}\right\}$; $B_\theta(t) = \frac{1}{1+t}\left[1-\left(1+t^{-\theta}\right)^{\frac{-1}{\theta}}\right]$	$\tilde{\delta}_C = 1 - 2^{1-2^{\frac{1}{\theta}}}$ $\bar{\delta}_C = 0.25$
Negative extension of logistic model (Joe, 1997):	
$C_{\theta_1,\theta_2}(u_1,u_2) = u_1u_2 \exp\left\{\left[\left(\tilde{u}_1^{\theta_1} + \tilde{u}_2^{\theta_1}\right) - \left(\tilde{u}_1^{-\theta_1\theta_2} + \tilde{u}_2^{-\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right\}$;	$\tilde{\delta}_C = 1 - 2^{1-\left(2-2^{\frac{1}{\theta_2}}\right)^{\frac{-1}{\theta_1}}}$ $\bar{\delta}_C = 0.125$
$B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left(\left[t^{-\theta_1} + 1 - \left(t^{\theta_1\theta_2}+1\right)^{\frac{-1}{\theta_2}}\right]^{\frac{-1}{\theta_1}} - 1\right)$; $\theta_1 > 1$; $\theta_2 > 0$	
BEV Gaussian model (Hüsler-Reiss, 1989):	
$C_\theta(u_1,u_2) = \exp\left\{-\tilde{u}_1\Phi\left[\frac{1}{\theta} + \frac{\theta}{2}\log\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right] - \tilde{u}_2\Phi\left[\frac{1}{\theta} + \frac{\theta}{2}\log\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right]\right\}$, $B_\theta(t) = \frac{1}{1+t}\left[\frac{1}{t}\Phi\left(\frac{2-\theta^2\log t}{2\theta}\right) - \Phi\left(-\frac{2+\theta^2\log t}{2\theta}\right)\right]$ With $\theta \ge 0$	$\tilde{\delta}_C = 1 - 2^{1-2\Phi\left(\frac{1}{\theta}\right)}$ $\bar{\delta}_C = 0.25$
Symmetric extension of logistic model (Tajvidi, 1996):	
$C_{\theta_1,\theta_2}(u_1,u_2) = \exp\left\{-\left[\left(\tilde{u}_1^{\theta_1} + \tilde{u}_2^{\theta_1}\right) + \theta_2\left(\tilde{u}_1\tilde{u}_2\right)^{\frac{\theta_1}{2}}\right]^{\frac{1}{\theta_1}}\right\}$;	
$B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left(\left[t^{-\theta_1} + 1 + \theta_2 t^{\frac{-\theta_1}{2}}\right]^{\frac{1}{\theta_1}}\right)$; $0 < \theta_2 \le 2(\theta_1 - 1)$; $\theta_2 \ge 2$	
Symmetric extension of logistic model proposed by Joe (1997):	
$C_{\theta_1,\theta_2}(u_1,u_2) = \exp\left\{-\left[\left(\tilde{u}_1^{\theta_1} + \tilde{u}_2^{\theta_1}\right) - \left(\tilde{u}_1^{-\theta_1\theta_2} + \tilde{u}_2^{-\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right\}$;	
$B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left(\left[t^{-\theta_1} + 1 - \left(t^{\theta_1\theta_2}+1\right)^{\frac{-1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right)$; $\theta_1 \ge 1, \theta_2 > 0$	$\tilde{\delta}_C = 1 - 2^{1-\left(2-2^{\frac{-1}{\theta_2}}\right)^{\frac{1}{\theta_1}}}$ $\bar{\delta}_C = 0.125$

Table 2: Discordance function of logistic BEV copulas and extensions (2)

Copula $C_\theta(u_1, u_2)$, $(u_1, u_2) \in I^2$, discordance $B_\theta(t)$, $t>0$	Median, average
Asymmetric extension of logistic model (Tawn, 1988):	
$C_{\theta_1,\theta_2,\theta_3}(u_1,u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \exp\left\{-\left[(\theta_1\tilde{u}_1)^{\theta_3} + (\theta_2\tilde{u}_2)^{\theta_3}\right]^{\frac{1}{\theta_3}}\right\};$ $B_{\theta_1,\theta_2,\theta_3}(t) = \frac{1}{1+t}\left[1-\theta_1+\theta_2 t + \left(\theta_1^{\theta_3} + (\theta_2 t)^{\theta_3}\right)^{\frac{1}{\theta_3}}\right],\ \theta_1 \geq 0; \theta_2 \leq 1; \theta_3 \geq 1$	$\tilde{\delta}_C = 1-2^{\theta_2-\theta_1-\left(\theta_1^{\theta_3}+\theta_2^{\theta_3}\right)^{\frac{1}{\theta_3}}-1}$ $\bar{\delta}_C = 0.125$
Negative and asymmetric extension of logistic model (Joe, 1997):	
$C_{\theta_1,\theta_2,\theta_3}(u_1,u_2) = u_1 u_2 \exp\left\{-\left[(\theta_1\tilde{u}_1)^{-\theta_3} + (\theta_2\tilde{u}_2)^{-\theta_3}\right]^{\frac{-1}{\theta_3}}\right\};$ $B_{\theta_1,\theta_2,\theta_3}(t) = \frac{1}{1+t}\left[1-\left(\theta_1^{-\theta_3} + (\theta_2 t)^{-\theta_3}\right)^{\frac{-1}{\theta_3}}\right],\ \theta_1 > 0; \theta_2 \leq 1; \theta_3 > 0$	$\tilde{\delta}_C = 1-2^{\left(\theta_1^{-\theta_3}+\theta_2^{-\theta_3}\right)^{\frac{-1}{\theta_3}}}$ $\bar{\delta}_C = 0.325$
Bilogistic and symmetric model proposed by Smith (Müler, 2006):	
$C_{\theta_1,\theta_2}(u_1,u_2) = \exp\left\{\tilde{u}_1 q^{1-\theta_1} + \tilde{u}_2(1-q)^{1-\theta_2}\right\},\ q = q(u_1,u_2,\theta_1,\theta_2)$ are the roots of equation : $(1-\theta_1)u_1(1-q)^{\theta_2} - (1-\theta_2)u_2 q^{\theta_1} = 0$ $B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left[q^{1-\theta_1} + t(1-q)^{1-\theta_2} + 1\right]$ with $\theta_1 > 0; \theta_2 \leq 1; \theta_3 > 0$	$\tilde{\delta}_C = 1-2^{1-2^{1-q^{1-\theta_1}-(1-q)^{1-\theta_2}}}$ $\bar{\delta}_C = \tilde{\delta}_C(q)$
Asymmetric model proposed by Coles and Tawn (Müler, 2006):	
$C_{\theta_1,\theta_2}(u_1,u_2) = \exp\left\{-\left[\tilde{u}_1[1-B(q,\theta_1+1,\theta_2)] + \tilde{u}_2 B(q,\theta_1,\theta_2+1)\right]\right\}$ where, $q = \frac{\theta_1 u_1}{\theta_1 u_1 + \theta_2 u_2}$; $B(q,\theta_1,\theta_2)$ being the beta distribution $B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left[\frac{1-B(q,\theta_1+1,\theta_2)}{t} + B(q,\theta_1,\theta_2+1) - 1\right];\ \theta_1,\theta_2 > 0$	$\tilde{\delta}_C = 1-2^{1-B(q,\theta_1+1,\theta_2)-B(q,\theta_1,\theta_2+1)}$ $\bar{\delta}_C = \tilde{\delta}_C(q)$
Bilogistic and negative model (Müler, 2006):	
$C_{\theta_1,\theta_2}(u_1,u_2) = u_1 u_2 \exp\left\{\tilde{u}_1 q^{1+\theta_1} + \tilde{u}_2(1-q)^{1+\theta_2}\right\};\ q = q(u_1,u_2,\theta_1,\theta_2)$ are the roots of equation : $(1+\theta_1)u_1 q^{\theta_1} - (1+\theta_2)u_2(1-q)^{\theta_2} = 0$ $B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left[1-q^{1+\theta_1} - t(1-q)^{1+\theta_2}\right]$ with $\theta_1 > 0; \theta_2 > 0$	$\tilde{\delta}_C = 1-2^{q^{1+\theta_1}+(1-q)^{1+\theta_2}-1}$ $\bar{\delta}_C = \tilde{\delta}_C(q)$

Parametric Logistic Copulas and Extensions

Definition 7

A bivariate function C is a logistic BEV copula if C has the representation:

$$C_\theta(u_1,u_2) = \exp\left\{-\left(\tilde{u}_1^\theta + \tilde{u}_2^\theta\right)^{\frac{1}{\theta}}\right\}$$

for all $(u_1, u_2) \in I^2$ here the parameter θ describes the association between the two random variables X_1 and X_2; $\tilde{u}_i = \ln H_i(x_i)$; $H_i(x_i)$ being the univariate margins of the distribution function associated to C by Eq. 2 (Michel, 2006).

For example, with the normal margins $H_i(xi) = \Phi(x_i)$, where Φ denotes the distribution function of standard Normal law N(0, 1), we get the BEV Gaussian copula (Hüsler-Reiss, 1989).

$$C_\theta(u_1,u_2) = \exp\left\{-\tilde{u}_1\Phi\left(\frac{1}{\theta}+\frac{\theta}{2}\log\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right) - \tilde{u}_2\Phi\left(\frac{1}{\theta}+\frac{\theta}{2}\log\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right)\right\}$$

Parametric Mixed Copulas and Extensions

Definition 8

A bivariate parametric function C is a BEV mixed copula if C has the representation:

$$C_\theta(u_1,u_2) = u_1 u_2 \exp\left\{\theta\left(\tilde{u}_1^{-1} + \tilde{u}_2^{-1}\right)^{-1}\right\}$$

where, the parameter $0 \le \theta \le 1$ describes the association between the two random variables X_1 and X_2 (Michel, 2006).

Table 3: Discordance function of BEV mixed copulas and extensions

Copula $C_\theta(u_1, u_2)$, $(u_1, u_2) \in I^2$, discordance $B_\theta(t)$, $t>0$	Median, average
Asymmetric and mixed model (Tawn, 1988): $C_{\theta_1,\theta_2}(u_1,u_2) = u_1u_2 \exp\left\{\tilde{u}_1\tilde{u}_2 \frac{\tilde{u}_1(\theta_1+\theta_2)+\tilde{u}_2(\theta_1+2\theta_2)}{(\tilde{u}_1\tilde{u}_2)^2}\right\}$; $B_\theta(t) = \frac{1}{(1+t)^3}\left[1+(2-\theta_1-2\theta_2)t+(1-\theta_1-\theta_2)t^2\right]$ where, $\theta_1 \ge 0; \theta_1+2\theta_2 \le 1, \theta_1+3\theta_2 \ge 0$	$\tilde{\delta}_C = 1-2^{\frac{\theta_1}{2}+\frac{3\theta_2}{4}-1}$ $\bar{\delta}_C = 0.46$
Symmetric and mixed model (Müler, 2006): $C_\theta(u_1,u_2) = u_1u_2 \exp\left\{\frac{\theta\tilde{u}_1\tilde{u}_2}{\tilde{u}_1+\tilde{u}_2}\right\}$; $B_\theta(t) = \frac{\theta}{(1+t)^2}+\frac{1-\theta}{1+t}$; $\theta \in [0,1]$	$\tilde{\delta}_C = 1-2^{\frac{\theta}{2}-1}$ $\bar{\delta}_C = 0.75$
Symmetric extension of mixed model (Tajvidi, 1996): $C_{\theta_1,\theta_2}(u_1,u_2) = u_1u_2 \exp\left\{\frac{-\theta_2}{\left(\tilde{u}_1^{-\theta_1}+\tilde{u}^{-\theta_1}\right)^{\frac{-1}{\theta_1}}}\right\}$ $B_{\theta_1,\theta_2}(t) = \frac{1}{1+t}\left(\left[1-\theta_2\left(1+t^{-\theta_1}\right)^{\frac{-1}{\theta_1}}\right]\right)$ with $\theta_1 \ge 0; 0 < \theta_2 \le 1$	$\tilde{\delta}_C = 1-2^{(\theta_2-1)2^{\frac{-1}{\theta_1}}}$ $\bar{\delta}_C = 0.75$
Symmetric and polynomial extension of mixed model proposed by Klüppelberg (Beirlant *et al*., 2005), where $\theta \in [0,1]$ $C_\theta(u_1,u_2) = u_1u_2 \exp\left\{\theta\tilde{u}_1 - \frac{\theta\tilde{u}_1}{\tilde{u}_1+\tilde{u}_2}\right\}$; $B_\theta(t) = \frac{1}{1+t}\left(\left[1-\theta+\frac{\theta}{1+t}\right]\right)$	$\tilde{\delta}_C = 1-2^{\frac{-1}{2}(2-\theta)}$ $\bar{\delta}_C = 0.4$
Asymmetric and polynomial extension of mixed model proposed by Klüppelberg (Beirlant *et al*., 2005): $C_{\theta_1,\theta_2}(u_1,u_2) = u_1u_2 \exp\left\{(\theta_1+\theta_2)\tilde{u}_1 + \frac{\theta_1\tilde{u}_1^2}{\tilde{u}_1+\tilde{u}_2} + \frac{\theta_2\tilde{u}_1^3}{(\tilde{u}_1+\tilde{u}_2)^2}\right\}$ where, $\theta_1 \ge 0, \theta_1+3\theta_2 \ge 0; \theta_1+\theta_2 \le 1$ and $\theta_1+2\theta_2 \le 1$ $B_\theta(t) = \frac{1}{(1+t)^2}\left(\left[(1+t)(1-\theta_1-\theta_2)+\theta_1+\frac{\theta_2}{1+t}\right]\right)$	$\tilde{\delta}_C = 1-2^{\frac{-1}{4}(4-2\theta_1-3\theta_2)}$ $\bar{\delta}_C = 0.79$

DISCUSSION

The results of the study show that the dependence of all BEV copula, under given conditions on the marginal variable is totally described by the discordance function. These results differ from the previous equivalent dependence measures for BEV structures because they characterize both the joint dependence of the model and the possible conditions made on its univariate margins.

The application of the results shows also that the logistic and mixed families are the main models of parametric BEV copulas. Their symmetric and asymmetric subfamilies have been defined by their discordance functions, respectively.

Moreover, the average values of the median discordance of the main parametric copula have been computed. We found that the results conform to the solution of the problem considered earlier. This is seen at all point $(u_1, u_2) \in I^2$. We also note that (Theorem 3) a connection is established between the new dependence structure and the Pickands dependence function.

CONCLUSION

In this study, we have investigated about characterization of a conditional dependence of BEV copulas and their underlying distributions. We have built a new measure and function which describe this conditional dependence. Basic bivariate subfamilies of parametric extreme value copulas was characterized by this function and the average value of the new measure have been computed for the main parametric BEV copulas.

APPENDIX

Proof of Theorem 2

Before giving the proof of Theorem 2, we require the following proposition proved in Resnick (1987).

Proposition 1

Let H be a continuous distribution function defined on $\mathbb{R}^2$ ith density function h. Then H is max-id if and only if for all $(x_1, x_2) \in R^2$ we have:

$$\frac{\partial H(x_1,x_2)}{\partial x} \times \frac{\partial H(x_1,x_2)}{\partial y} \le h(x_1,x_2)H(x_1,x_2) \tag{11}$$

Proof of the Theorem

Let suppose that δ_C is non-decreasing in the second variable i.e.,

$$\delta_C(x_1,x_2) \le \delta_C(x_1,x_2+\varepsilon) \text{ with } (x_1,x_2) \in R^2,\ \varepsilon > 0 \tag{12}$$

Furthermore, for all $(x_1, x_2) \in R^2$ and all ε>0:

$$\delta_C(x_1+\varepsilon,x_2) = P(X_1 > x_1 / X_2 \le x_2) - P(x_1 < X_1 < x_1 + \varepsilon / X_2 \le x_2)$$

It follows that:

$$\delta_C(x_1+\varepsilon,x_2) \le \delta_C(x_1,x_2) \text{ with } (x_1,x_2) \in R^2,\ \varepsilon > 0 \tag{13}$$

Using both Eq. 12 and 13 yields that the function δ_C verifies the rectangular property:

$$\delta_C(x_1,x_2) - \delta_C(x'_1,x_2) - \delta_C(x_1,x'_2) + \delta_C(x'_1,x'_2) \ge 0 \tag{14}$$

Provided that $(x_1, x_2) \in R^2$ and $(x'_1, x'_2) \in R^2$ with $x_1 \le x_2$ and $x'_1 \le x'_2$ (15)

Then, under the condition (15)

$$[1-\delta_C(x_1,x_2)][1-\delta_C(x'_1,x'_2)] \ge [1-\delta_C(x_1,x'_2)][1-\delta_C(x'_1,x'_2)] \tag{16}$$

Since, from the Eq. 8 we have:

$$H(x_1,x_2) = [1-\delta_C(x_1,x_2)]H_2(x_2) \text{ for all} (x_1,x_2) \in \mathbb{R}^2$$

by multiplying the two members of Eq. 16 by the non-negative product $H_2(x_2)H_2(x'_2)$ yields:

$$H(x_1,x_2)H(x'_1,x'_2) \ge H(x'_1,x_2)H(x_1,x'_2) \tag{17}$$

i.e., that H is TP_2. Finaly, we conclude that H is max-id on since it has been shown (Joe, 1997) that Eq. 11 implies Eq. 17.

Proof of Theorem 3

The canonical representation of C gives that, for all $(u_1, u_2) \in I^2$:

$$C(u_1,u_2) = \exp\left\{-(\tilde{u}_1+\tilde{u}_2)A_C\left(\frac{\tilde{u}_1}{\tilde{u}_1+\tilde{u}_2}\right)\right\}$$

Where:

$$\tilde{u}_i = \mu_i - \frac{\sigma_i}{\xi_i}\left[1-(-\ln u_i)^{-\xi_i}\right]$$

the parameters $\{\mu_i \in R\}$, $\{\xi_i \in R\}$ and $\{\sigma_i>0\}$ verifying Eq. 5; i =1, 2.

Moreover, writing Eq. 2 equivalently yields:

$$C(u_1,u_2) = H\left(H_1^{-1}(u_1); H_2^{-1}(u_2)\right) \tag{18}$$

Where:

$$H_i^{-1}(u_i) = \tilde{u}_i; i=1,2$$

Then, it follows that the distribution function H in Eq. 18 with generalized margin:

$$H_i(x_i) = \exp\{-y_i(x_i)\} = \exp\left\{-\left[1+\xi_i\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right]^{\frac{-1}{\xi_i}}\right\}; i=1,2$$

is expressed by:

$$H(x_1,x_2) = \exp\left\{-\left(y_1(x_1)+y_2(x_2)\right)A_C\left(\frac{y_1(x_1)}{y_1(x_1)+y_2(x_2)}\right)\right\}; (x_1,x_2)\in R^2$$

where, A_C is the Pickands dependence function of C.

Furthermore, from Eq. 8 we have:

$$\delta_C(x_1,x_2) = 1-\frac{H(x_1,x_2)}{H_2(x_2)}; (x_1,x_2)\in R^2$$

Then,

$$\begin{aligned}\delta_C(x_1,x_2) &= 1-\exp\left\{-\left(y_1(x_1)+y_2(x_2)\right)A_C\left(\frac{y_1(x_1)}{y_1(x_1)+y_2(x_2)}\right)+y_2(x_2)\right\}\\ &= 1-\exp\left\{-\left(y_1(x_1)+y_2(x_2)\right)\left[A_C\left(\frac{y_1(x_1)}{y_1(x_1)+y_2(x_2)}\right)-\frac{y_2(x_2)}{y_1(x_1)+y_2(x_2)}\right]\right\}\end{aligned}$$

It follows that, for all $(x_1, x_2) \in R^2$ we have:

$$\delta_C(x_1,x_2) = 1-\exp\left\{-\left(y_1(x_1)+y_2(x_2)\right)B_C\left(\frac{y_2(x_2)}{y_1(x_1)}\right)\right\}$$

where, the function B_C is defined from R^+ to $\left[\frac{-1}{2}, 1\right]$ by $B_C(t) = A_C\left(\frac{1}{1+t}\right) - \frac{t}{1+t}$.

Proof of Theorem 4

Before giving the proof of Theorem 4, we require the following results (Fougères, 1996).

Remark 1

Each of the three univariate extreme value distribution are linked to the others by a simple functional transformation. Indeed,

$$X_i \to \Phi_{\frac{1}{\xi_i}} \Leftrightarrow \ln X_i \to \Lambda \Leftrightarrow \frac{-1}{X_i} \to \Psi_{\frac{1}{\xi_i}} \tag{19}$$

Proposition 2

The univariate margins $\{H_i;\, i = 1,...,n\}$ of a multivariate extreme value distribution H are univariate extreme value distribution. Moreover, if $F \in MDA(H)$ then $F_i \in MDA(H_i)$; $i = 1,...,n$.

Proof of the Theorem

- The univariate margins $\{H_i;\, i = 1, 2\}$ of H are expressed as:

$$H_i(x_i) = \exp\left\{-\left[1 + \xi_i\left(\frac{x_i - \mu_i}{\sigma_i}\right)\right]^{\frac{-1}{\xi_i}}\right\}; i = 1, 2$$

where, $\{\mu_i \in R\}$, $\{\xi_i \in R\}$ and $\{\sigma_i > 0\}$ are the parameters of X_i

From Eq. 2 and 18, respectively, we have:

$$H(x_1, x_2) = C_H(H_1(x_1), H_2(x_2));\ (x_1, x_2) \in R^2$$

and

$$C(u_1, u_2) = H(H_1^{-1}(u_1); H_2^{-1}(u_2))$$

Where:

$H_i^{-1}(u_i) = \tilde{u}_i;\ i = 1, 2$

Moreover Eq. 19 means also that the discordance function is independent from the choice of margins. That allows to associate to H_i any of the forms $\Phi_{\frac{1}{\xi_i}}; \Lambda$ and $\Psi_{\frac{1}{\xi_i}}$. Then the arbitrary choice $H_i \equiv \Lambda$ i.e., $\mu_i = \ln u_i$, $\xi_i = 0$ and $\sigma_i = 1$ yields C (u_1, u_2) = C $(-\ln u_1 - \ln u_2)$.

Then we conclude by expressing A_C in terms of B_C in the canonical representation of C.

- Since, C is the copula of H, for all $(u_1, u_2) \in I^2$

$$C(u_1, u_2) = \lim_{n \to +\infty} H^n(H_1^{-1}(u_1); H_2^{-1}(u_2))$$

Moreover, if $\{\tilde{H}_i; i=1,2\}$ denote the univariate margins of $\tilde{H}$ Then,

$$\tilde{H}(x_1,x_2)=C_{\tilde{H}}\left(\tilde{H}_1(x_1),\tilde{H}_2(x_2)\right);\ (x_1,x_2)\in R^2$$

Furthermore, $\tilde{H}\in MDA(H)$ implies that there exists normalizing sequences $\{\sigma_{i,n}>0\}$ and $\{\mu_{i,n}\in R\}$; i = 1, 2; such that:

$$\lim_{n\to+\infty}\tilde{H}^n\left(\sigma_{1,n}x_1+\mu_{1,n};\sigma_{2,n}x_2+\mu_{2,n}\right)=H(x_1;x_2);\ (x_1,x_2)\in R^2$$

which gives in Eq. 4.

$$\lim_{n\to+\infty}C_{\tilde{H}}{}^n\left(\tilde{H}_1\left(\sigma_{1,n}x_1+\mu_{1,n}\right);\tilde{H}_2\left(\sigma_{2,n}x_2+\mu_{2,n}\right)\right)=C\left(H_1(x_1);H_2(x_2)\right),\ (x_1,x_2)\in R^2$$

Then,

$$\lim_{n\to+\infty}C_{\tilde{H}}\left(\tilde{H}_1^n\left(\sigma_{1,n}x_1+\mu_{1,n}\right);\tilde{H}_2^n\left(\sigma_{2,n}x_2+\mu_{2,n}\right)\right)=C\left(H_1(x_1);\ H_2(x_2)\right)$$

It follows from proposition 2 that, for all $(x_1, x_2)\in R^2$.

$$C_{\tilde{H}}\left(H_1(x_1);H_2(x_2)\right)=C\left(H_1(x_1);H_2(x_2)\right)$$

Then, the uniqueness of the copula of a continuous distribution function implies that $C_{\tilde{H}}=C$.

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