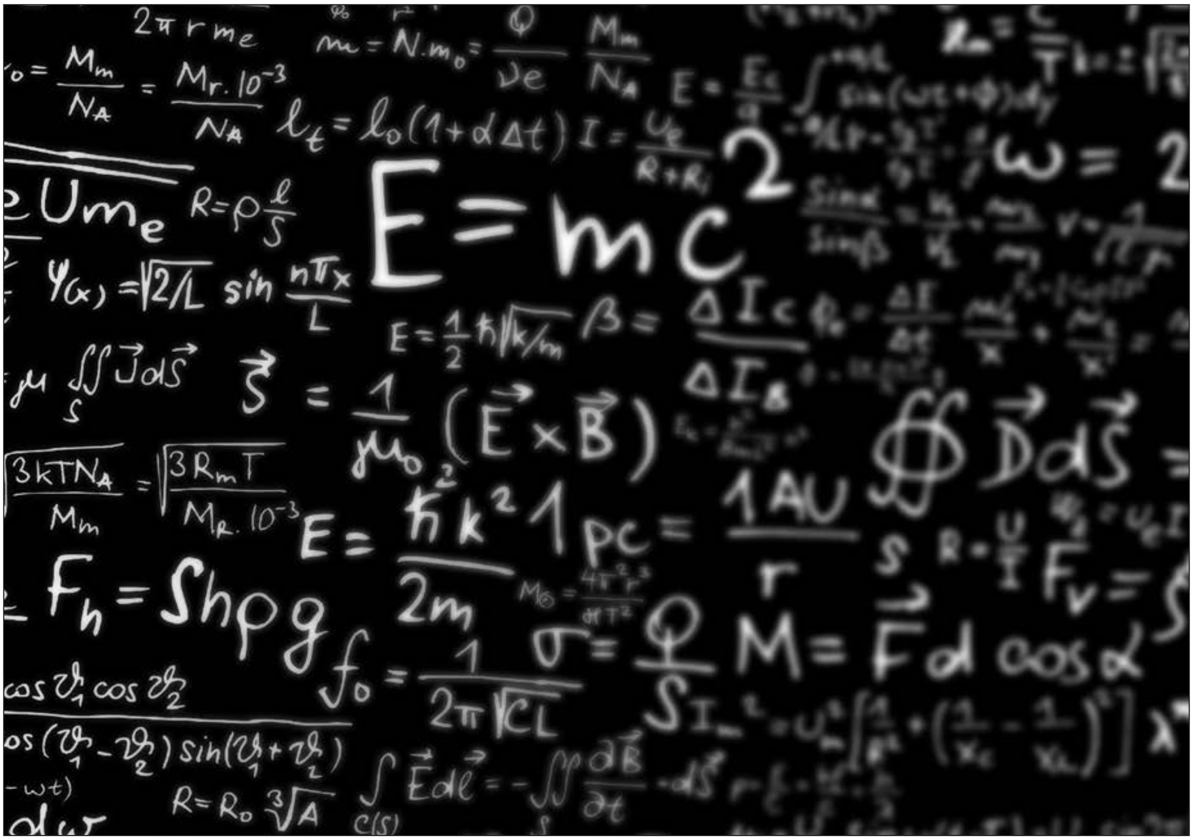

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An Analytic Proof of Bugeaud-Mignotte Theorem

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Abstract: The Bugeaud-Mignotte theorem concerns the Diophantine equation $\frac{X^n-1}{X-1}=Y^q$, when X = 10. It includes the fact that any integer greater than 2 and with all digits equal to 1 in base ten cannot be a pure power. It means that does not exist Y an integer strictly greater than 1 and q an integer strictly greater than 1 for which Y^q is a number with all digits equal to 1 in base ten.

Key words: Diophantine equations, analysis, sequences, series

INTRODUCTION

11, 111, 1111 in base ten are not pure powers, we can easily verify it, but is it true for a number of n digits equal 1 in base ten, for all n. This study is an answer to this question. A number with n digits equal to 1 (111…1) in base ten is equal to a number with n digits equal to 9 (999…9) in base ten divided by 9, which means $\frac{10^n-1}{10-1}$, the theorem stipulates that $\forall Y>1, q>1, n>1$, $\frac{10^n-1}{10-1}\neq Y^q$. This research proposes an original and analytic proof of this theorem called Bugeaud and Mignotte theorem (Bugeaud and Mignotte, 1999).

THE PROOF

Let Bugeaud-Mignotte equation:

$$\frac{10^n-1}{10-1}=Y^q$$

Then with $j^2 = -1$

$$10^n-9Y^q = 1 = 10^n+j(9jY^q)$$

We pose:

$$10^n = x,\ 9jY^q = y,\ u = 1,\ z = 9jY^q10^n$$

Lemma 1

$$u = x+jy \quad (1)$$

$$\frac{1}{z}=\frac{j}{x}+\frac{1}{y} \quad (2)$$

Proof of Lemma 1

$$u = 1 = 10^n + j(9jY^q) = x + jy$$

And:

$$\frac{1}{z}=\frac{1}{9jY^q10^n}=\frac{u}{xy}=\frac{x+jy}{xy}=\frac{j}{x}+\frac{1}{y}$$

We pose:

$x_1 = x,\ y_1 = y,\ u_1 = u,\ z_1 = z$

$\forall i \geq 1,\ x_i + jy_i = u_i,\ \frac{j}{x_i} + \frac{1}{y_i} = \frac{1}{z_i}$

$(x_i + jy_i)z_i = x_i y_i$

$x_i(y_i - z_i) = jy_i z_i = x_i y_{i+1}$

$y_i(x_i - jz_i) = x_i z_i = y_i x_{i+1}$

$x_i y_i x_{i+1} y_{i+1} = jx_i y_i z_i^2 \Rightarrow z_i = \sqrt{\frac{x_{i+1}y_{i+1}}{j}}$

$y_i = y_{i+1} + z_i = y_{i+1} + \sqrt{\frac{x_{i+1}y_{i+1}}{j}} = \sqrt{\frac{y_{i+1}}{j}}(\sqrt{x_{i+1}} + \sqrt{jy_{i+1}})$

$x_i = x_{i+1} + jz_i = x_{i+1} + j\sqrt{\frac{x_{i+1}y_{i+1}}{j}} = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{jy_{i+1}})$

$u_i = x_i + jy_i = (\sqrt{x_{i+1}} + \sqrt{jy_{i+1}})^2$

$\frac{1}{z_{i+1}} = \frac{j}{x_{i+1}} + \frac{1}{y_{i+1}}$

until infinty.

Lemma 2

The expression of the sequences is

$$x_i = \frac{x^{2^{i-1}}}{\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k})},\ y_i = j^{2^{i-1}-1}\frac{y^{2^{i-1}}}{\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k})}$$

Proof of Lemma 2

For I = 2, $x_2 = \frac{x^2}{x + yj},\ y_2 = j\frac{y^2}{x + yj}$, it is verified. We suppose the expressions true for I

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{jy_{i+1}}) = \sqrt{x_{i+1}}(x_i + jy_i)^{\frac{1}{2}}$$

$$x_{i+1} = \frac{x_i^2}{x_i + jy_i} = \frac{x^{2^i}}{\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k})^2}\left(\frac{1}{x^{2^{i-1}} + (jy)^{2^{i-1}}}\right)\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k}) = \frac{x^{2^i}}{\prod_{k=0}^{k=i-1}(x^{2^k} + (jy)^{2^k})}$$

$$y_i = \sqrt{\frac{y_{i+1}}{j}}(\sqrt{x_{i+1}} + \sqrt{jy_{i+1}}) = \sqrt{\frac{y_{i+1}}{j}}(x_i + jy_i)^{\frac{1}{2}}$$

$$y_{i+1} = j\frac{y_i^2}{x_i + jy_i} = j^{2^{i-1}-1}\frac{y^{2^i}}{\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k})^2}\left(\frac{1}{x^{2^{i-1}} + (jy)^{2^{i-1}}}\right)\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k}) = j^{2^{i-1}-1}\frac{y^{2^i}}{\prod_{k=0}^{k=i-1}(x^{2^k} + (jy)^{2^k})}$$

But:

$$\prod_{k=0}^{k=i-2}(x^{2^k} + (jy)^{2^k}) = \frac{x^{2^{i-1}} - (jy)^{2^{i-1}}}{x - jy}$$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}}(x - jy),\ y_i = j^{-1}\frac{(jy)^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}}(x - jy)$$

Lemma 3

x_i-jy_i = x-jy

Lemma 4

The lemmas 1, 2, 3 imply that jy-x = 0

Proof of Lemma 4

$$\sqrt{jx_k y_k} = jy_{k-1} - jy_k = x_{k-1} - x_k$$

$$\sum_{k=2}^{k=i+1} (\sqrt{jx_k y_k}) = x - x_2 + x_2 - x_3 + x_3 - x_4 + ... + x_i - x_{i+1} = x - x_{i+1}$$

$$\sum_{k=2}^{k=\infty} (\sqrt{jx_k y_k}) = \lim_{i\to\infty}(x - x_{i+1})$$

$$x = 10^n = 1 + 9Y^q > jy = 9Y^q \Rightarrow \lim_{i\to\infty}(jy_i) = \lim_{i\to\infty}\left(\frac{(jy)^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}}(x - jy)\right) = 0$$

$$\lim_{i\to\infty}(x_i) = \lim_{i\to\infty}\left(\frac{x^{2^{i-1}}}{x^{2^{i-1}} - (jy)^{2^{i-1}}}(x - jy)\right) = x - jy$$

$$\sum_{k=2}^{k=\infty} (\sqrt{jx_k y_k}) = \lim_{i\to\infty}(x - x_{i+1}) = x - (x - jy) = jy$$

$$\sum_{k=2}^{k=i} ((-1)^k \sqrt{jx_k y_k}) = x - x_2 - x_2 + x_3 + ... + (-1)^i (x_{i-1} - x_i)$$

$$= x - 2x_2 + 2x_3 - ... + 2(-1)^i x_{i-1} + (-1)^{i+1} x_i$$

$$= 2\sum_{k=1}^{k=i} ((-1)^{k+1} x_k) - x - (-1)^{i+1} x_i = 2\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k) + x + (-1)^{i+1} x_i$$

$$= \sum_{k=1}^{k=i} ((-1)^{k+1} x_k) + \sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)$$

$$\sum_{k=2}^{k=i} ((-1)^k \sqrt{jx_k y_k}) = jy - jy_2 - jy_2 + jy_3 + ... + (-1)^i (jy_{i-1} - jy_i)$$

$$= jy - 2jy_2 + 2jy_3 - ... + 2(-1)^i jy_{i-1} + (-1)^{i+1} jy_i$$

$$= 2\sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) - jy - (-1)^{i+1} jy_i = 2\sum_{k=2}^{k=i-1} ((-1)^{k+1} jy_k) + jy + (-1)^{i+1} jy_i$$

$$= \sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) + \sum_{k=2}^{k=i-1} ((-1)^{k+1} jy_k)$$

$$\Rightarrow L = \sum_{k=2}^{k=\infty} ((-1)^k \sqrt{jx_k y_k}) = 2\lim_{i\to\infty}\left(\sum_{k=1}^{k=i} ((-1)^{k+1} x_k)\right) - x - \lim_{i\to\infty}((-1)^{i+1} x_i)$$

$$= 2\lim_{i\to\infty}\left(\sum_{k=1}^{k=i} ((-1)^{k+1} jy_k) - jy - \lim_{i\to\infty}((-1)^{i+1} jy_i)\right) = 2\sum_{k=1}^{k=\infty} ((-1)^{j+1} jy_j) - jy = 2\sum_{k=2}^{k=\infty} ((-1)^{j+1} jy_j) + jy$$

$$= 2\lim_{i\to\infty}\left(\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)\right) + x + \lim_{i\to\infty}((-1)^{i+1} x_i)$$

$$= \lim_{i\to\infty}\left(\sum_{k=1}^{k=i} ((-1)^{k+1} x_k)\right) + \lim_{i\to\infty}\left(\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)\right) \text{ is convergent}$$

We pose:

$$L_1 = \lim_{i\to\infty}\left(2\sum_{k=1}^{k=i} ((-1)^{k+1} x_k)\right) - x + a\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$L_2 = \lim_{i\to\infty}\left(2\sum_{k=2}^{k=i-1} ((-1)^{k+1} x_k)\right) + x - b\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$, with $(a+1)(b+1) \neq 0$

$$L_1 = L + (a+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$L_2 = L - (b+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = L_1 - (a + b + 2)\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$L = \frac{L_1 + L_2 + (b - a)\lim_{i\to\infty}((-1)^{i+1}(x - jy))}{2}$$

We pose:

$$L = \alpha_1 L_1 = \alpha_2 L_2 = \frac{L_1 + L_2 + (b - a)\lim_{i\to\infty}((-1)^{i+1}(x - jy))}{2}$$

$$(\alpha_1-1)L_1 = L - L_1 = \frac{L_1 + L_2 + (b-a)\lim_{i\to\infty}((-1)^{i+1}(x-jy))}{2} - L_1$$

$$= \frac{L_2 - L_1 + (b-a)\lim_{i\to\infty}((-1)^{i+1}(x-jy))}{2}$$

$$= L_2 - L + (b-a)\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = (1-\alpha_2)L_2 + (b-a)\lim_{i\to\infty}((-1)^{i+1}(x-jy))$$

$$= -(a+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy))$$

$$\Rightarrow \lim_{i\to\infty}((-1)^{i+1}(x-jy)) = \frac{(\alpha_1-1)L_1}{-(a+1)} = \frac{(\alpha_2-1)L_2}{(b+1)}$$

$$\alpha_1 = \frac{L}{L_1} = \frac{\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x - \lim_{i\to\infty}((-1)^{i+1}(x-jy))}{\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x + a\lim_{i\to\infty}((-1)^{i+1}(x-jy))}$$

$$= 1 + \frac{-(a+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy))}{\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x + a\lim_{i\to\infty}((-1)^{i+1}(x-jy))}$$

$$\alpha_2 = \frac{L}{L_2} = \frac{\lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x + \lim_{i\to\infty}((-1)^{i+1}(x-jy))}{\lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x - b\lim_{i\to\infty}((-1)^{i+1}(x-jy))}$$

$$= 1 + \frac{(b+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy))}{\lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x - b\lim_{i\to\infty}((-1)^{i+1}(x-jy))}$$

Let $a \mid -(a+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = \alpha L_1$

$$= \alpha\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - \alpha x + \alpha a\lim_{i\to\infty}((-1)^{i+1}(x-jy)) \Rightarrow \alpha_1 = 1+\alpha$$

$b \mid (b+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = \beta L_2$

$$= \beta\lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + \beta x - \beta b\lim_{i\to\infty}((-1)^{i+1}(x-jy)) \Rightarrow \alpha_2 = 1+\beta$$

$$-((1+\frac{1}{\alpha})a + \frac{1}{\alpha})\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x$$

And:

$$((\frac{1}{\beta}+1)b + \frac{1}{\beta})\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = \lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x$$

$$\Rightarrow -(\frac{1}{\alpha}+1)(a+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = L$$

And:

$$(\frac{1}{\beta}+1)(b+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy)) = L$$

If we make the hypothesis that $\lim_{i\to\infty}((-1)^{i+1}(x-jy)) \neq 0$

$$\lim_{i\to\infty}((-1)^{i+1}(x-jy)) \neq 0 \Rightarrow a = \frac{-\alpha L}{(\alpha+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy))} - 1$$

And:

$$b = \frac{\beta L}{(\beta+1)\lim_{i\to\infty}((-1)^{i+1}(x-jy))} - 1$$

$$L_1 = \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x - \lim_{i\to\infty}((-1)^{i+1}(x-jy)) - \frac{\alpha L}{(\alpha+1)} = L - \frac{\alpha L}{\alpha+1} = \frac{1}{\alpha+1}L$$

$$L_2 = \lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x + \lim_{i\to\infty}((-1)^{i+1}(x - jy)) - \frac{\beta L}{(\beta+1)} = L - \frac{\beta L}{\beta+1} = \frac{1}{\beta+1}L$$

But:

$$-(a+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = \alpha L_1$$

$$= \alpha\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - \alpha x + \alpha a\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$\Rightarrow a\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = -\lim_{i\to\infty}((-1)^{i+1}(x - jy)) - \alpha L_1$$

$$= L_1 - \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + x = aL - a\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + ax$$

$$\Rightarrow (\alpha+1)L_1 = \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) - x - \lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$= L = (\alpha+1)aL - a(\alpha+1)\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + a(\alpha+1)x$$

$$+(\alpha+1)\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) - (\alpha+1)x$$

$$\Rightarrow (1 - a(\alpha+1))L = -a(\alpha+1)\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + a(\alpha+1)x$$

$$+(\alpha+1)\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) - (\alpha+1)x$$

$$= (1 - a(\alpha+1))\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + (-1 + a(\alpha+1))x$$

$$+(-1 + a(\alpha+1))\lim_{i\to\infty}((-1)^{i+1}(x - jy))$$

$$\Rightarrow -\alpha\lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k))) + \alpha x + (a(\alpha+1) - 1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0$$

$$= -\alpha L + (a(\alpha+1) - \alpha - 1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0$$

$$= -\alpha L + (a - 1)(\alpha+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0$$

$$= -\alpha L - \alpha(\alpha+1)L_1 - 2(\alpha+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0$$

$$= -2\alpha L - 2(\alpha+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0$$

$$\Rightarrow \alpha L = -(\alpha+1)\lim_{i\to\infty}((-1)^{i+1}(x - jy)) \text{ is convergent}$$

$$\forall\alpha \mid \alpha(\alpha+1)(\alpha-1) \neq 0$$

$$\Rightarrow -\lim_{i\to\infty}((-1)^{i+1}(x - jy)) = \frac{\alpha-1}{\alpha+1}L = 0$$

$$\Rightarrow \lim_{i\to\infty}((-1)^{i+1}(x - jy)) = 0 \Rightarrow x - jy = 0$$

The hypothesis $\lim_{i\to\infty}((-1)^{i+1}(x - jy)) \neq 0$ is false and

$$L = \sum_{k=2}^{k=\infty}((-1)^k\sqrt{jx_k y_k}) = \sum_{k=1}^{k=\infty}(\sqrt{jx_{2k}y_{2k}}) - \sum_{k=1}^{k=\infty}(\sqrt{jx_{2k+1}y_{2k+1}}) = 0,$$

$$\sum_{k=2}^{k=\infty}(\sqrt{jx_k y_k}) = \sum_{k=1}^{k=\infty}(\sqrt{jx_{2k}y_{2k}}) + \sum_{k=1}^{k=\infty}(\sqrt{jx_{2k+1}y_{2k+1}}) = 2\sum_{k=1}^{k=\infty}(\sqrt{jx_{2k}y_{2k}}) = 2\sum_{k=1}^{k=\infty}(\sqrt{jx_{2k+1}y_{2k+1}}) = jy = x$$

In all cases

$$L_1 = \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) - x = L_2 = \lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + x = 0$$

$$\Rightarrow \sum_{k=1}^{k=\infty}((-1)^{k+1}x_k) = \sum_{k=1}^{k=\infty}((-1)^{k+1}jy_k) = \frac{x}{2} = \frac{jy}{2}$$

$$\Rightarrow \lim_{i\to\infty}(2\sum_{k=1}^{k=i}((-1)^{k+1}x_k)) = \lim_{i\to\infty}(2\sum_{k=2}^{k=i-1}((-1)^{k+1}x_k)) + 2x = \lim_{i\to\infty}(2\sum_{k=1}^{k=i-1}((-1)^{k+1}x_k))$$

$$= \lim_{i\to\infty}(2\sum_{k=1}^{k=i-1}((-1)^{k+1}x_k)) + 2\lim_{i\to\infty}((-1)^{i+1}x_i) \Rightarrow \lim_{i\to\infty}((-1)^{i+1}x_i) = 0 = \lim_{i\to\infty}((-1)^{i+1}(x-jy))$$

$$\Rightarrow x - jy = 0$$

$\sum_{k=1}^{k=\infty}((-1)^{j+1}x_j)$ is convergent, then the general term of the series tends to zero

$$\Rightarrow \lim_{i\to\infty}((-1)^{i+1}x_i) = \lim_{i\to\infty}((-1)^{i+1}(x-jy)) = 0 \Rightarrow x = jy$$

$$\Rightarrow x - jy = 0 = 10^n + 9Y^q = 2(10^n) - 1 = 18Y^q + 1$$

And it is impossible : the existence of Y and q is impossible!

CONCLUSION

Bugeaud-Mignotte equation has effectively no solution, and an analytic proof exists. The generalization is $\frac{X^n-1}{X-1} = Y^q$ is it possible? It seems that there are only three solutions:

$$\frac{3^5-1}{3-1} = 11^2, \frac{7^4-1}{7-1} = 20^2, \frac{18^3-1}{18-1} = 7^3$$

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